Math 114, Problem Set 2 (due Monday, September 23)

September 16, 2013

(1) Let $S \subseteq \mathbb{R}^n$ be a measurable set with $\mu(S) < \infty$, and let $\epsilon > 0$ be a positive real number. Show that there exists a compact subset $K \subseteq S$ such that

$$\mu(S) - \epsilon \leq \mu(K) \leq \mu(S).$$

(2) Let X be a set and let $\mathcal{M}$ be a σ-algebra of subsets of X. Suppose that $m : \mathcal{M} \to [0, \infty]$ is a function satisfying the following axioms:

(a) The function m is finitely additive. That is, if $S, T \subseteq X$ are disjoint sets belonging to $\mathcal{M}$, then $m(S \cup T) = m(S) + m(T)$.

(b) The function m is countably subadditive. That is, for every sequence of subsets $S_0, S_1, S_2, S_3, \ldots \subseteq X$ which belong to $\mathcal{M}$, we have an inequality

$$m(\bigcup_{n \geq 0} S_n) \leq \sum_{n \geq 0} m(S_n).$$

Show that m is countably additive. That is, if $S_0, S_1, S_2, \ldots \subseteq X$ is a sequence of pairwise disjoint subsets of X which belong to $\mathcal{M}$, show that $m(\bigcup_{n \geq 0} S_n) = \sum_{n \geq 0} m(S_n)$.

(3) Let E be a subset of $\mathbb{R}^n$. Show that E is measurable if and only if $\mu^*(B \cap E) + \mu^*(B \cap E^c) = \mu^*(B)$ for every open box $B \subseteq \mathbb{R}^n$.

(4) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Recall that f is *differentiable* at a point $x \in \mathbb{R}$ if the expression $\frac{f(x+h) - f(x)}{h}$ approaches a limit as $h \to 0$. Show that $S = \{x \in \mathbb{R} \mid f \text{ is differentiable at } x\}$ is a Borel set.