Math 114, Problem Set 6 (due Monday, October 28)

October 21, 2013

(1) Let X be a metric space, and let $\{x_n\}_{n \geq 0}$ be a sequence of points in X which satisfies the following conditions:

  (a) For every subsequence $\{x_{i_0}, x_{i_1}, \ldots\}$ of $\{x_n\}_{n \geq 0}$, there exists a further subsequence $\{x_{i_{j_0}}, x_{i_{j_1}}, \ldots\}$ which converges.

  (b) For any pair of convergent subsequences $\{x_{i_0}, x_{i_1}, x_{i_2}, \ldots\}$, $\{x_{j_0}, x_{j_1}, x_{j_2}, \ldots\}$ of $\{x_n\}_{n \geq 0}$, the limits $\lim\{x_{i_n}\}$ and $\lim\{x_{j_n}\}$ are the same.

Show that the sequence $\{x_n\}_{n>0}$ converges.

(2) Let E be a measurable subset of $\mathbb{R}^m \times \mathbb{R}^n$. For each $x \in \mathbb{R}^m$, let $E_x = \{y \in \mathbb{R}^n : (x, y) \in E\}$. Show that E has measure zero if and only if the sets E_x have measure zero for almost every x.

(3) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function given by

$$f(x,y) = \begin{cases} 1 & \text{if } (\exists n \in \mathbf{Z}_{\geq 0})[n \leq x, y < n+1] \\ -1 & \text{if } (\exists n \in \mathbf{Z}_{\geq 0})[n \leq x < n+1 \leq y < n+2] \\ 0 & \text{otherwise.} \end{cases}$$

For each $x \in \mathbb{R}$, let f_x denote the function given by $f_x(y) = f(x, y)$. For each $y \in \mathbb{R}$, let f_y denote the function given by $f_y(x) = f(x, y)$. Show that the functions

$$x \mapsto \int_{\mathbb{R}} f_x \qquad y \mapsto \int_{\mathbb{R}} f_y$$

are integrable, and compute their integrals (in other words, compute the double integrals $\int(\int f(x,y)dx)dy$ and $\int(\int f(x,y)dy)dx$.) Why does the result not contradict Fubini's theorem?

(4) Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the inequality

$$\phi(\frac{x+y}{2}) \leq \frac{\phi(x) + \phi(y)}{2}$$

for all $x, y \in \mathbb{R}$. Show that ϕ is convex: that is, for each real number $\lambda \in [0, 1]$, we have

$$\phi(\lambda x + (1-\lambda)y) \leq \lambda\phi(x) + (1-\lambda)\phi(y)$$

for all $x, y \in \mathbb{R}$.