Math 155 (Lecture 17)

October 14, 2011

Our goal in this lecture is to study the behavior of the cycle index under composition of species. More precisely, we have the following result:

Theorem 1. *For every species S and every integer $i \geq 1$, let $Z_S^{(i)}(s_1, s_2, \ldots) = Z_S(s_i, s_{2i}, s_{3i}, \ldots)$. If T is a species with $T[\emptyset] = \emptyset$, then we have*

$$Z_{S \circ T}(s_1, s_2, \ldots) = Z_S(Z_T(s_1, s_2, \ldots), Z_T(s_2, s_4, \ldots), \ldots) = Z_S(Z_T^{(1)}, Z_T^{(2)}, Z_T^{(3)}, \ldots).$$

Before giving a proof of Theorem 1, let us see that it generalizes several things that we already know.

Example 2. Let S and T be species with $T[\emptyset] = \emptyset$. Then $Z_T(0, 0, \ldots) = 0$, so that

$$\begin{aligned} F_{S \circ T}(x) &= Z_{S \circ T}(x, 0, \ldots) \\ &= Z_S(Z_T(x, 0, \ldots), Z_T(0, 0, \ldots), \ldots) \\ &= Z_S(F_T(x), 0, 0, \ldots) \\ &= F_S(F_T(x)). \end{aligned}$$

In other words, Theorem 1 generalizes our formula

$$F_{S \circ T} = F_S \circ F_T$$

for the composition of species.

Example 3. Let G be a subgroup of the permutation group of a finite set X, and let H be a subgroup of the permutation group of a finite nonempty set Y¿ Consider the species $T = S_{(G,X)} \circ S_{(H,Y)}$. By definition, a T-structure on a finite set J consists of the following data:

(i) An equivalence relation $\sim$ on J.

(ii) A bijection of $J/\sim \simeq X$, which is well-defined up to the action of G.

(iii) A bijection of each equivalence class $E \in J/\sim$ with Y, which is well-defined up to the action of H.

In particular, every T-structure is isomorphic to $(X \times Y, \eta)$, where η is the T-structure on $X \times Y$ corresponding to the equivalence relation of "having the same X-coordinate", where we take the bijections described in (ii) and (iii) to be the identity maps. It follows that T is a molecular species.

Let us compute the automorphism group of $(X \times Y, \eta)$. By definition, it is the group of all permutations π of $X \times Y$ which have the following properties:

- π preserves the equivalence relation of "having the same X-coordinate". That is, we can write $\pi(x, y) = (\sigma(x), \tau_x(y))$ for some permutation $\sigma : X \to X$. Note that for each $x \in X$, we have

$$\tau_x(y) = \tau_x(y') \Leftrightarrow \pi(x, y) = \pi(x, y') \Leftrightarrow y = y',$$

so that τ_x is a permutation of X.

- The permutation $\sigma : X \to X$ preserves the $S_{(G,X)}$-structure given by the identity bijection $\mathrm{id} : X \to X$. That is, $\sigma \in G$.
- For each $x \in X$, the permutation $\tau_x : Y \to Y$ belongs to the group H.

In other words, we can identify elements of the automorphism group $\mathrm{Aut}(X \times Y, \eta)$ with pairs $(\sigma, \{\tau_x\}_{x \in X})$ where $\sigma \in G$ and each τ_x belongs to H. It is not difficult to show that this group coincides with the wreath product $H \wr G$ we studied in a previous lecture, with its canonical action on $X \times Y$. Theorem 1 then gives

$$\begin{aligned} Z_{H \wr G}(s_1, s_2, \ldots) &= Z_T(s_1, s_2, \ldots) \\ &= Z_{S_{(G,X)}}(Z_{S_{(H,Y)}}(s_1, s_2, \ldots) Z_{S_{(H,Y)}}(s_2, s_4, \ldots), \ldots) \\ &= Z_G(Z_H(s_1, s_2, \ldots), Z_H(s_2, s_4, \ldots), \ldots). \end{aligned}$$

which recovers our formula for the cycle index of a wreath product of two group actions.

