Math 155 (Lecture 16)

October 10, 2011

We now pick up where we left off in the previous lecture. Let S be the species of finite sets (so that $S[I] = \{*\}$ for every finite set I). We have computed the cycle index of S to be the power series

$$Z_S(s_1, s_2, \cdots) = e^{s_1 + s_2/2 + s_3/3 + \cdots}.$$

From this formula, we recover the answer to a question raised in a previous lecture:

Question 1. Let G be the symmetric group Σ_n, acting on the set $\{1, 2, \ldots, n\}$. What is the cycle index of G?

Up to isomorphism, the species S of finite sets has one structure of each cardinality n, with symmetric group Σ_n. It follows that

$$Z_S(s_1, \ldots,) = \sum_{n \geq 0} Z_{\Sigma_n}(s_1, s_2, \ldots).$$

We can therefore recover the answer to Question 1 by extracting the degree n part of the answer to Question **??**. That is, the cycle index of Σ_n is just the degree n part of the power series

$$e^{s_1 + s_2/2 + s_3/3 + s_4/4 + \cdots}$$

(where we count each variable s_k as having degree k).

Remark 2. For any species S, we have

$$Z_S(x, 0, 0, 0, \ldots) = F_S(x).$$

Taking S to be the species of finite sets, we recover the formula

$$F_S(x) = Z_S(x, 0, 0, \ldots) = e^{x + 0 + \cdots} = e^x.$$

We also recall that $Z_S(x, x^2, x^3, \cdots)$ can be interpreted as the *ordinary* generating function whose coefficient of x^n is given by the number of isomorphism classes of S-structures having size n. If S is the species of finite sets, we get

$$\begin{aligned} Z_S(x, x^2, x^3, \ldots) &= e^{x + x^2/2 + x^3/3 + \cdots} \\ &= e^{\log(\frac{1}{1-x})} \\ &= \frac{1}{1-x} \\ &= 1 + x + x^2 + x^3 + \cdots. \end{aligned}$$

This reflects the observation that, up to isomorphism, there is exactly one S-structure of each cardinality.

Let's now consider another example.

Question 3. Let S denote the species of permutations (so that, for each finite set I, we let $S[I]$ denote the collection of all permutations of I). What is the cycle index $Z_S(s_1, s_2, \ldots)$?

To address Question 3, we use the formula

$$Z_S(s_1, s_2, \ldots) = \sum_{n \geq 0} \frac{1}{n!} \sum_{\sigma \in \Sigma_n} |S[\langle n \rangle]^\sigma| Z_\sigma$$

where Z_σ denote the cycle monomial $s_1^{k_1} s_2^{k_2} \cdots$, where k_i is the number of i-cycles in σ.

If S is the species of permutations, then $S[\langle n \rangle]$ is the symmetric group Σ_n of permutations of $\{1, 2, \ldots, n\}$. Moreover, we have seen that the action of Σ_n on $S[\langle n \rangle]$ is via conjugation. Consequently, we can identify the set $S[\langle n \rangle]^\sigma$ with the group $\{\tau \in \Sigma_n : \tau\sigma = \sigma\tau\}$ consisting of permutations which commute with σ.

Note that if σ and σ' are conjugate permutations, then the monomials $S[\langle n \rangle]^\sigma Z_\sigma$ and $S[\langle n \rangle]^{\sigma'} Z_{\sigma'}$ are the same. We may therefore write

$$Z_S(s_1, s_2, \ldots) = \sum_{n \geq 0} \frac{1}{n!} \sum_{\sigma \in \Sigma_n / \text{ conj}} C_\sigma D_\sigma Z_\sigma(s_1, s_2, \ldots)$$

where the sum is taken over all *conjugacy classes* of permutations, C_σ denotes the number of permutations conjugate to σ, and D_σ denotes the number of permutations which commute with σ.

Lemma 4. *For every permutation $\sigma \in \Sigma_n$, we have $C_\sigma D_\sigma = n!$.*

Proof. Let $X \subseteq \Sigma_n$ be the conjugacy class of σ: that is, the set of all permutations which commute with σ. Then Σ_n acts transitively on X (via conjugation). It follows that $|\Sigma_n| = |X| \, |\operatorname{Stab}(\sigma)|$. Here $|\operatorname{Stab}(\sigma)| = D_\sigma$, $|X| = C_\sigma$, and $|\Sigma_n| = n!$, so $C_\sigma D_\sigma = n!$. □

Taking Lemma 4 into account, we get

$$Z_S(s_1, s_2, \ldots) = \sum_{n \geq 0} \frac{1}{n!} \sum_{\sigma \in \Sigma_n / \text{ conj}} n! Z_\sigma(s_1, s_2, \ldots) = \sum_{n \geq 0} \sim_{\sigma \in \Sigma_n} Z_\sigma(s_1, s_2, \ldots).$$

