# Math 155 (Lecture 15)

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Recall that species $S$ is said to be *molecular* if there is exactly one $S$-structure, up to isomorphism. Equivalently, $S$ is molecular if there exists an integer $n$ such that $S[\langle m\rangle]$ is empty for $m \neq n$, and $S[\langle n\rangle]$ is acted on transitively by the symmetric group $\Sigma_n$. Every species $S$ can be decomposed uniquely as a sum $\sum_\alpha S_\alpha$ of molecular species. Furthermore, there is a one to one correspondence between (isomorphism classes of) molecular species and (isomorphism classes of) pairs $(G, X)$, where $G$ is a finite group acting faithfully on a finite set $X$. This correspondence assigns to a pair $(G, X)$ the species $S_{(G,X)}$, where $S_{(G,X)}[I] = \mathrm{Bij}(X, I)/G$.

If $S$ is any species, the *cycle index* of $S$ is given by

$$Z_S(s_1, s_2, \ldots) = \sum_{(I,\eta)} Z_{\mathrm{Aut}(I,\eta)}(s_1, \ldots, )$$

where the sum is over all isomorphism classes of $S$-structures $(I, \eta)$. If $G$ is a finite group acting faithfully on a set $X$ and $S = S_{(G,X)}$ is the corresponding molecular species, then there is only one isomorphism class of $S$-structures, and its automorphism group is given by $G$. We therefore have

$$Z_S(s_1, s_2, \ldots) = Z_G(s_1, s_2, \ldots) :$$

in other words, we can regard the cycle index of a group $G$ as a special case of the cycle index of a species.

**Remark 1.** Let $G$ be a finite group acting on a set $X$. The definition of the cycle index $Z_G(s_1, s_2, \ldots)$ does not require that the action of $G$ on $X$ is faithful. However, there is no harm in assuming that. Suppose that we are given an arbitrary action of a finite group $G$ on a finite set $X$, given by a map $\rho : G \to \mathrm{Perm}(X)$. Let $N = \ker(\rho)$ be the kernel of $\rho$: that is, the subgroup of $G$ consisting of elements which fix every $x \in X$. Then $N$ is a normal subgroup of $G$, and the quotient group $G/N$ acts on $X$. Moreover, we have

$$Z_G(s_1, \ldots, ) = Z_{G/N}(s_1, \ldots, ).$$

Here is a more direct description of the cycle index of a species:

**Proposition 2.** *Let $S$ be a species. Then the cycle index of $S$ is given by*

$$Z_S(s_1, s_2, \ldots) = \sum_{n \geq 0} \frac{1}{n!} \sum_{\sigma \in \Sigma_n} |S[\langle n\rangle]^\sigma| s_1^{k_1} s_2^{k_2} \cdots ,$$

*where $k_m$ denotes the number of $m$-cycles in the permutation $\sigma$.*

*Proof.* We can rewrite the right hand side as

$$\sum_{n \geq 0} \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \sum_{\eta \in S[\langle n\rangle]} \begin{cases} s_1^{k_1} s_2^{k_2} \cdots & \text{if } S[\sigma](\eta) = \eta \\ 0 & \text{otherwise.} \end{cases}$$

Rearranging the order of summation, this is given by

$$\sum_{n\geq 0}\sum_{\eta\in S[\langle n\rangle]}\frac{1}{n!}\sum_{\sigma\in G}Z_{\sigma},$$

where $G = \mathrm{Stab}(\eta)$ denote the stabilizer of the point $\eta$ and $Z_\sigma$ is the cycle monomial of $\sigma$ (regarded as a permutation of the set $\{1, 2, \ldots, n\}$). We can rewrite this as

$$\sum_{n\geq 0}\sum_{\eta\in S[\langle n\rangle]}\frac{|G|}{n!}Z_G(s_1,\ldots).$$

Note that the contribution coming from a particular element of $S[\langle n\rangle]$ is the same for all other $\eta'$ belonging to the $\Sigma_n$ orbit of $\eta$. The number of elements in this orbit is given by $\frac{n!}{|G|}$. We may therefore write our sum as

$$\sum_{n\geq 0}\sum_{S[\langle n\rangle]/\Sigma_n}Z_G(s_1,\ldots),$$

which reproduces the definition of $Z_S$. □

**Example 3.** Let $S$ be any species, and consider the power series

$$Z_S(x, 0, 0, \ldots).$$

Writing

$$Z_S=\sum_{n\geq 0}\frac{1}{n!}\sum_{\sigma\in\Sigma_n}|S[\langle n\rangle]^{\sigma}|s_1^{k_1}s_2^{k_2}\cdots,$$

we note that the contribution from any non-identity permutation vanishes. We therefore obtain

$$Z_S(x,0,0,\ldots)=\sum_{n\geq 0}\frac{|S[\langle n\rangle]|}{n!}x^n,$$

thereby recovering the exponential generating function of $S$.