Example 4. Let G be a finite group acting on a finite set X of size n and let $S = S_{(G,X)}$. Fix a finite set $\{c_1, c_2, \ldots, c_t\}$, and define a species T by the formula

$$T[I] = \begin{cases} \{c_1, \ldots, c_t\} & \text{if } |I| = 1 \\ \emptyset & \text{if } |I| \neq 1. \end{cases}$$

The cycle index of T is given by the formula

$$Z_T = \sum_{n \geq 0} \frac{1}{n!} \sum_{\sigma \in \Sigma_n} |T[\langle n \rangle]^\sigma| Z_\sigma = |T[\langle 1 \rangle]| Z_{\mathrm{id}_{\langle 1 \rangle}} = t s_1.$$

For every finite set Y, an element of $(S \circ T)[Y]$ consists of the following data:

(i) An equivalence relation $\sim$ on Y.

(ii) An element of the set $S[Y/\sim] = G \backslash \mathrm{Bij}(X, Y/\sim)$.

(iii) For each equivalence class $E \in Y/\sim$, an element of $T[E]$.

Data (iii) can be supplied only when each equivalence class E has size 1, in which case the equivalence relation $\sim$ is trivial (that is, $Y/\sim$ can be identified with Y). In this case, data (ii) consists of an element of $G \backslash \mathrm{Bij}(X, Y)$, while data ($iii$) is a map $Y \to \{c_1, \ldots, c_t\}$: that is, a coloring of Y with the set of colors $\{c_1, \ldots, c_t\}$.

Without loss of generality, we can take $X = \{1, \ldots, n\}$, so that X is acted on by the group Σ_n. Then the isomorphism classes of $S \circ T$-structures are parametrized by the quotient

$$((G \backslash \mathrm{Bij}(X, X)) \times \{c_1, \ldots, c_t\}^X)/\Sigma_n \simeq G \backslash ((\mathrm{Bij}(X, X) \times \{c_1, \ldots, c_t\}^X)/\Sigma_n) \simeq G \backslash \{c_1, \ldots, c_t\}^X.$$

We therefore obtain

$$\sum_{m \geq 0} |(S \circ T)[\langle m \rangle]/\Sigma_m| x^m = |G \backslash \{c_1, \ldots, c_t\}^X| x^n$$

Using Theorem 1, we see that this is given by

$$\begin{aligned} Z_{S \circ T}(x, x^2, x^3, \ldots) &= Z_S(Z_T(x, x^2, \ldots), Z_T(x^2, x^4, \ldots), \ldots) \\ &= Z_S(tx, tx^2, tx^3, \ldots) \\ &= Z_G(tx, tx^2, tx^3, \ldots). \end{aligned}$$

Taking $x = 1$, we recover Polya's enumeration theorem

$$|G \backslash \{c_1, \ldots, c_t\}^X| = Z_G(t, t, t, \ldots).$$

Remark 5. We can recover the more refined version of Polya's theorem (which keeps track of the number of times each color is used) using an analogue of Theorem 1 in the setting of *multivariate species*: that is, species which depend on several variables, rather than just one. We will not pursue this point further.

Proof of Theorem 1. Let us examine the cycle index of the composition $S \circ T$. If X is a finite set, we will identify the elements of $(S \circ T)[X]$ with triples $(\sim, \eta, \rho)$, where $\sim$ is an equivalence relation on X, $\eta \in S[X/\sim]$, and $\rho \in \prod_{E \in S/\sim} T[E]$. We can write

$$Z_{S\circ T} = \sum_{n \geq 0} \frac{1}{n!} \sum_{\pi \in \Sigma_n} \sum_{\pi(\sim,\eta,\rho)=(\sim,\eta,\rho)} Z_\pi.$$

Let $Z_{S\circ T,m}$ be the sum of all those terms where the quotient $\langle n \rangle / \sim$ has size exactly m, so that $Z_{S\circ T} = \sum_{m \geq 0} Z_{S \circ T,m}$. Fix a bijection $\alpha : \langle m \rangle \simeq \langle n \rangle / \sim$, giving an enumeration $E_1, E_2, \ldots, E_m$ of the equivalence classes of $\langle n \rangle$ (there are $m!$ such enumerations). Then giving a permutation π of $\langle n \rangle$ fixing $\sim$ is equivalent to supplying the following data:

- (i) A permutation σ of $\{1, \ldots, m\}$.
- (ii) For each $i \in \{1, \ldots, m\}$, a bijection $u_i : E_i \to E_{\sigma(i)}$.

