Math 155 (Lecture 10)

September 22, 2011

In the last lecture, we proved the following result:

Theorem 1 (Polya Enumeration Theorem). *Let G be a finite group, X a finite G-set, and T a finite set with t elements. Then the quotient $G \backslash T^X$ has cardinality $\frac{1}{|G|} \sum_{g \in G} t^{o(g)}$, where $o(g)$ denotes the number of g-orbits of the set X.*

In this lecture, we will describe some simple applications.

Question 2. Suppose we are given a circular wheel with n spokes. How many ways can the wheel be colored with t colors (up to rotational symmetries)?

In the situation of Question 2, the relevant symmetry group is the cyclic group $G = \mathbf{Z}/n\mathbf{Z}$ of order n, acting on itself by translation.

Example 3. Let n be a prime number p. Then every non-identity element of G generates G. It follows that

$$o(g) = \begin{cases} 1 & \text{if } g \neq e \\ p & \text{if } g = e. \end{cases}$$

Applying Polya's formula, we deduce that the number of colorings is

$$\frac{1}{p}((p-1)t + t^p) = t + \frac{t^p - t}{p}$$

. In particular, this reproduces our proof from the first lecture that $t^p - t$ is divisible by p.

Example 4. Let $n = p^2$ for some prime number p. The elements of the group $\mathbf{Z}/p^2\mathbf{Z}$ have three types:

- If d is not divisible by p, then the residue class of d in $G = \mathbf{Z}/p^2\mathbf{Z}$ is a generator, and has only one orbit on G. There are $p^2 - p$ such residue classes.
- If d is divisible by p but not by p^2, then d generates a subgroup of $\mathbf{Z}/p^2\mathbf{Z}$ having size p, and therefore has exactly p orbits. There are $p - 1$ residue classes of such integers in $\mathbf{Z}/p^2\mathbf{Z}$.
- The identity element of $\mathbf{Z}/p^2\mathbf{Z}$ has p^2 orbits.

We conclude that the number of colorings is given by

$$\frac{1}{p^2}((p^2 - p)t + (p-1)t^p + t^{p^2}) = t + \frac{t^p - t}{p} + \frac{t^{p^2} - t^p}{p^2}.$$

Since the first two terms are integers, we conclude that $\frac{t^{p^2} - t^p}{p^2}$ is also an integer.

Example 5. More generally, suppose that $n = p^k$ for some integer k. Every element of $\mathbf{Z}/n\mathbf{Z}$ is given by the residue class of some integer d. Let p^a be the largest power of p that divides d. If $a \geq k$, then the residue class of d is the identity of G, which has exactly p^k orbits. Otherwise, the residue class of d has order p^{k-a}, and therefore has exactly p^a orbits. For fixed $a < k$, there are exactly $p^{k-a} - p^{k-a-1}$ such residue classes. It follows that the answer to Question 2 is given by

$$\frac{1}{p^k}(t^{p^k} + \sum_{a<k}(p^{k-a} - p^{k-a-1})t^{p^a})) = t + \frac{t^p - t}{p} + \frac{t^{p^2} - t^p}{p^2} + \frac{t^{p^3} - t^{p^2}}{p^3} + \cdots + \frac{t^{p^k} - t^{p^{k-1}}}{p^k}.$$

Subtracting the analogous expression for the cyclic group $\mathbf{Z}/p^{k-1}\mathbf{Z}$, we obtain the following generalization of Fermat's Little Theorem:

Proposition 6. *Let $t \geq 0$ and $k \geq 1$ be integers and let p be a prime number. Then $t^{p^k} - t^{p^{k-1}}$ is divisible by p^k.*

Question 7. In how many ways can one paint the faces of a tetrahedron with t colors, up to rotational symmetry?

The rotational symmetries of a tetrahedron form a subgroup of the group Σ_4 of all permutations of the faces of the tetrahedron. It is not difficult to see that this subgroup is the group of all *even* permutations of four elements. It contains three types of elements:

(a) The identity permutation, which has four orbits on the set of faces.

