Math 155 (Lecture 13)

September 30, 2011

In this lecture, we will study some of the formal properties of the cycle index Z_G of a finite group G acting on a set X. Ideally, we would like to have some recipe for computing the cycle index of X in terms of the structure of X as a G-set. For example, we know that X decomposes uniquely as a disjoint union of orbits $X_1 \cup \cdots \cup X_m$. Let $Z_{G,i}$ denote the cycle index of G acting on the set X_i. We know that $Z_G = Z_G(s_1, s_2, \ldots)$ is a polynomial of degree $|X|$, where each s_i is regarded as having degree i. Similarly, each $Z_{G,i}$ has degree $|X_i|$. This raises the following possibility:

Question 1. In the situation above, is the cycle index Z_G given by the product $\prod_{1 \leq i \leq m} Z_{G,i}$?

Let's test this out in a simple example. Let G be the group $\mathbf{Z}/p\mathbf{Z}$. Let X be a disjoint union $G \coprod G$, with G acting on each summand by left translation. The identity element of G has $2p$ fixed points, and every nontrivial element of G has two orbits of size p. It follows that the cycle index Z_G is given by

$$Z_G = \frac{s_1^{2p} + (p-1)s_p^2}{p}.$$

On the other hand, we computed the cycle index Z'_G for the group G acting on itself in the last lecture: it is given by

$$Z'_G = \frac{s_1^p + (p-1)s_p}{p}.$$

It is easy to see that $Z_G \neq Z'^2_G$.

There *is* a formula of the form suggested by Question 1. However, it requires looking at a pair of group actions, rather than a single group acting on a pair of sets.

Construction 2. Let G and H be finite groups, and let X and Y be finite sets acted on by G and H, respectively. We let the product group $G \times H$ act on the disjoint union $X \coprod Y$ by the formula

$$(g,h)x = gx \qquad (g,h)y = hy.$$

Proposition 3. *In the situation of Construction 2, we have*

$$Z_{G \times H}(s_1, s_2, \ldots) = Z_G(s_1, s_2, \ldots) Z_H(s_1, s_2, \ldots).$$

Proof. We have

$$\begin{aligned} Z_G Z_H &= \frac{1}{|G|}\frac{1}{|H|}(\sum_{g \in G} Z_g)(\sum_{h \in H} Z_h) \\ &= \frac{1}{|G \times H|} \sum_{(g,h) \in G \times H} Z_g Z_h. \end{aligned}$$

It therefore suffices to observe that there is an identity of cycle monomials

$$Z_g Z_h = Z_{(g,h)}.$$

This arises from the following simple observation: the number of orbits of (g, h) on $X \coprod Y$ having size m is the sum of the number of orbits of g on X having size m, and the number of orbits of h on Y having size m. $\square$

We now consider a more subtle question. What if we are given a finite group G acting on X, a finite group H acting on a finite set Y, and we want to study the product $X \times Y$? Here we have cycle indices Z_G and Z_H, which are polynomials of degree $|X|$ and $|Y|$ respectively. To relate these to a polynomial of degree $|X \times Y| = |X|\,|Y|$, it is natural to guess that some form of composition is involved.

First, let's ask what group acts on the set $X \times Y$. Of course, there is a natural action of $G \times H$, given by the formula

$$(g, h)(x, y) = (gx, hy).$$

Another way of saying this is that there are actions of G and H on $X \times Y$, and these actions commute with one another. But there are much larger groups that also act on $X \times Y$. For example, let H^X denote the product $\prod_{x \in X} H$ of several copies of H, indexed by H. We will denote elements of H^X by tuples $(h_x)_{x \in X}$. The group H^X acts on $X \times Y$ by the formula

$$(h_x)_{x \in X}(x', y') = (x', h_{x'} y').$$

Similarly, the product G^Y acts on $X \times Y$ by the formula

$$(g_y)_{y \in Y}(x', y') = (g_{y'}(x'), y').$$

These actions do *not* commute with each other, and so do not define an action of $G^Y \times H^X$ on $X \times Y$. In general, it may be very difficult to describe what sorts of permutations of $X \times Y$ can be obtained by combining the actions of G^Y and H^X. However, the action of the smaller group G on $X \times Y$ *almost* commutes with the action of H^X. We have

$$g((h_x)_{x \in X}(x', y')) = g(x', h_{x'} y') = (gx', h_{x'} y')$$

$$(h_x)_{x \in X} g(x', y') = (h_x)_{x \in X}(gx', y') = (gx', h_{gx'} y').$$

This suggests the possibility that G and H^X can be combined into a larger group that acts on $X \times Y$.

Construction 4. Let G be a group acting on a set X and let H be another group. We define a new group $H \wr G$, called the *wreath product* of G and H. As a set, $H \wr G$ coincides with the direct product $G \times H^X$. However, the multiplication is slightly twisted: it is given by

$$(g, (h_x)_{x \in X})(g', (h'_x)_{x \in X} = (gg', (h_{g'x} h'_x)_{x \in X}.$$

Exercise 5. Show that the multiplication above makes $H \wr G$ into a group (that is, show that it is associative and that each element has an inverse).

