# Math 155 (Lecture 11)

September 25, 2011

In the last lecture, we explained how to use Polya's theorem to count the number if isomorphism classes of graphs of size $n$. The answer was given by the sum

$$\frac{1}{n!} \sum_{\sigma \in \Sigma_n} 2^{o(\sigma)},$$

where $o(\sigma)$ denotes the number of orbits of the permutation $\sigma$ on the set $E$ of all two-element subsets of the set $\{1, 2, \ldots, n\}$. Let's ask a more refined question:

**Question 1.** Let $n$ and $k$ be nonnegative integers. Up to isomorphism, how many graphs are there with exactly $n$ vertices and exactly $k$ edges?

For the moment, let's regard $n$ as fixed and ask how the answer to this question depends on $k$. For each $k \geq 0$, let's denote the number of isomorphism classes of graphs with $k$ edges (and $n$ vertices) by $C_k$. Of course, $\sum_{k \geq 0} C_k$ is just the total number of graphs with $n$ vertices, given by the sum

$$\frac{1}{n!} \sum_{\sigma \in \Sigma_n} 2^{o(\sigma)}.$$

Now we're asking not for a single number but for several. However, we can package the different answers together in a convenient way. Consider the generating function

$$F(t) = \sum_{k \geq 0} C_k t^k.$$

This is just a polynomial in $t$, with integer coefficients. We have $F(1) = \sum_{k \geq 0} C_k = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} 2^{o(\sigma)}$. Let's try to describe the entire polynomial $F$ in a similar way.

We begin by describing $C_k$ in a convenient way. Let $X_k$ denote the collection of all graphs with vertex set $\{1, \ldots, n\}$ which have exactly $k$ vertices. By definition, $C_k$ is the number of elements of the quotient $|\Sigma_n \backslash X_k|$. Note that the cardinality of $X_k$ is given by the binomial coefficient $\binom{\binom{n}{2}}{k}$, which is given by the coefficient of $t^k$ in the expression $(1+t)^{\binom{n}{2}}$. We can count the number of elements of $\Sigma_n \backslash X_k$ using Burnside's formula: it is given by

$$\frac{1}{n!} \sum_{\sigma \in \Sigma_n} |X_k^\sigma|,$$

where $X_k^\sigma$ denotes the number of graphs with vertex set $\{1, \ldots, n\}$ which are have exactly $k$ vertices and are fixed by the permutation $\sigma$. Let $E$ be the collection of all two-element subsets of the set $\{1, \ldots, n\}$. Then $E$ decomposes as a union $E_1 \cup E_2 \cup \cdots \cup E_{o(\sigma)}$ of orbits under the permutation $\sigma$. To give a graph which is invariant under the permutation $\sigma$,we need to give a subset of $E$ which is a union of orbits: there are $2^{o(\sigma)}$ such subsets in all. For our more refined question, we want to count only those subsets which have size exactly $k$. This is given by the coefficient of $t^k$ in the polynomial

$$(1 + t^{|E_1|})(1 + t^{|E_2|}) \cdots (1 + t^{|E_{o(\sigma)}|}).$$

In particular, the contribution depends not only on the number $o(\sigma)$ of orbits of $\sigma$, but also on the sizes of the different orbits.

We now introduce a convenient device for keeping track of this sort of information.

**Definition 2.** Let $G$ be a finite group and let $X$ be a finite $G$-set. Fix an element $g \in G$. For each $n \geq 1$, let $c_n$ denote the number of $g$-orbits of $X$ having size exactly $n$. We define the *cycle monomial* $Z_g$ to be the expression

$$s_1^{c_1} s_2^{c_2} s_3^{c_3} \cdots$$

This is a monomial in the sequence of formal variables $s_1, s_2, \ldots$. Note that although we have infinitely many variables here, only finitely many of them actually appear in our expression (since $c_n = 0$ for all sufficiently large $n$).

We define the *cycle index* of $G$ to be the sum

$$Z_G = Z_G(s_1, s_2, s_3, \ldots) = \frac{1}{|G|} \sum_{g \in G} Z_g.$$

This is a polynomial in the infinite sequence of variables $s_1, s_2, \ldots$ (though we should again note that only finitely many of these variables actually appear).

**Warning 3.** Although we refer to $Z_G$ as the *cycle index of* $G$, it depends not only on $G$ but also on the $G$-set $X$.

**Example 4.** Let $t$ be a nonnegative integer. If we evaluate the cycle index $Z_G(s_1, s_2, \ldots)$ at the point $s_1 = s_2 = \cdots = t$, we obtain the number

$$Z_G(t, t, \ldots) = \frac{1}{|G|} \sum_{g \in G} t^{o(g)} = |G \backslash T^X|$$

appearing in Polya's enumeration theorem.

**Example 5.** Suppose we fix a variable $t$ and evaluate the cycle index on the sequence $s_1 = t, s_2 = t^2, s_3 = t^3, \cdots$. We then get

$$Z_G(s_1, s_2, \cdots) = Z_G(t, t^2, t^3, \cdots) = \frac{1}{|G|} \sum_{g \in G} t^{c_1 + 2c_2 + 3c_3 + \cdots} = t^{|X|}.$$

If we take $t$ to be the number of elements in some set $T$, this is counting the size of of $T^X$, where symmetry is *not* taken into account.

