Math 155 (Lecture 19)

October 18, 2011

We begin this lecture with a review of partially ordered sets.

Definition 1. Let A be a set. A *partial ordering* of A is a binary relation $\leq$ satisfying the following axioms:

($Reflexivity$) For each $a \in A$, we have $a \leq a$.

($Transitivity$) If $a \leq b$ and $b \leq c$, then $a \leq c$.

($Antisymmetry$) If $a \leq b$ and $b \leq a$, then $a = b$.

A *partially ordered set* is a pair $(A, \leq)$, where A is a set and $\leq$ is a partial ordering on A.

Example 2. Let A be the set of real numbers (or rational numbers, or integers). Define $a \leq b$ if $b - a$ is nonnegative. Then $\leq$ is a partial ordering on A.

Example 3. Let S be a set, and let $P(S)$ denote the collection of all subsets of S. For $T, T' \in P(S)$, write $T \leq T'$ if $T \subseteq T'$. This relation makes $P(S)$ into a partially ordered set.

Example 4. Let $\mathbf{Z}_{>0}$ be the set of positive integers. Then $\mathbf{Z}_{>0}$ is partially ordered by *divisibility*: we can define $a \leq b$ if and only if $a|b$.

Definition 5. Let A be a partially ordered set. We say that an element $a \in A$ is a *least* element of A if $a \leq b$ for all $b \in A$. We say that a is a *minimal* element of A if $b \leq a$ implies $b = a$.

We say that $a \in A$ is a *greatest element* if $b \leq a$ for all $b \in A$. We say that $a \in A$ is a *maximal element* if $b \geq a$ implies $b = a$.

Let A be a partially ordered set. If A has a least element a, then a is unique, and is also a minimal element of A. However, the converse fails: a minimal element of A is generally not a least element of A, and a partially ordered set A can have many minimal elements (in which case none of them can be least elements).

Example 6. Let A be an arbitrary set. For $a, b \in A$, write $a \leq b$ if $a = b$. Then $\leq$ is a partial ordering on A, which is called the *discrete ordering*. Every element of A is minimal (and maximal). However, A has no least (or greatest) element unless it has only a single element.

Since this is a course in combinatorics, we will be mostly interested in the case of *finite* linearly ordered sets.

Lemma 7. *Let A be a finite partially ordered set. If A is nonempty, then A has at least one minimal element.*

Proof. Since A is nonempty, we can choose an element $a_0 \in A$. If A is minimal, then we are done. Otherwise, there exists an element a_1 such that $a_1 \leq a_0$ and $a_1 \neq a_0$. If a_1 is minimal, then we are done. Otherwise we can choose an element a_2 such that $a_2 \leq a_1$ and $a_2 \neq a_1$. Proceeding in this way, we produce a sequence

$$a_0 \geq a_1 \geq a_2 \geq \cdots$$

Since A is finite, this sequence must have some repeated terms: that is, we must have $a_i = a_j$ for some $j \neq i$. Without loss of generality we may assume that $j > i$. Then $a_i = a_j \leq a_{i+1}$ and $a_{i+1} \leq a_i$. Using antisymmetry we deduce that $a_{i+1} = a_i$, which contradicts our choice of a_{i+1}. □

Remark 8. The proof of Lemma 7 actually shows something stronger. Namely, if A is finite, then for every $a \in A$ we can find a minimal element $b \in A$ such that $b \leq a$.

Definition 9. Let $(A, \leq)$ be a partially ordered set. We say that $\leq$ is a *linear ordering* on A if, for every pair of elements $a, b \in A$, we have either $a \leq b$ or $b \leq a$. In this case, we also say that $\leq$ is a *total ordering* of A, that A is a *linearly ordered* set, or that A is a *totally ordered* set.

Example 10. The usual ordering on the real numbers (or rational numbers, or integers) is a linear ordering. The partial ordering of Examples 3 and 4 are not linear orderings.

Remark 11. If A is a linearly ordered set, then every minimal element of A is a least element of A. Using Lemma 7, we deduce that if A is finite and nonempty, then it contains a least element. The same argument shows that A has a greatest element.

We have already considered finite linearly ordered sets in our study of species. These have a very simple structure:

Proposition 12. *Let A be a finite linearly ordered set. Then there is a unique order-preserving bijection $\pi : \{1, 2, \ldots, n\} \to A$, for some integer n.*

Proof. Take n to be the number of elements of A, and work by induction on n. If $n > 0$, then A has a greatest element a (Remark 11), and the bijection π must clearly satisfy $\pi(n) = a$. Now apply the inductive hypothesis to the set $A - \{a\}$. □

Definition 13. Let A be a partially ordered set. Then any subset $A_0 \subseteq A$ inherits the structure of a partially ordered set. We say that A_0 is a *chain* if it is linearly ordered, and that A_0 is an *antichain* if the ordering on A_0 is discrete (Example 6).

Up to isomorphism, we can produce all partially ordered sets by combining Example 3 with the preceding observation.

Proposition 14. *Let A be a partially ordered set. Then A is isomorphic (as a partially ordered set) to a subset of $P(S)$, for some set S.*

Proof. For each $a \in A$, let $A_{\leq a} = \{b \in A : b \leq a\}$. The construction $a \mapsto A_{\leq a}$ determines a map $\phi : A \to P(A)$. We claim that ϕ is an isomorphism of partially ordered sets from A onto a subset of $P(A)$. In other words, we claim that:

(i) The map ϕ is injective.

