PROBLEM SET X: PROBLEMS (3, 4)

PATRICK RYAN

Problem 1. Let V be a Banach space. Show that the dimension of V is either finite or uncountable (that is, V does not have a countably infinite basis).

Proof. Let V be a Banach space, and suppose that it has a countably infinite basis $\{v_i\}_{i>0}$. Next, consider the subsets $V_n = \mathcal{S}(v_1, \ldots, v_n)$, which are finite dimensional subspaces of V. It is clear that each subspace is closed in V.

Now we show that V_n is nowhere dense in V. Consider any $\alpha \in V_n$ and $\epsilon > 0$. Then the open ball $B_\epsilon(\alpha)$ contains the element

$$\left(\alpha + \frac{\epsilon}{2\|v_{n+1}\|} v_{n+1}\right),$$

which is not an element of V_n. Hence, each $V_n = \overline{V_n}$ is nowhere dense in V.

Since each V_n is closed an nowhere dense in V, the complements, $(V_n)^c$, are open and dense in V. Applying the Baire Category Theorem, the set $\bigcap_{n>0}(V_n)^c$ is dense in V, i.e.,

$$\left(\bigcap_{n>0}(V_n)^c\right)^c = \bigcup_{n>0} V_n = V$$

is nowhere dense in V. Thus we have a contradiction, and we are done. □

Problem 2. Let $E \subseteq \mathbb{R}^n$ be a measurable set with $0 < \mu(E) < \infty$. Let us regard $L^1(E)$ as a metric space, and $L^2(E)$ as a subset of $L^1(E)$. Show that $L^2(E)$ is meagre (that is, it is a countable union of nowhere dense subsets of $L^1(E)$).

Proof. Consider the sets

$$\mathcal{S}_n = \left\{ f \in L^1(E) : \int_E |f|^2 > n \right\}$$

and let $(\mathcal{S}_n)^c$ be its complement. Then we may write

$$L^2(E) = \bigcup_n (\mathcal{S}_n)^c,$$

as every element of $L^2(E)$ must have square integral bounded by n. It suffices to prove that $\mathcal{S}_n$ is dense and open for every n, as this implies that each $(\mathcal{S}_n)^c$ is nowhere dense.

Let $h \in L^1(E)/\mathcal{S}_n$ such that

$$\int_E |h|^2 \leq n.$$

Date: December 1, 2013.

We wish to show that h is in the closure of $\mathcal{S}_n$ with respect to the L^1-norm. That is, there is a sequence $\{f_k\}$ in $\mathcal{S}_n$ such that

$$\lim_{k\to\infty} \|f_k - h\|_{L^1} = 0.$$

Since we have $\mu(E) > 0$, E has a sequence of subsets $\{E_k\}$ such that $\mu(E_k) = \mu(E) / \left(k^{3/2}\right)$. Define

$$f_k = \begin{cases} h + \sqrt{n} \cdot \sqrt{k} & \text{for } x \in E_k, \\ h & \text{for } x \in E - E_k \end{cases}.$$

Then we have $f_k \in L^1(E)$ since

$$\int_E |f_k| = \int_E |h| + \sqrt{n} \cdot \left(\mu(E) / \sqrt{k}\right).$$

Since $k^{-1/2} \to 0$, we have $f_k \to h$ with respect to the L^1-norm. However,

$$\int_E |f_k|^2 = \int_E |h|^2 + \left(\mu(E) \sqrt{n}/\sqrt{k}\right) \int_E |h| + n \left(\mu(E) \sqrt{k}\right) > n.$$

Thus, $f_k \in \mathcal{S}_n$. Hence, $\mathcal{S}_n$ dense in $L^1(E)$.

Now we show the openness of $\mathcal{S}_n$ to complete the proof. Consider f bounded. For any $f \in \mathcal{S}_n$, write

$$f_\xi = \begin{cases} f & \text{for } |f| < \xi, \\ 0 & \text{else} \end{cases}.$$

Then there exists some ξ such that

$$\int_E |f|^2 > n \Longrightarrow \int_E |f_\xi|^2 > n.$$

Consider the support of f_ξ. For all $h \in L^1(E)$ such that $\left\|h\,|_{\text{supp}(f_\xi)} - f_\xi\right\|_{L^1} < \epsilon$, we will have $h\,|_{\text{supp}(f_\xi)} \in \mathcal{S}_n$. This allows us to demonstrate

$$\|h - f\|_{L^1} < \epsilon \Longrightarrow \left\|h\,|_{\text{supp}(f_\xi)} - f_\xi\right\|_{L^1} < \epsilon \Longrightarrow h\,|_{\text{supp}(f_\xi)} \in \mathcal{S}_n \Longrightarrow h \in \mathcal{S}_n.$$

Thus, we replace f by f_ξ and E by $\text{supp}(f_\xi)$. That is, we take f bounded by ξ. Finally, write h such that $\|h - f\| < \epsilon$, hence

$$\int_E |h|^2 \geq \int_E |f|^2 - 2\int_E |f(h-f)| + \int_E |h-f|^2 \geq \int_E |f|^2 - 2\int_E |f(h-f)| > n - 2\xi\epsilon.$$

Thus, we have $\mathcal{S}_n$ is dense and open for every n, which implies that

$$L^2(E) = \bigcup_n (\mathcal{S}_n)^c$$

is nowhere dense, i.e., $L^2(E)$ is meagre. □