PROBLEM SET IV: PROBLEMS III, IV

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Problem 1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a measurable function. Show that for each $\epsilon > 0$, there exists a continuous function $g : \mathbb{R}^n \to \mathbb{R}$ such that the set $\{x \in \mathbb{R}^n : f(x) \neq g(x)\}$ has measure $< \epsilon$.

Proof. In this problem we extend the case of Lusin's Theorem for $\mu(E) < \infty$ to the case where $E = \mathbb{R}^n$. Let $\epsilon > 0$ be given and partition $\mathbb{R}^n$ into countably many disjoint unit boxes E_i. We see that the measure of the closure of the E_i is finite, so we apply Lusin's Theorem for the finite measure case to obtain closed sets $K_i \subseteq \overline{E}_i$ and continuous functions $g_i : \mathbb{R}^n \to \mathbb{R}$ such that $f = g_i$ on K_i and

$$\mu\left(\overline{E}_i - K_i\right) < \frac{\epsilon}{2^{i+1}}.$$

Taking $F_i \subseteq \overline{E}_i$ where

$$\mu\left(\overline{E}_i - F_i\right) < \frac{\epsilon}{2^{i+1}},$$

define a continuous bump function $\beta_i : \overline{E}_i \to \mathbb{R}$ that is precisely one on $\overline{E}_i - F_i$ and goes to zero on the boundaries of $\overline{E}_i$. Clearly then, the function $g_i\beta_i : \overline{E}_i \to \mathbb{R}$ is continuous, since it is the product of two continuous functions. Consider the set

$$S_i = \left\{x \in \overline{E}_i : f(x) \neq g_i(x)\,\beta_i(x)\right\}.$$

By construction, we have

$$\mu(S_i) < \mu\left(\overline{E}_i - K_i\right) + \mu\left(\overline{E}_i - F_i\right) = \frac{\epsilon}{2^i}.$$

Since each $g_i\beta_i$ is zero at the boundaries of the box over which it is defined, $g : \mathbb{R}^n \to \mathbb{R}$, given by $g(x) = g_i(x)\,\beta_i(x)$ for $x \in \overline{E}_i$, is a continuous function. Furthermore, g and f disagree on the union of the S_i. Finally,

$$\mu\left(\bigcup_{i>0} S_i\right) \leq \sum_{i>0} \mu(S_i) < \sum_{i>0} \frac{\epsilon}{2^i} = \epsilon.$$

Thus, we conclude g is our desired function. □

Problem 2. Let $E \subseteq \mathbb{R}^m$ and $E' \subseteq \mathbb{R}^n$ be measurable sets. Show that $E \times E'$ is a measurable subset of $\mathbb{R}^{m+n}$, and that $\mu_{\mathbb{R}^{m+n}}(E \times E') = \mu_{\mathbb{R}^m}(E)\,\mu_{\mathbb{R}^n}(E')$. Here $\mu_{\mathbb{R}^k}$ denotes the Lebesgue measure on $\mathbb{R}^k$. Hint: reduce to the case where $\mu_{\mathbb{R}^m}(E) < \infty$ and study the function $S \mapsto \mu_{\mathbb{R}^{m+n}}(E \times S)$.

Proof. First we will prove two brief lemmas: □

Lemma. *Let $B_1 \subset \mathbb{R}^m, B_2 \subset \mathbb{R}^n$ be open boxes. Then, $B_1 \times B_2$ is an open box in $\mathbb{R}^{m+n}$ and*

$$\left|(B_1 \times B_2)\right|_{\mathbb{R}^{m+n}} = \left|(B_1)\right|_{\mathbb{R}^m}\left|(B_2)\right|_{\mathbb{R}^n}.$$

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Proof. The result should be clear by writing B_1, B_2, and $B_1 \times B_2$ as products of intervals. $\square$

Lemma. *Let $E \subset \mathbb{R}^m$ and $E' \subset \mathbb{R}^n$ be such that $\mu^\star_{\mathbb{R}^m}(E) < \infty, \mu^\star_{\mathbb{R}^n}(E') < \infty$. Then*

$$\mu^\star_{\mathbb{R}^{m+n}}(E \times E') = \mu^\star_{\mathbb{R}^m}(E)\,\mu^\star_{\mathbb{R}^n}(E').$$

Proof. Let $\epsilon > 0$. By definition, we can find open boxes $\{Q_i\}_{i=1}^\infty$ and $\{Q_j\}_{j=1}^\infty$ in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively such that

$$E \subset \bigcup_{i=1}^\infty Q_i, \qquad E' \subset \bigcup_{j=1}^\infty Q_j$$

and

$$\sum_{i=1}^\infty |Q_i| = \mu^\star(E) + \epsilon, \qquad \sum_{j=1}^\infty |Q_j| = \mu^\star(E') + \epsilon.$$

