PROBLEM SET VIII: PROBLEMS III, IV

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Problem 1. Let $f : V \to W$ be a linear map between normed vector spaces. Show that if V is finite-dimensional, then f is continuous.

Proof. Let $\{e_1, ..., e_n\}$ be a basis for V. Then

$$\|f(v)\|_W = \|f(v_1 e_1 + \cdots + v_n e_n)\|_W = \|v_1 f(e_1) + \cdots + v_n f(e_n)\|_W$$

$$\leq |v_1| \|f(e_1)\|_W + \cdots + |v_n| \|f(e_n)\|_W \leq \sum_{k=1}^{n} |v_k| \left(\max_{1 \leq k \leq n} \|f(e_k)\|_W \right).$$

Define the 1-norm, $\|\cdot\|_1 : V \to \mathbb{R}$, by

$$\|v\|_1 = \sum_{k=1}^{n} |v_k|, \quad \text{with } v = \sum_{k=1}^{n} v_k e_k.$$

It suffices to show that $\|v\|_1 \leq \alpha \|v\|_V$ for arbitrary constant α and V a finite dimensional vector space. We have the following

$$\|v\|_V = \left\| \sum_{k=1}^{n} v_k e_k \right\|_V \leq \sum_{k=1}^{n} |v_k| \|e_k\|_V \leq \left(\max_{1 \leq k \leq n} \|e_k\|_V \right) \|v\|_1.$$

This implies $\|v\|_V \leq \beta \|v\|_1$ for some constant β. Thus, our norm on V, $\|\cdot\|_V : V \to \mathbb{R}$, is continuous with respect to the topology induced by the 1-norm. Taking $\|v - w\|_1 \leq \varepsilon$, we see that

$$|\|v\|_V - \|w\|_V| \leq \|v - w\|_V \leq M\varepsilon,$$

by the Triangle Inequality. Consider the compact set $\Omega = \{v \in V : \|v\|_1 = 1\}$. By the compactness of our set and the continuity of the norm on V, $\|\cdot\|_V$ achieves a minimum on Ω. Denoting this minimum by ξ, we have $0 < \xi \leq \|v\|_V$, for any $v \in V$ where $\|v\|_1 = 1$. Thus, $\xi \|v\|_1 \leq \|v\|_V$. Selecting our constant α appropriately yields our desired result. □

Problem 2. Let P be a partially ordered set and suppose that every linearly ordered subset of P has an upper bound. Prove that P has a maximal element by completing the argument outlined in class. Assume (for a contradiction) that P has no maximal element.

(a) Show that for each linearly ordered subset $Q \subseteq P$, there exists an element $\lambda(Q) \in P$ such that $q < \lambda(Q)$ for each $q \in Q$.

We will say that a subset $Q \subseteq P$ is a *good chain* if Q is well-ordered and each element $x \in Q$ satisfies the formula $x = \lambda(\{q \in Q : q < x\})$.

(b) Show that there is no largest good chain in P.

(c) Show that if Q and Q' are good chains, then exactly one of the following conditions holds: (i) $Q = Q'$; (ii) There exists an element $q_0 \in Q$ such

Date: November 11, 2013.

that $Q' = \{q \in Q : q < q_0\}$; (iii) There exists an element $q_0' \in Q'$ such that $Q = \{q' \in Q' : q' < q_0'\}$.

(d) Show that if $\{Q_\alpha\}$ is a collection of good chains, then the union $\bigcup Q_\alpha$ is also a good chain.

(e) Find a contradiction between (b) and (d).

Proof. (a) Let $x \in Q$ be an upper bound for Q. Since x is not maximal, there exists some $\tilde{x} \in P$ such that $\tilde{x} > x$ (by the Axiom of Choice). Set $\lambda(Q) = \tilde{x}$, so that for any $y \in Q$, we have $y \leq x < \lambda(Q)$.

(b) Let us assume that Q is the largest good chain in P. Let $Q^+ = Q \cup \{\lambda(Q)\}$. For each $x \in Q$, we have that $x = \lambda(\{q \in Q^+ : q < x\})$, since $\lambda(Q) > x$ for all $x \in Q$. Definitionally, $\lambda(Q) = \lambda(\{q \in Q^+ : q < x\})$, so Q^+ is a good chain.

(c) If $Q = Q'$ we are done. Thus, let us suppose that Q, Q' are good chains and $Q \neq Q'$. We show that either $Q \subset Q'$ or $Q' \subset Q$. Let Λ be the union of all subsets of P such that $P \subseteq Q$ and $P \subseteq Q'$. Then Λ is the largest such set, which is also well-ordered by the well-ordering of Q and Q'. Suppose that $\Lambda \neq Q$ and $\Lambda \neq Q'$. Then we may select $q \in Q$ and $q' \in Q'$ such that q is the minimal element of Q and $q \notin S$. Define q' analogously. Thus, $\Lambda \subseteq \{x \in Q : x < q\}$ and $\Lambda \subseteq \{x \in Q' : x < q'\}$. Then let x_m be the smallest element of Q such that $x_m < q$ and $x_m \notin \Lambda$. Then we have

$$\{x \in Q : x < x_m\} \subseteq \Lambda \subseteq \{x \in Q' : x < q'\} \subseteq Q.$$

Thus, $x_m = \lambda(\{x \in Q : x < x_m\}) \subseteq Q'$ so $x_m \in Q \cap Q'$. However, x_m could have been appended to Λ, thereby contradicting the maximality of Λ. Thus

$$\Lambda = \{x \in Q : x < q\} = \{x \in Q' : x < q'\} \Rightarrow \lambda(\Lambda) = q = q'.$$

However, we could have appended q to Λ, again contradicting the maximality of Λ. Thus, either $Q \subset Q'$ or $Q' \subset Q$.

WLOG suppose that $Q \subset Q'$, and let $q' \subset Q'$ be the smallest element of Q' such that $q' \notin Q$. We have $\{x \in Q' : x < q'\} \subseteq Q$. Suppose that $Q \neq \{x \in Q' : x < q'\}$, and let x_m be the smallest element in Q such that $x_m > q'$. However, by definition and total ordering, we have

$$x_m = \lambda(\{x \in Q : x < x_m\}) = \lambda(\{x \in Q : x < q'\}) = q'.$$

Contradiction. Thus, $Q = \{x \in Q' : x < q'\}$. The second case follows analogously if $Q' \subset Q$.

(d) Let $\{Q_\alpha\}$ be a collection of good chains. From (c), we know that for any two elements Q_θ, Q_ω of $\{Q_\alpha\}$, we have $Q_\theta \subset Q_\omega, Q_\omega \subset Q_\theta$, or $Q_\theta = Q_\omega$. Thus, the union of any number of good chains will be equal to a good chain, and $\bigcup Q_\alpha$ is a good chain.

(e) The above implies that $\bigcup Q_\alpha$ is the largest good chain. This contradicts (b), so we conclude that P has a maximal element. $\square$