Math 114, Problem Set 4 (due Monday, October 7)

September 29, 2013

(1) Let $E \subseteq \mathbb{R}^n$ be a measurable set and let $f : E \to \mathbb{R}$ be a nonnegative measurable function. Show that the set $\{x \in E : f(x) \neq 0\}$ has measure zero if and only if $\int_E f = 0$.

(2) Let $E \subseteq \mathbb{R}^n$ be a measurable set, and let $f_1, f_2, \ldots$ be a sequence of measurable functions on E which converges pointwise to another function $f : E \to \mathbb{R}^n$. Show that there exists a sequence of subsets

$$E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots \subseteq E$$

such that $\mu(E - \bigcup E_i) = 0$ and the sequence $\{f_i|_{E_j}\}$ converges uniformly to $f|_{E_j}$ for each j.

(3) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a measurable function. Show that for each $\epsilon > 0$, there exists a continuous function $g : \mathbb{R}^n \to \mathbb{R}$ such that the set $\{x \in \mathbb{R}^n : f(x) \neq g(x)\}$ has measure $< \epsilon$.

(4) Let $E \subseteq \mathbb{R}^m$ and $E' \subseteq \mathbb{R}^n$ be measurable sets. Show that $E \times E'$ is a measurable subset of $\mathbb{R}^{m+n}$, and that $\mu_{\mathbb{R}^{m+n}}(E \times E') = \mu_{\mathbb{R}^m}(E)\mu_{\mathbb{R}^n}(E')$. Here $\mu_{\mathbb{R}^k}$ denotes Lebesgue measure on $\mathbb{R}^k$ (hint: reduce to the case where $\mu_{\mathbb{R}^m}(E) < \infty$ and study the function $S \mapsto \mu_{\mathbb{R}^{m+n}}(E \times S)$).