**Example 6.** Let $G = \Sigma_n$, acting on the set $E$ of two-element subsets of $\{1, \ldots, n\}$. For a given element $\sigma \in G$, we have seen that the number of $\sigma$-invariant graphs with vertex set $\{1, \ldots, n\}$ and $k$ edges is given by the coefficient of $t^k$ in the cycle monomial

$$Z_\sigma(s_1, s_2, \ldots),$$

evaluated at $s_1 = 1 + t$, $s_2 = 1 + t^2$, and so forth. Summing over $\sigma$ and dividing by $n!$, we see (using Burnside's formula) that the number of isomorphism classes of graphs with $n$ vertices and $k$ edges is given by the coefficient of $t^k$ in the expression

$$Z_G(1 + t, 1 + t^2, 1 + t^3, \ldots).$$

The moral of the above examples is that the cycle index of a group action contains a lot of useful enumerative information.

**Theorem 7** (Polya Enumeration Theorem)**.** *Let $G$ be a finite group, $X$ a finite $G$-set, and $T = \{y_1, \ldots, y_t\}$ another finite set. Fix a finite set of nonnegative integers $\vec{e} = (e_1, \ldots, e_t)$, and let $T^X_{\vec{e}}$ denote the collection of all maps $f : X \to T$ such that each of the inverse images $f^{-1}\{y_i\}$ has size exactly $e_i$. (If we think of $f$ as a coloring of the set $X$ using the set of colors $\{y_1, \ldots, y_t\}$, then we are requiring that the color $y_i$ is used exactly $e_i$ times.)*

*The cardinality of the set $G\backslash T^X_{\vec{e}}$ is given by the coefficient of $Y_1^{e_1} \cdots Y_t^{e_t}$ in the polynomial*

$$Z_G(Y_1 + \cdots + Y_t, Y_1^2 + \cdots + Y_t^2, Y_1^3 + \cdots + Y_n^3, \cdots).$$

**Example 8.** If we want to compute the cardinality of $G\backslash T^X$, then we should sum the sizes of $T^X_{\vec{e}}$ over all tuples $(e_1, \ldots, e_t)$ of nonnegative integers. According to the Theorem, this is just given by the sum of all the coefficients which appear in the polynomial

$$Z_G(Y_1 + \cdots + Y_t, Y_1^2 + \cdots + Y_t^2, Y_1^3 + \cdots + Y_t^3, \ldots).$$

In other words, it is given by evaluating this polynomial at the point $Y_1 = Y_2 = \cdots = Y_t = 1$. This is just given by

$$Z_G(t, t, \ldots),$$

which recovers the version of Polya's theorem we proved last week.

The proof of this new version of Polya's theorem is much like the proof of the old version. We can compute the size of the set $G\backslash T^X_{\vec{e}}$ using Burnside's formula. It is given by

$$\frac{1}{|G|} \sum_{g \in G} |(T^X_{\vec{e}})^g|.$$

To prove the formula, it suffices to show that for each $g \in G$, the size of the set $(T^X_{\vec{e}})^g$ is given by the coefficient of the expression $Y_1^{e_1} \cdots Y_t^{e_t}$ in the cycle monomial

$$Z_g(Y_1 + \cdots + Y_t, Y_1^2 + \cdots + Y_t^2, Y_1^3 + \cdots + Y_t^3, \cdots).$$

If we let $c_i$ denote the number of $g$-orbits on $X$ having size $i$, this expression is given by

$$(Y_1 + \cdots + Y_t)^{c_1} (Y_1^2 + \cdots + Y_t^2)^{c_2} (Y_1^3 + \cdots + Y_t^3)^{c_3} \cdots,$$

from which the desired interpretation can read off easily.

**Example 9.** Let's return to the problem of coloring the faces of a regular tetrahedron, up to rotational symmetry. This time, we will try to keep track of the number of times that each color is used. The relevant group is the alternating group $A_4$ of even permutations of the set $\{1, 2, \ldots, 4\}$, acting on the set of faces of the tetrahedron. Up to conjugacy in $\Sigma_4$, the group $A_4$ has three types of elements:

- The identity element, which has cycle monomial $s_1^4$.
- Permutations which break into a pair of two cycles. There are three of these, each of which has two orbits of size 2, and therefore cycle monomial $s_2^2$.
- Permutations given by a 3-cycle and a fixed point. There are eight of these, each of which has cycle monomial $s_1 s_3$.

It follows that the cycle index of the alternating group $A_4$ (with its standard action on $\{1, 2, 3, 4\}$) is given by

$$Z_G(s_1, s_2, \ldots) = \frac{s_1^4 + 3s_2^2 + 8s_1 s_3}{12}$$

To figure out the number of colorings, we substitute the power sum $Y_1^i + Y_2^i + \cdots + Y_t^i$ for the variable $s_i$, to obtain the sum

$$\frac{1}{12}(Y_1 + \cdots + Y_t)^4 + \frac{1}{4}(Y_1^2 + \cdots + Y_t^2)^2 + \frac{2}{3}(Y_1 + \cdots + Y_t)(Y_1^3 + \cdots + Y_t^3).$$

For example, suppose that $t = 4$, and we ask how many ways we can color the faces of a tetrahedron so that each color is used exactly once. Then we are looking for the coefficient of $Y_1Y_2Y_3Y_4$ in the above expression. We may therefore ignore the last two terms in the sum: we are looking for the coefficient of $Y_1Y_2Y_3Y_4$ in the expression $\frac{1}{12}(Y_1 + Y_2 + Y_3 + Y_4)^4$, which is $\frac{4!}{12} = 2$. (Of course, we're just rediscovering that the alternating group has index 2 in the group $\Sigma_4$ of *all* permutations of the faces of the tetrahedron.)