Math 114, Problem Set 9 (due Monday, November 18)

November 11, 2013

(1) Let V be a Banach space, and let $f : V \to \mathbb{R}^n$ be a linear map. Show that f is bounded if and only if the kernel $\ker(f)$ is a closed subset of V.

(2) Let V be a Banach space with norm $|| \bullet ||_V$, let $V_0 \subseteq V$ be a subspace, and let W denote the quotient V/V_0. Define a map $|| \bullet ||_W : W \to \mathbb{R}$ by the formula

$$||x||_W = \inf\{||\widetilde{x}||_V : \widetilde{x} \in V \text{ represents } x \}.$$

Show that if V_0 is closed in V, then $|| \bullet ||_W$ is a norm which makes W into a Banach space.

(3) Let $E \subseteq \mathbb{R}^n$ be a measurable set. Let $M(E)$ be the collection of all finite signed measures on E. For $\nu \in M(E)$, define

$$||\nu|| = \sup\{\nu(S) - \nu(T)\}$$

where the supremum is taken over all pairs of disjoint measurable subsets $S, T \subseteq E$ such that $\nu(S) \geq 0$ and $\nu(T) \leq 0$. Show that the construction $\nu \mapsto ||\nu||$ is a norm on the vector space $M(E)$ which makes $M(E)$ into a Banach space.

(4) Let $E \subseteq \mathbb{R}^n$ be a measurable set. Given a function $f \in L^1(E)$, define ν_f by the formula $\nu_f(S) = \int_S f|_S$. Show that ν_f is a finite signed measure on E, and that the construction

$$f \mapsto \nu_f$$

determines an isometry (that is, a norm-preserving map) from $L^1(E)$ onto a closed subspace of $M(E)$.