We observe that $E \times E' \subset \cup_{i,j=1}^\infty Q_i \times Q_j$, and $\mu^\star(E \times E')$ is the infimum of

$$\Big|\bigcup_{i,j=1}^\infty Q_i \times Q_j\Big| \leq \sum_{i,j=1}^\infty |Q_i \times Q_j|$$

$$= \left(\sum_{i=1}^\infty |Q_i|\right)\left(\sum_{j=1}^\infty |Q_j|\right) = (\mu^\star(E) + \epsilon)(\mu^\star(E') + \epsilon),$$

so we have our result.

Now, if some $E \subset \mathbb{R}^m$ is measurable, then we can write $E = G_\delta - S, G_\delta, S \subset \mathbb{R}^m$, and $\mu(S) = 0$. Let $E = G_\delta - S, E' = G'_\delta - S'$. Then

$$E \times E' = (G_\delta - S) \times (G'_\delta - S') = (G_\delta \times G'_\delta - S \times G'_\delta) - (G_\delta \times S' - S \times S').$$

Since $G_\delta \times G'_\delta$ is a countable intersection of open sets, $\{U_i \times V_j\}$, with $G_\delta = \cap U_i, G'_\delta = \cap V_j$, then it is measurable. We now wish to show that the remaining terms in the above decomposition $(S \times G'_\delta, G_\delta \times S', S \times S')$ have outer measure zero, thus ensuring the measurability of $E \times E'$. It suffices to show that for $S \in \mathbb{R}^m$ with $\mu(S) = 0, \mu^\star(S \times \mathbb{R}^n) = 0$ (by the monotonicity of outer measure). Moreover, by the countable subadditivity of outer measure, it suffices to show that $\mu^\star(S \times [0,1]^n) = 0$. But this proceeds from the second lemma, and we have that $E \times E'$ is measurable.

First, assume $\mu_{\mathbb{R}^m}(E) = \infty, \mu_{\mathbb{R}^n}(E') < \infty$. Then take $E = \sqcup_{i\geq 0} S_i$ with the S_i measurable and $\sum_{i\geq 0} \mu_{\mathbb{R}^m}(S_i)$ divergent. Then

$$\mu_{\mathbb{R}^{m+n}}(E \times E') = \mu_{\mathbb{R}^{m+n}}(\sqcup_{i\geq 0} S_i \times E') = \sum_{i\geq 0} \mu_{\mathbb{R}^{m+n}}(S_i \times E')$$

$$= \mu_{\mathbb{R}^n}(E') \sum_{i\geq 0} \mu_{\mathbb{R}^m}(S_i) = \infty.$$

Thus, it suffices to prove the case where E, E' are of finite measure, as any case with infinite measure collapses.

Consider the map $\mathcal{M}_E(S) = \mu_{\mathbb{R}^{m+n}}(E \times S)$ for any $S \subset \mathbb{R}^n$. We check translation invariance and additivity for $\mathcal{M}_E(S)$ and conclude it is a measure.

For any $v \in \mathbb{R}^n$,

$$\begin{aligned} \mathcal{M}_E (S+v) &= \mu_{\mathbb{R}^{m+n}} (E \times (S+v)) = \mu_{\mathbb{R}^{m+n}} ((E \times S) + v') \\ &= \mu_{\mathbb{R}^{m+n}} (E \times S) = \mathcal{M}_E (S), \end{aligned}$$

where v' denotes the inclusion of v in $\mathbb{R}^{m+n}$.

For additivity,

$$\begin{aligned} \mathcal{M}_E (\sqcup_{i\geq 0} S_i) &= \mu_{\mathbb{R}^{m+n}} (E \times \sqcup_{i\geq 0} S_i) = \mu_{\mathbb{R}^{m+n}} (\sqcup_{i\geq 0} (E \times S_i)) \\ &= \sum_{i\geq 0} \mu_{\mathbb{R}^{m+n}} (E \times S_i) = \sum_{i\geq 0} \mathcal{M}_E (S_i). \end{aligned}$$

Finally, consider $\mathcal{M}_E ([0,1]^n)$. Since any covering of the set E by boxes can be extended to a covering of $E \times [0,1]^n$ of equivalent volume, the two sets have the same measure in $\mathbb{R}^{m+n}$. Thus, $\mathcal{M}_E$ is a measure, that is, $\mathcal{M}_E (S) = \mu_{\mathbb{R}^m} (E) \mu_{\mathbb{R}^n} (S)$, and we are done. □