**Example 4.** Let $S$ be any species, and consider the power series $Z_S(x, x^2, x^3, \ldots)$. This is given by

$$Z_S(x,x^2,\ldots)=\sum_{n\geq 0}\frac{1}{n!}\sum_{\sigma\in\Sigma_n}|S[\langle n\rangle]^{\sigma}|x^n.$$

Applying Burnside's formula, we see that this is given by

$$\sum_{n\geq 0}|S[\langle n\rangle]/\Sigma_n|x^n.$$

This is the ordinary generating function for the unlabelled enumeration problem of counting $S$-structures, up to isomorphism (this formula also follows immediately from Definition **??**).

Let's now compute an example.

**Question 5.** Let $S$ be the species of sets with no structure: that is, $S[I] = \{*\}$ for every finite set $I$. What is the cycle index of $S$?

According to Proposition 2, the answer is given by

$$\sum_{n\geq 0} \frac{1}{n!} \sum_{\sigma\in\Sigma_n} s_1^{k_1} s_2^{k_2} \cdots$$

where $k_i$ denotes the number of $i$-cycles of $\sigma$. We can rewrite this as

$$\sum_{n\geq 0} \frac{1}{n!} \sum_{k_1+2k_2+\cdots=n} C_{\vec{k}} s_1^{k_1} s_2^{k_2} \cdots ,$$

where $C_{\vec{k}}$ denotes the number of permutations having exactly $k_i$ $i$-cycles. Let's first determine the numbers $C_{\vec{k}}$.

Fix a decomposition $n = k_1 + 2k_2 + 3k_3 + \cdots$. Suppose $\sigma$ is a permutation with $k_1$ 1-cycles, $k_2$ 2-cycles, and so forth. How many possibilities are therefore $\sigma$? First, let's count the number of ways to partition $\sigma$ into labelled subsets, $k_1$ of which have size 1, $k_2$ of which have size 2, and so forth. This is given by the multinomial coefficient

$$\frac{n!}{(1!)^{k_1}(2!)^{k_2}(3!)^{k_3}\cdots}.$$

Our counting problem has a slightly different answer. The cycles of $\sigma$ are not labelled, so we must divide by the product $k_1!k_2!\cdots$. Also, a permutation is not determined by the decomposition of $\{1, 2, \ldots, n\}$ into orbits: we must also specify a cyclic permutation of each orbit. Consequently, we should multiply by $(0!)^{k_1}(1!)^{k^2}(2!)^{k_3}\cdots$ We therefore obtain

$$\begin{aligned} C_{\vec{k}} &= \frac{n!}{k_1!k_2!\cdots} \frac{(0!)^{k_1}(1!)^{k_2}(2!)^{k_3}\cdots}{(1!)^{k_1}(2!)^{k_2}(3!)^{k_3}\cdots} \\ &= \frac{n!}{(k_1!k_2!\cdots)(1^{k_1}2^{k_2}\cdots)}. \end{aligned}$$

Plugging this in, we get

$$\begin{aligned} Z_S &= \sum_{n\geq 0} \frac{1}{n!} \sum_{n=k_1+2k_2+3k_3+\cdots} \frac{n!}{(k_1!k_2!\cdots)(1^{k_1}2^{k_2}\cdots)} s_1^{k_1} s_2^{k_2} \\ &= \sum_{k_1,k_2,k_3,\ldots} \prod_{i\geq 1} \frac{s_i^{k_i}}{k_i! i^{k_i}} \\ &= \prod_{i\geq 1} \sum_{k\geq 0} \frac{1}{k!} (\frac{s_i}{i})^k \\ &= \prod_{i\geq 1} e^{s_i/i} \\ &= e^{s_1+s_2/2+s_3/3+\cdots} \end{aligned}$$