In this case π fixes a triple $(\sim, \eta, \rho) \in (S \circ T)[\langle n \rangle]$ if and only if $\sigma(\eta) = \eta$ and $T[u_i](\rho_i) = \rho_{\sigma(i)}$ for $1 \leq i \leq m$. Let's now regard σ as fixed and decompose the set $\langle m \rangle$ into cycles $C_1, C_2, \ldots, C_b$ under σ. Choose a representative i_j from each cycle C_j, and let C_j have length l_j. Let $\tau_j : E_{i_j} \to E_{i_j}$ be the composition of the maps

$$E_{i_j} \stackrel{u_{i_j}}{\to} E_{\sigma(i_j)} \stackrel{u_{\sigma(i_j)}}{\to} \cdots \to E_{\sigma^{l_j}(i_j)} = E_{i_j}.$$

If ρ is fixed by π, then ρ_{i_j} determines $\rho_{\sigma^p(i_j)}$ for each $p \geq 0$, and ρ_{i_j} is fixed by τ_j. We may therefore write $Z_{S\circ T,m}$ as a sum

$$\frac{1}{m!} \sum_{n \geq 0, \sim, \alpha} \sum_{\sigma \in \Sigma_m} \frac{|S[\langle m \rangle]^\sigma|}{n!} \sum_{u_i \in \mathrm{Bij}(E_i, E_{\sigma(i)})} (\prod_{1 \leq j \leq b} |T[E_{i_j}|^{\tau_j}|) Z_\pi.$$

Note that each cycle of π corresponds to a cycle of one of the permutations τ_j, but has l_j times the length. Also note that, provided that E_i and $E_{\sigma(i)}$ have the same cardinality e_j for each $i \in C_j$, each of sets $\mathrm{Bij}(E_i, E_{\sigma(i)})$ has size $e_j!$. We can therefore write this sum as

$$\frac{1}{m!} \sum_{n \geq 0, \sim, \alpha} \frac{|S[\langle m \rangle]^\sigma|}{n!} (\prod_{1 \leq j \leq b} \sum_{\tau_j \in \Sigma_{e_j}} (e_j!)^{l(j)-1} |T[\langle e_j \rangle|^{\tau_j}|) Z_{\tau_j}^{(l_j)}.$$

Once the sizes e_j (and therefore the integer $n = \sum l_j e_j$) have been fixed, the number of choices for $(\sim, \alpha)$ is given by the multinomial coefficient $\frac{n!}{\prod (e_j!)^{l_j}}$. We may therefore write instead

$$\begin{aligned} Z_{S\circ T,m} &= \frac{1}{m!} \sum_{\sigma \in \Sigma_m} \frac{n! |S[\langle m \rangle]^\sigma|}{n! (e_1!)^{l_1} \cdots (e_b!)^{l_b}} \prod_{1 \leq j \leq b} \sum_{\tau_j \in \Sigma_{e_j}} (e_j!)^{l(j)-1} |T[\langle e_j \rangle|^{\tau_j}|) Z_{\tau_j}^{(l_j)} \\ &= \sum_{\sigma \in \Sigma_m} \frac{|S[\langle m \rangle]^\sigma|}{m!} \prod_{1 \leq j \leq b} \sum_{\tau_j \in \Sigma_{e_j}} \frac{|T[\langle e_j \rangle|^{\tau_j}|)}{e_j!} Z_{\tau_j}^{(l_j)} \\ &= Z_S(Z_T^{(1)}, Z_T^{(2)}, \ldots). \end{aligned}$$

as desired. □