Math 114, Problem Set 8 (due Monday, November 11)

November 4, 2013

(1) Let $K \subseteq \mathbb{R}^n$ be a set with the following properties:

- (a) The set K is convex. That is, given points $x, y \in K$, the line segment $\{\lambda x + (1-\lambda)y : \lambda \in [0,1]\}$ is contained in K.
- (b) The set K is compact.
- (c) The set K contains an open neighborhood of $0 \in \mathbb{R}^n$.
- (d) The set K is symmetric: that is, if $x \in K$, then $-x \in K$.

Show that there is a unique norm $|| \bullet || : \mathbb{R}^n \to \mathbb{R}$ such that $K = \{x \in \mathbb{R}^n : ||x|| \leq 1\}$.

(2) Let V be a vector space (possibly of infinite dimension) and let $V_0 \subseteq V$ be a subspace. Show that every linear functional $f_0 : V_0 \to \mathbb{R}$ can be extended to a linear functional $f : V \to \mathbb{R}$ (use problem (4)).

(3) Let $f : V \to W$ be a linear map between normed vector spaces. Show that if V is finite-dimensional, then f is continuous.

(4) Let P be a partially ordered set and suppose that every linearly ordered subset of P has an upper bound. Prove that P has a maximal element (Zorn's lemma) by completing the argument outlined in class. Assume (for a contradiction) that P has no maximal element.

- (a) Show that for each linearly ordered subset $Q \subseteq P$, there exists an element $\lambda(Q) \in P$ such that $q < \lambda(Q)$ for each $q \in Q$.

We will say that a subset $Q \subseteq P$ is a *good chain* if Q is well-ordered and each element $x \in Q$ satisfies the formula $x = \lambda(\{q \in Q : q < x\})$.

- (b) Show that there is no largest good chain in P (hint: show that if Q is a good chain, then $Q \cup \{\lambda(Q)\}$ is also a good chain).
- (c) Show that if Q and Q' are good chains, then exactly one of the following conditions holds:
  - (i) $Q = Q'$.
  - (ii) There exists an element $q_0 \in Q$ such that $Q' = \{q \in Q : q < q_0\}$.
  - (iii) There exists an element $q'_0 \in Q'$ such that $Q = \{q' \in Q' : q' < q'_0\}$.
- (d) Show that if $\{Q_\alpha\}$ is a collection of good chains, then the union $\bigcup Q_\alpha$ is also a good chain.
- (e) Find a contradiction between (b) and (d).