Math 155 (Lecture 14)

October 5, 2011

In the last few lectures, we have discussed the cycle index of a finite group G acting on a set X. The cycle index contains a lot of enumerative information in a conveniently accessible form. However, it is not always easy to compute, even in relatively simple cases.

Question 1. Let G be the group of permutations of the set $X = \{1, 2, \ldots, n\}$, and regard G as acting on X. What is the cycle index $Z_G(s_1, \ldots)$?

We will return to Question 1 in the next lecture. First, we need to make a large digression back to the theory of species.

Let S be a species. By our definition, S is a rule which assigns to each finite set I another finite set $S[I]$, and to each bijection $\pi : I \to J$ another bijection $S[\pi] : S[I] \to S[J]$. However, there is another way of thinking about a species.

Definition 2. Let S be a species. An *S-structure* is a finite set I and an element $\eta \in S[I]$. Given two S-structures (I, η) and (I', η'), an *isomorphism* between (I, η) and (I', η') is a bijection $\pi : I \to I'$ such that $S[\pi](\eta) = \eta'$.

Example 3. Let S be the species of linearly ordered sets. Then an S-structure is just a finite set equipped with a linear ordering (and an isomorphism of S-structures is just an order-preserving bijection).

Example 4. Let S be the species of graphs. Then an S-structure is just a graph with finitely many vertices.

By definition, every S-structure has an underlying finite set. Conversely, from the theory of S-structures (and knowledge of the underlying finite set) we can recover the species S: $S[I]$ is just the collection of all S-structures on I. We can summarize the situation by saying that a species is just a type of structure that can live on a finite set (this idea can be formalized with a little bit of category theory, but we will not need this).

Now suppose that S is a species and that (I, η) is an S-structure. The collection of all isomorphisms of (I, η) with itself forms a group: namely, it is the group of all permutations $\pi : I \to I$ which satisfy $S[\pi](\eta) = \eta$. We will refer to this group as the *automorphism group* of (I, η), and denote it by $\mathrm{Aut}(I, \eta)$. Note that the group $\mathrm{Perm}(I)$ of all permutations of I acts on the set $S[I]$; the group $\mathrm{Aut}(I, \eta)$ is just the stabilizer of the element $\eta \in S[I]$.

Definition 5. Let S be a species. The *cycle index* of S is the formal power series

$$Z_S(s_1, s_2, \ldots) = \sum_{(I,\eta)} Z_{\mathrm{Aut}(I,\eta)}(s_1, s_2, \ldots).$$

Here the sum is taken over all isomorphism classes of S-structures, and each summand is the cycle index of the group $\mathrm{Aut}(I, \eta)$ acting on the set I.

Example 6. Let S be the species of linearly ordered sets, so that an S-structure is just a finite linearly ordered set. Up to isomorphism, there is exactly one S-structure of size n, for each $n \geq 0$. Moreover, the automorphism group of every S-structure is trivial. The cycle index of the trivial group acting on a set of size n is just s_1^n. It follows that the cycle index $Z_S(s_1, s_2, \ldots)$ is given by

$$\sum_{n \geq 0} s_1^n = \frac{1}{1 - s_1}.$$

We have defined the cycle index of a species in terms of the cycle index of a group action. In fact, the relationship between the two is quite close: the cycle index of a group action can be viewed as a special case of the cycle index of a species.

Construction 7. Let G be a finite group acting on a finite set X. Assume that the action of G is faithful: that is, if an element $g \in G$ fixes every element of X, then g is the identity (this is equivalent to saying that the action of G on X is given by an *injective* map $G \to \mathrm{Perm}(X)$). We define a species $S_{(G,X)}$ as follows. For each finite set I, let $\mathrm{Bij}(X, I)$ denote the set of all bijections $I \to X$, and set $S_{(G,X)}[I] = \mathrm{Bij}(X, I)/G$ be the quotient of $\mathrm{Bij}(X, I)$ by the action of G. Then $S_{(G,X)}$ is a species. Up to isomorphism, there is only one $S_{(G,X)}$-structure: it is given by the identity map $\mathrm{id} : X \to X$.

Suppose that S is any species and that there is only one isomorphism class of S-structures. Choose a representative of this isomorphism class (X, η), with $\eta \in S[X]$. Let $G = \mathrm{Aut}(X, \eta)$. Then G is a subgroup of $\mathrm{Perm}(X)$, so we can regard G as acting faithfully on the set X. We claim that S is isomorphic to the species $S_{(G,X)}$. To prove this, let I be any finite set. To every bijection $\pi : X \to I$ we can assign an element $S[\pi](\eta) \in S[I]$. Since η is fixed by the group G, the map $\pi \mapsto S[\pi](\eta)$ determines a map

$$S_{(G,X)}[I] = \mathrm{Bij}(X, I)/G \to S[I].$$

This map is injective: if we are given a pair of bijections $\pi, \pi' : X \to I$ such that $S[\pi](\eta) = S[\pi'](\eta)$, then $\pi^{-1}\pi'$ is an automorphism of (X, η) and therefore belongs to G. It is also surjective, because of our assumption that all S-structures are isomorphic. We haven proven:

Proposition 8. *Let S be a species. The following conditions are equivalent:*

(1) *There is exactly one isomorphism class of S-structures.*

(2) *S is isomorphic to $S_{(G,X)}$, for some finite group G acting faithfully on a finite set X.*

A species S is called *molecular* if it satisfies the conditions of Proposition 8. Note that the group G and the set X are well-defined up to isomorphism: if we choose an S-structure (I, η), we can identify X with the underlying set I and G with the automorphism group $\mathrm{Aut}(I, \eta)$. We can summarize the situation as follows: giving a molecular species is equivalent to giving a finite group G, and an action of G on a finite set X.

Now let S be an arbitrary species. Then we can write S as a (possibly infinite) sum $\sum S_\alpha$, where each S_α is a molecular species. Here there is one summand for each isomorphism class of S-structures (more concretely, there is one summand for each integer $n \geq 0$ and each orbit of the group Σ_n on the set $S(\langle n \rangle)$).

We can now explain Definition 5 as follows: if $S = S_{(G,X)}$ is a molecular species, the the cycle index Z_S is just defined to be the cycle index Z_G. In general, if we write S as a sum of molecular species S_α, then $Z_S = \sum_\alpha Z_{S_\alpha}$.