Math 114 Midterm (with solutions)

October 16, 2013

(1) Let $C = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ be the unit circle. Show that C has measure zero (when regarded as a subset of the Euclidean plane $\mathbb{R}^2$).

For each positive real number r, define a function μ_r on measurable subsets of $\mathbb{R}^2$ by the formula $\mu_r(E) = \mu(rE)$. Then μ_r is countably additive and translation invariant, so (as we saw in class) there exists a real number λ such that $\mu_r(E) = \lambda\mu(E)$ for every measurable subset $E \subseteq \mathbb{R}^2$. Taking $E = (0,1)^2$, we deduce that

$$\lambda = \lambda\mu(E) = \mu_r(E) = \mu(rE) = \mu((0,r)^2) = r^2.$$

Let $B = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. Then B an open subset of $\mathbb{R}^2$, hence measurable. It is contained in an open box, and therefore has finite measure. Note that for each $r > 1$, rB contains C and B as disjoint subsets. We therefore have

$$\mu(C) + \mu(B) \leq \mu(rB) = \mu_r(B) = r^2\mu(B).$$

Since B contains an open box, it has positive measure. Dividing by $\mu(B)$, we obtain an inequality

$$\frac{\mu(C)}{\mu(B)} + 1 \leq r^2.$$

Since this inequality holds for an arbitrary real number $r^2 \geq 1$, we must have $\frac{\mu(C)}{\mu(B)} = 0$, so that $\mu(C) = 0$.

(2) Let $E \subseteq \mathbb{R}^n$ be a measurable set, and let $f_1, f_2, \ldots$ be a sequence of measurable functions from E into $\mathbb{R}$. Show that the set $S = \{x \in E : \{f_i(x)\}_{i>0} \text{ converges }\}$ is a measurable subset of E.

Note that the sequence of real numbers $\{f_i(x)\}_{i>0}$ converges if and only if it is a Cauchy sequence: that is, if and only if

$$(\forall k > 0)(\exists m)(\forall i,j > m)|f_i(x) - f_j(x)| < \frac{1}{k}.$$

Since each f_i is a measurable function, the differences $f_i - f_j$ are measurable functions, and therefore the sets

$$S_{i,j,k} = \{x \in E : |f_i(x) - f_j(x)| < \frac{1}{k}\}$$

are measurable. Because the collection of measurable sets is closed under countable unions and intersections, it follows also that

$$S = \bigcap_{k>0} \bigcup_m \bigcap_{i,j>m} S_{i,j,k}$$

is measurable.

(3) Let $E \subseteq \mathbb{R}^n$ be a measurable set and let $f : E \to \mathbb{R}$ be an integrable function. Prove that for each $\epsilon > 0$, there exists a positive real number δ with the following property: for every measurable subset $S \subseteq E$ with $\mu(S) < \delta$, we have

$$-\epsilon < \int_S f|_S < \epsilon.$$

For each integer m, let $f_m : E \to \mathbb{R}^n$ be the function given by

$$f_m(x) = \begin{cases} |f(x)| & \text{if } |f(x)| \leq m. \\ m & \text{otherwise.} \end{cases}$$

Then $f_1, f_2, \ldots$ is a monotone sequence of functions converging pointwise to $|f|$. Using the Monotone Convergence Theorem, we can choose an integer m such that

$$\int_E f_m > (\int_E |f|) - \frac{\epsilon}{2}.$$

Set $\delta = \frac{\epsilon}{2m}$. If $S \subseteq E$ is a measurable set with $\mu(S) < \delta$, then we have

$$\begin{aligned} |\int_S f| &\leq \int_S |f| \\ &\leq \int_S f_m + \int_S (|f| - f_m) \\ &\leq \int_S f_m + \int_E (|f| - f_m) \\ &\leq m\mu(S) + \frac{\epsilon}{2} \\ &< m\delta + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$