The wreath product $H \wr G$ contains both G and H^X as subgroups. However, these subgroups do not commute: by construction, we have

$$g^{-1}(h_x)_{x \in X} g = (h_{gx})_{x \in X}.$$

Now suppose that H acts on a set Y. We define an action of $H \wr G$ on the product $X \times Y$ by the formula

$$(g, (h_x)_{x \in X})(x', y') = (gx', h_{x'} y').$$

Exercise 6. Show that this gives an action of $H \wr G$ on the product $X \times Y$.

Let's now assume that all of our groups and sets are finite, and try to compute the cycle index $Z_{H\wr G}$ of the wreath product (which we regard as acting on the set $X \times Y$). First, let's compute the cycle monomials $Z_{(g,(h_x))}$ associated to an element $(g,(h_x)) \in H \wr G$. Choose a set of representatives $x_1, \ldots, x_m$ for the g-orbits on X, and let d_i denote the smallest positive integer such that $g^{d_i} x_i = x_i$. Thus

$$X = \{x_1, gx_1, \ldots, g^{d_1-1}x_1\} \cup \cdots \cup \{x_m, gx_m, \ldots, g^{d_m-1}x_m\}.$$

Let's try to describe the orbits of $(g,(h_x))$ on the product $X \times Y$. Take the $(g,(h_x))$ orbit of an element $(x', y') \in X \times Y$. Then x' belongs to some G-orbit on X. Applying some power of $(g,(h_x))$, we can assume that $x' = x_i$ for some $1 \leq i \leq m$. For every exponent k, we have

$$(g,(h_x))^k(x', y') = (g^k x', h_{g^{k-1}x} \cdots h_x y').$$

In particular, if $(g,(h_x))^k(x', y') = (x', y'')$, then k must be divisible by d_i. Write $k = d_i q$ and $\vec{h}_i = h_{g^{d_i-1}x} \cdots h_x$, so that

$$(g,(h_x))^k(x', y') = (x', \vec{h}_i^q y').$$

This analysis proves:

(∗) There is a bijection between orbits of $\vec{h}_i$ on Y with orbits of $(g,(h_x))$ on $X \times Y$ lying over $\{x_i, \cdots, g^{d_i-1}x_i\}$. This bijection carries orbits of size t to orbits of size $d_i t$.

From (∗) we deduce the following formula for cycle monomials:

$$Z_{(g,(h_x))}(s_1, s_2, \ldots) = \prod_{1 \leq i \leq m} Z_{\vec{h}_i}(s_{d_i}, s_{2d_i}, s_{3d_i}, \ldots).$$

Now let's try to evaluate the entire cycle index. We have

$$\begin{aligned} Z_{H\wr G} &= \frac{1}{|G||H|^{|X|}} \sum_{(g,(h_x))} Z_{(g,h_x)} \\ &= \frac{1}{|G||H|^{|X|}} \sum_{g \in G} \sum_{(h_x) \in H^X} \prod_{1 \leq i \leq m} Z_{\vec{h}_i}(s_{d_i}, \ldots,) \end{aligned}$$

where m and $\vec{h}_i$ are defined as above. Note that the expression in the product depends only on the elements $\vec{h}_i$. Consequently, each term in the sum over H^X is repeated $|H|^{|X|-m}$ times. We can rewrite our expression as

$$\frac{1}{|G||H|^{|X|}} \sum_{g \in G} \sum_{(\vec{h}_i) \in H^m} |H|^{|X|-m} \prod_{1 \leq i \leq m} Z_{\vec{h}_i}(s_{d_i}, \ldots).$$

Rearranging this, we get

$$\frac{1}{|G|} \sum_{g \in G} \sum_{(\vec{h}_i) \in H^m} \frac{\prod_{1 \leq i \leq m} Z_{\vec{h}_i}(s_{d_i}, \ldots)}{|H|^m}.$$

Applying the distributive law, we get

$$\frac{1}{|G|} \sum_{g \in G} \prod_{1 \leq i \leq m} \frac{1}{|H|} \sum_{h \in H} Z_h(s_{d_i}, s_{2d_i}, \ldots)$$

or

$$\frac{1}{|G|} \sum_{g \in G} \prod_{1 \leq i \leq m} Z_H(s_{d_i}, s_{2d_i}, \ldots).$$

This polynomial can obtained from the cycle index for G by the replacement

$$s_d \mapsto Z_H(s_d, s_{2d}, \ldots).$$

We have proven the following:

Theorem 7. *Let G be a finite group acting on a set X, let H be a finite group acting on a set Y, and regard the wreath product $H \wr G$ as acting on the set $X \times Y$. Then we have an equality of polynomials*

$$Z_{H \wr G}(s_1, s_2, \ldots) = Z_G(t_1, t_2, t_3, \ldots)$$

where $t_d = Z_H(s_d, s_{2d}, s_{3d}, \ldots)$.