PROBLEM SET VII: PROBLEMS (1, 2)

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Problem 1. Let $V_0, V_1, V_2, V_3, \ldots$ be real vector spaces with norms

$$\|\bullet\|_n : V_n \to \mathbb{R}_{\geq 0}.$$

Given an element $\vec{v} = (v_n)_{n\geq 0} \in \Pi_{n\geq 0} V_n$, let

$$\|\vec{v}\| = \sum_{n\geq 0} \|v_n\|_n \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

Let $V \subseteq \Pi_{n\geq 0} V_n$ be the subset consisting of those elements $\vec{v}$ such that $\|\vec{v}\| < \infty$. Show that V is a real vector space and that $\vec{v} \mapsto \|\vec{v}\|$ is a norm on V. If each V_n is a Banach space, show that V is a Banach space. We will refer to V as the ℓ^1-*sum* of the Banach spaces $\{V_n\}_{n\geq 0}$.

Proof. First we show that V is a real normed vector space. Let $\vec{v}, \vec{w} \in V$ and $\lambda \in \mathbb{R}$. We may define

$$\vec{v} + \vec{w} = (v_n + w_n)_{n\geq 0}, \quad \lambda\vec{v} = (\lambda\vec{v}_n)_{n\geq 0},$$

which clearly induce a vector space structure on V.

Next we show that $\|\cdot\|$ is a norm. Definitionally, we have that $\|\vec{v}\| = 0$ if and only if $\|v_n\|_n = 0$ for all n, which is precisely when $\vec{v} = 0$. Next, we see

$$\lambda\|\vec{v}\| = \sum_{n\geq 0} \lambda\|v_n\|_n = \sum_{n\geq 0} \|\lambda v_n\|_n = \|\lambda\vec{v}\|.$$

Similarly, the Triangle Inequality holds, since it holds on each component of the sum. Thus, we have that $\|\cdot\|$ is a norm.

Now, if each V_n is a Banach space, then we wish to prove that V is a Banach space. Let v_i be a Cauchy sequence in V. Namely, for all $\epsilon > 0$, there exists some N such that for all $j, k > N$ implies $\|v_j - v_k\| < \epsilon$. This implies that for $v_i = (v_{n,i})_{n\geq 0}$, each sequence $v_{n,i}$ is Cauchy. Since each V_n is a Banach space,

$$\lim_{i\to\infty} v_{n,i} = v_n \in V_n;$$

then

$$v = (v_n)_{n\geq 0} = \lim_{i\to\infty} v_i.$$

Now we must show that $v \in V$; equivalently, $\|v\| < \infty$. However, we have, for $i > N$,

$$\|v_{n,i} - v_n\|_n < \epsilon_n$$

and $\sum_n \epsilon_n < \epsilon$ as the sequence is Cauchy. Thus, for $i > N$,

$$\|v\| - \|v_i\| \leq \|v - v_i\| < \epsilon \Longrightarrow \|v\| < \|v_i\| + \epsilon < \infty,$$

as desired. □

Date: November 4, 2013.

Problem 2. Suppose we are given a sequence $E_0, E_1, E_2, ... \subseteq \mathbb{R}^m$ of pairwise disjoint measurable subsets of $\mathbb{R}^m$. Let $E = \bigcup E_n$. Show that $L^1(E)$ is isomorphic to the ℓ^1-sum of the Banach spaces $L^1(E_n)$.

Proof. For brevity we denote the ℓ^1-sum of the $L^1(E_n)$ by Λ. We define the map $\varphi : L^1(E) \to \Lambda$ by

$$\varphi(f) = (f|_{E_n})_{n\geq 0}.$$

Then we have

$$\|\varphi(f)\|_{\ell^1} = \sum_{n\geq 0} \|(f|_{E_n})\|_{L^1(E_n)} = \sum_{n\geq 0} \int_{E_n} |(f|_{E_n})| = \int_E |f| = \|f\|_{L^1(E)} < \infty,$$

so our map is well-defined and preserves norms on $L^1(E)$ and Λ.

Now let us define $\rho : \Lambda \to L^1(E)$ by

$$\rho\left((f_n)_{n\geq 0}\right) = f_n(x)$$

for $x \in E_n$. This map is well-defined by the disjointness of the E_n and

$$\left\|\rho\left((f_n)_{n\geq 0}\right)\right\|_{L^1(E)} = \int_E \left|\rho\left((f_n)_{n\geq 0}\right)\right| = \sum_{n\geq 0} \int_{E_n} |f_n| = \sum_{n\geq 0} \|f_n\|_{L^1(E_n)} = \left\|(f_n)_{n\geq 0}\right\|_{\ell^1} < \infty.$$

Thus, ρ is an inverse to φ, so φ is an isomorphism of Banach spaces. In particular, it is an isometric isomorphism ∎