(b) Permutations which fix one face and cyclically permute the other three. These have two orbits on the set of faces, and there are eight such permutations.

(c) Permutations which exchange two pairs of faces. There are three such permutations, and each has two orbits on the set of faces.

It follows from Polya's theorem that the answer to Question 7 is given by

$$\frac{1}{12}(t^4 + 8t^2 + 3t^2) = t^2 + \frac{t^4 - t^2}{12}.$$

Question 8. Up to isomorphism, how many graphs are there with n vertices?

Equivalently, how large is the set $\Sigma_n \backslash S$, where Σ_n denotes the symmetric group on n letters and S is the set of graphs with vertex set $\{1, 2, \ldots, n\}$? Let E denote the set of unordered pairs of distinct elements of $\{1, \ldots, n\}$: then E is a set with $\binom{n}{2}$ elements, acted on by Σ_n. The set S is just the collection of all subsets of E. Equivalently, we can identify S with the set of all functions $E \to \{0, 1\}$. This identification places our problem in the context of Polya's theorem: we are trying to determine the number of elements of the quotient

$$\Sigma_n \backslash \{0, 1\}^E.$$

Using Polya's theorem, we see that this number is given by

$$\frac{1}{n!} \sum_{\sigma \in \Sigma_n} 2^{o(\sigma)},$$

where $o(\sigma)$ denotes the number of σ-orbits on the set E.

Example 9. Let's take $n = 6$. The set E has $\binom{6}{2} = 15$ elements. There are several types of permutations to consider:

(a) The identity permutation, with cycle structure (1)(2)(3)(4)(5)(6). There is exactly one such permutation and it has 15 orbits on E, and therefore contributes $2^15 = 32768$ to the sum.

(b) Permutations with cycle structure (12)(3)(4)(5)(6). There are exactly 15 of these, and each has 11 orbits on E. The total contribution of these terms is therefore $15 \times 2^1 1 = 30720$.

(c) Permutations with cycle structure (12)(34)(5)(6). There are 45 of these, and each has 9 orbits on E. The total contribution of these terms is therefore $45 \times 2^9 = 23040$.

(d) Permutations with cycle structure (12)(34)(56). There are 15 of these, and each has 9 orbits on E. The total contribution of these terms is therefore $15 \times 2^9 = 7680$.

(e) Permutations with cycle structure (123)(4)(5)(6). There are 40 of these, and each has 7 orbits on E. The total contribution of these terms is therefore $40 \times 2^7 = 5120$.

(f) Permutations with cycle structure (123)(45)(6). There are 120 of these, and each has 5 orbits on E. The total contribution of these terms is therefore $120 \times 2^5 = 3840$.

(g) Permutations with cycle structure (123)(456). There are 40 of these, and each has 5 orbits on E. The total contribution of these terms is therefore $40 \times 2^5 = 1280$.

(h) Permutations with cycle structure (1234)(5)(6). There are 90 of these, and each has 5 orbits on E. The total contribution of these terms is therefore $90 \times 2^5 = 2880$.

(i) Permutations with cycle structure (1234)(56). There are also 90 of these, and each has 5 orbits on E. We again get a contribution of $90 \times 2^5 = 2880$.

(j) Permutations with cycle structure (12345)(6). There are 144 of these, and each has 3 orbits on E. We get a contribution of $144 \times 2^3 = 1152$.

(k) Permutations with cycle structure (123456). There are 120 of these, and each has 3 orbits on E. We get a contribution of $120 \times 2^3 = 960$.

Summing these contributions up, we get

$$\sum_{\sigma \in \Sigma_6} 2^{o(\sigma)} = 112320.$$

It follows that there are exactly $\frac{112320}{720} = 156$ isomorphism classes of graphs with six vertices.