PROBLEM SET VI SOLUTIONS (3,4)

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Problem 1. Let $f:\mathbb{R}^2\to\mathbb{R}$ be the function given by

$$f(x,y)=\begin{cases}1 & \text{if } (\exists n\in\mathbb{Z}_{\geq 0})\,[n\leq x,y<n+1]\\ -1 & \text{if } (\exists n\in\mathbb{Z}_{\geq 0})\,[n\leq x<n+1\leq y<n+2]\\ 0 & \text{otherwise.}\end{cases}$$

For each $x\in\mathbb{R}$, let f_x denote the function given by $f_x(y)=f(x,y)$. For each $y\in\mathbb{R}$, let f_y denote the function given by $f_y(x)=f(x,y)$. Show that the functions

$$x\mapsto\int_{\mathbb{R}}f_x,\quad y\mapsto\int_{\mathbb{R}}f_y$$

are integrable, and compute their integrals (in other words, compute the double integrals $\int\left(\int f(x,y)\,dx\right)dy$ and $\int\left(\int f(x,y)\,dy\right)dx$.) Why does the result not contradict Fubini's Theorem?

Proof. We may write

$$f_x(y)=\begin{cases}1, & x\geq 0 \text{ and } \lfloor x\rfloor\leq y\leq\lfloor x\rfloor+1,\\ -1, & x\geq 0 \text{ and } \lfloor x\rfloor+1\leq y\leq\lfloor x\rfloor+2,\\ 0, & \text{else,}\end{cases}$$

and

$$f_y(x)=\begin{cases}1, & y\geq 0 \text{ and } \lfloor y\rfloor\leq x\leq\lfloor y\rfloor+1,\\ -1, & y\geq 1 \text{ and } \lfloor y\rfloor-1\leq x\leq\lfloor y\rfloor,\\ 0, & \text{else.}\end{cases}$$

From this it is clear that

$$\int_{\mathbb{R}}f_x=0,\qquad\int_{\mathbb{R}}f_y=\begin{cases}1, & y\in[0,1),\\ 0, & \text{else.}\end{cases}$$

Thus, we have that

$$\int\left(\int f(x,y)\,dx\right)dy=0,\qquad\int\left(\int f(x,y)\,dy\right)dx=1.$$

This does not contradict Fubini's Theorem because $f(x,y)$ is not an integrable function. Explicitly, observe that $|f|\geq\chi_{\mathscr{S}}$ where $\mathscr{S}=\{(x,y):n\leq x,y<n+1,n\in\mathbb{Z}_{\geq 0}\}$, and $\mu(\mathscr{S})=\infty$. The measure of $\mathscr{S}$ is infinite because

$$\mathscr{S}=\bigsqcup_{n=0}^{\infty}\{(x,y):n\leq x,y<n+1\}$$

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and each $\{(x,y) : n \le x, y < n+1\}$ has measure one. Therefore,

$$\int_{\mathbb{R}^2} |f| \ge \int_{\mathbb{R}^2} \chi_{\mathscr{S}} = \mu(\mathscr{S}) = \infty.$$

□

Problem 2. Let $\phi : \mathbb{R} \to \mathbb{R}$ be the continuous function satisfying the inequality

$$\phi\left(\frac{x+y}{2}\right) \le \frac{\phi(x) + \phi(y)}{2}$$

for all $x, y \in \mathbb{R}$. Show that ϕ is convex; that is, for each real number $\lambda \in [0,1]$, we have

$$\phi(\lambda x + (1-\lambda) y) \le \lambda\phi(x) + (1-\lambda)\phi(y)$$

for all $x, y \in \mathbb{R}$.

Proof. We will demonstrate the convexity condition for every dyadic $\lambda = \alpha/2^i \in [0,1]$ by induction on the power of two in the denominator. We have the base case by assumption. Now suppose that convexity holds for dyadic λ with the exponent in the denominator i or less. Then, taking $\alpha = 2\beta + 1$ to be odd, we have

$$\begin{aligned} &\phi\left(\left(\frac{\alpha}{2^{i+1}}\right)x + \left(1 - \frac{\alpha}{2^{i+1}}\right)y\right) \\ \le \frac{1}{2}&\left(\phi\left(\left(\frac{\beta}{2^i}\right)x + \left(1-\frac{\beta}{2^i}\right)y\right) + \phi\left(\left(\frac{\beta+1}{2^i}\right)x + \left(1 - \frac{\beta+1}{2^i}\right)y\right)\right) \\ &\le \frac{1}{2}\left(\left(\frac{\beta}{2^i} + \frac{\beta+1}{2^i}\right)\phi(x) + \left(2 - \frac{\beta+1}{2^i} - \frac{\beta+1}{2^{i+1}}\right)\phi(y)\right) \\ &= \left(\frac{\alpha}{2^{i+1}}\right)\phi(x) + \left(1 - \frac{\alpha}{2^{i+1}}\right)\phi(y). \end{aligned}$$

More generally, suppose that $\lambda \in [0,1]$, and select a convergent sequence $\{\alpha_i/2^i\} \to \lambda$. Fix $x, y \in \mathbb{R}$, and let $\epsilon > 0$. Then, let

$$f(z) = \phi(zx + (1-z)y).$$

By continuity and convergence, there exists some $n > 0$ such that

$$\left|\frac{\alpha_n}{2^n} - \lambda\right| < \frac{\epsilon}{2\max|\phi(x)|, |\phi(y)|}$$

and

$$\left|f\left(\frac{\alpha_n}{2^n}\right) - f(\lambda)\right| \le \frac{\epsilon}{2}.$$

Thus,

$$f(\lambda) \le f\left(\frac{\alpha_n}{2^n}\right) + \frac{\epsilon}{2} \le \frac{\alpha_n}{2^n}\phi(x) + \left(1 - \frac{\alpha_n}{2^n}\right)\phi(y) + \frac{\epsilon}{2} \le \lambda\phi(x) + (1-\lambda)\phi(y) + \epsilon.$$

Since our ϵ was arbitrarily chosen, we are done. □