Note that a permutation of $\{1, \ldots, n\}$ is determined up to conjugacy by specifying its number of i-cycles for each integer i: these are integers k_i satisfying $n = \sum i k_i$. We therefore obtain

$$\begin{aligned} Z_S(s_1, s_2, \ldots) &= \sum_{n \geq 0} \sum_{k_1 + 2k_2 + \cdots = n} s_1^{k_1} s_2^{k_2} \cdots \\ &= \sum_{k_1, k_2, \cdots} \prod_{i \geq 1} s_i^{k_i} \\ &= \prod_{i \geq 1} \sum_{k \geq 0} s_i^{k_i} \\ &= \prod_{i \geq 1} \frac{1}{1 - s_i} \\ &= \frac{1}{(1 - s_1)(1 - s_2)(1 - s_3) \cdots}. \end{aligned}$$

Remark 5. Let's run a few reality checks on this calculation. For any species S, we have

$$F_S(x) = Z_S(x, 0, 0, \ldots).$$

Taking S to be the species of permutations, we recover the formula

$$F_S(x) = \frac{1}{(1 - x)(1 - 0)(1 - 0) \cdots} = \frac{1}{1 - x} = 1 + x + x^2 + x^3 + \cdots.$$

More concretely, this tells us that $S[\langle n \rangle]$ has size $n!$ for each n.

We also know that for any species S, $Z_S(x, x^2, x^3, \cdots)$ is the ordinary generating function which counts isomorphism classes of S-structures. In the case where S is the species of permutations, we get

$$Z_S(x, x^2, x^3, \cdots) = \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots}.$$

The coefficient of x^n in this expression is the number of *partitions* of the integer n: that is, the number of ways to write n as a sum of positive integers c_α (where the order of the summands does not matter).

Let's now return to some general remarks.

Proposition 6. *Let S and T be molecular species. Then the product $S \times T$ is a molecular species.*

Proof. Recall that $S \times T$ is defined by the formula

$$(S \times T)[I] = \coprod_{I = I_0 \cup I_1} S[I_0] \times T[I_1].$$

This has a more intuitive formulation in the language of structures: to give an $S \times T$-structure on a set I, we have must give a decomposition of I into pieces I_0 and I_1, an S-structure on I_0, and a T-structure on I_1. This has an even simpler formulation if we neglect the underlying sets: giving the data of an $S \times T$-structure is equivalent to the data of an S-structure and a T-structure individually.

If S and T are molecular, then up to isomorphism there is a unique S-structure and a unique T-structure. Taken together, these give the unique $S \times T$-structure. □

Note that the proof of Proposition 6 gives more. Suppose that S is molecular. Then, up to isomorphism, there is a unique S-structure (X, η), where $\eta \in S[X]$. We have seen that in this case, S is isomorphic to the species $S_{(G,X)}$, where G is the automorophism group of (X, η). Similarly, if T is molecular, we can write $T = S_{(H,Y)}$, where (Y, η') is a T-structure and H is its automorphism group. Then the pair (η, η') determines an $(S \times T)$-structure on the disjoint union $X \amalg Y$. Any automorphism of this $(S \times T)$-structure must preserve the decomposition of $X \amalg Y$ into X and Y, and must reduce to automorphisms of (X, η) and (Y, η'). We deduce that the automorphism group of $(X \amalg Y, (\eta, \eta'))$ is isomorphic to $G \times H$, so that $S \times T$ is isomorphic to the molecular species $S_{(G \times H, X \amalg Y)}$. In other words, there is a canonical isomorphism of species

$$S_{(G,X)} \times S_{(H,Y)} \simeq S_{(G \times H, X \amalg Y)}.$$

Combining this with our cycle index formula

$$Z_{G \times H}(s_1, s_2, \cdots) = Z_G(s_1, s_2, \cdots) Z_H(s_1, s_2, \ldots),$$

we obtain a proof of the following:

Lemma 7. *Let S and T be molecular species. Then we have*

$$Z_{S \times T}(s_1, s_2, \ldots) = Z_S(s_1, s_2, \cdots) Z_T(s_1, s_2, \cdots).$$

Since every species can be obtained as a (possibly infinite) sum of molecular species, Lemma 7 gives the following:

Proposition 8. *Let S and T be species. Then we have*

$$Z_{S \times T}(s_1, s_2, \ldots) = Z_S(s_1, s_2, \cdots) Z_T(s_1, s_2, \cdots).$$

Remark 9. When S and T are molecular species, Proposition 8 specializes to our product formula for the cycle indices of group actions. We obtain another specialization by making the substitution $s_1 = x$, $s_2 = s_3 = \cdots = 0$: in this case, we recover the product formula

$$F_{S \times T}(x) = F_S(x) F_T(x)$$

for exponential generating functions of species.