(ii) For $a, b \in A$, we have $a \leq b$ if and only if $\phi(a) \subseteq \phi(b)$.

Note that (i) is just a special case of (ii): if (ii) is satisfied and $\phi(a) = \phi(b)$, then $a \leq b$ and $b \leq a$ so that $a = b$ by antisymmetry.

To prove (ii), we first note that if $a \leq b$ and $c \in A_{\leq a}$, then $c \leq a$. By transitivity we get $c \leq b$ so that $c \in A_{\leq b}$. This proves that $\phi(a) \subseteq \phi(b)$. Conversely, suppose that $a, b \in A$ are arbitrary and that $A_{\leq a} \subseteq A_{\leq b}$. Since $a \in A_{\leq a}$, we deduce that $a \in A_{\leq b}$, which means that $a \leq b$. □

Definition 15. Let $(A, \leq_A)$ and $(B, \leq_B)$ be partially ordered sets. We say that a map $\phi : A \to B$ is *order-preserving*, or *monotone*, if $a \leq_A a'$ implies $\phi(a) \leq_B \phi(a')$.

Proposition 16. *Let A be a partially ordered set. Then there exists an order-preserving bijection $\phi : A \to B$, where B is a linearly ordered set.*

In other words, any partial ordering on a set A can be refined to a linear ordering.

Proof. We will give the proof when A is finite (the result is still true when A is infinite, at least if you are willing to assume the axiom of choice). Let $n = |A|$, and proceed by induction on n. The case $n = 0$ is trivial. Assume therefore that $n > 0$, so that A is nonempty. Let $a \in A$ be a maximal element (Lemma 7). The inductive hypothesis (together with Proposition 12, say) imply that there exists an order-preserving bijection $\phi_0 : A - \{a\} \to \{1, 2, \ldots, n-1\}$. We now extend ϕ_0 to a map $\phi : A \to \{1, \ldots, n\}$ by setting $\phi(a) = n$. Since a was chosen maximal, this map is order-preserving. □

If A is a finite partially ordered set, then the *width* of A is defined to be the maximum size of an antichain of A. Note that A is linearly ordered if and only if it has width ≤ 1 (that is, it has no 2-element antichains).

Question 17. Let S be a finite set of size n. What is the width of $P(S)$?

To get an answer to this question, we should try to find some antichains in $P(S)$. There are some obvious choices. For example, fix an integer $k \leq n$, and let $P_k(S) = \{T \subseteq S : |T| = k\}$. Then $P_k(S)$ is an antichain: if T and T' are two different subsets of S both having size k, then neither can contain the other. The number of elements of $P_k(S)$ is given by the binomial coefficient $\binom{n}{k}$. This gives a lower bound for the width of $P(S)$: it must be at least $\binom{n}{k}$, for each $0 \leq k \leq n$.

Theorem 18 (Sperner)**.** *Let S be a set of size n. Then the width of $P(S)$ is* $\max\{\binom{n}{k}\}_{0 \leq k \leq n}$.

We can be more precise: it is not hard to see that the binomial coefficient $\binom{n}{k}$ is maximized when k is as close to $\frac{n}{2}$ as possible. We therefore see that the width of $P(S)$ is $\binom{n}{m}$, where

$$m = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Proof of Theorem 18. Let $C = \max\{\binom{n}{k}\}_{0 \leq k \leq n}$, and let $K_1, K_2, \ldots, K_q$ be an antichain in $P(S)$. We wish to show that $q \leq C$.

Let X be the set of all bijections $\pi : \{1, \ldots, n\} \to S$, so that X has $n!$ elements. For each $1 \leq i \leq q$, let $X_i \subseteq X$ be the collection of those bijections for which

$$K_i = \pi^{-1}\{1, \ldots, |K_i|\}.$$

To give an element of X_i, we must give separate bijections

$$\{1, \ldots, |K_i|\} \to K_i \qquad \{|K_i| + 1, \ldots, n\} \to S - K_i.$$

It follows that $|X_i| = k_i!(n - k_i)!$, where $k_i = |K_i|$.

We claim that the sets X_i are disjoint. To prove this, consider a permutation $\pi \in X_i \cap X_j$. We may assume without loss of generality that $k_i \leq k_j$. Then $\{1, \ldots, k_i\} \subseteq \{1, \ldots, k_j\}$, so that

$$K_i = \pi^{-1}\{1, \ldots, k_i\} \subseteq \pi^{-1}\{1, \ldots, k_j\} = K_j.$$

Since $K_1, \ldots, K_q$ is an antichain, we deduce that $i = j$.

We now deduce

$$\begin{aligned} n! &= |X| \\ &\geq \sum_{1 \leq i \leq q} |X_i| \\ &= \sum_{1 \leq i \leq q} k_i!(n-k_i)! \\ &= \sum_{1 \leq i \leq q} \frac{n!}{\binom{n}{k_i}} \\ &\geq \sum_{1 \leq i \leq q} \frac{n!}{C} \\ &= n!\frac{q}{C}. \end{aligned}$$

Multiplying by C and dividing by $n!$, we get $q \leq C$. $\square$