PROBLEM SET V: PROBLEMS 1, 2

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Problem 1. Let $E \subseteq \mathbb{R}^n$ be a measurable set, and let $f_0 \leq f_1 \leq f_2 \leq \cdots$ be an increasing sequence of integrable functions on E for which the sequence of integrals $\left\{\int_E f_i\right\}_{i\geq 0}$ is bounded. Show that the sequence $\{f_i\}$ converges almost everywhere to an integrable function, and that $\int_E f$ is a limit of the sequence $\left\{\int_E f_i\right\}_{i\geq 0}$.

Proof. The sequence $\left\{\int_E f_i\right\}_{i\geq 0}$ is nondecreasing by the monotonicity of the Lebesgue integral. Since we also have that this sequence is bounded, it converges to some real number α. Let $\epsilon > 0$, and write, for each $k \in \mathbb{N}$:

$$\sum_{i=1}^{k} \|f_i - f_{i-1}\|_{L^1(E)} = \sum_{i=1}^{k} \int_E |f_i - f_{i-1}|$$

$$= \sum_{i=1}^{k} \int_E (f_i - f_{i-1}) = \int_E f_k - \int_E f_0.$$

Hence, we have that

$$\sum_{i=1}^{k} \|f_i - f_{i-1}\|_{L^1(E)} \longrightarrow \lim_{k\to\infty} \int_E f_k - \int_E f_0 = \alpha - \int_E f_0 < \infty,$$

so we have the quick convergence of $\left\{\int_E f_i\right\}_{i\geq 0}$. From lecture we know that $\left\{\int_E f_i\right\}_{i\geq 0}$ converges pointwise a.e. to a function f. Now we show the integrability of this function. We have

$$\int_E |f| \leq \int_E \sum_{i=1}^{k} |f_i - f_{i-1}| = \lim_{k\to\infty} \int_E \sum_{i=1}^{k} |f_i - f_{i-1}|$$

$$= \lim_{k\to\infty} \int_E (f_k - f_0) = \lim_{k\to\infty} \int_E f_k - \lim_{k\to\infty} \int_E f_0 < \infty,$$

where we justify the first inequality by the Triangle Inequality, and the following equality by the Monotone Covergence Theorem.

Now that we have the integrability of f, we need only show that $\int_E f$ is a limit of $\left\{\int_E f_i\right\}_{i\geq 0}$. Since this sequence converges a.e. pointwise to f and the $|f_i|$ are bounded above by our nonnegative integrable $|f|$, we apply the Dominated Convergence Theorem and conclude

$$\int_E f = \lim_{i\to\infty} \int_E f_i.$$

□

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Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function. Show that $\int_{\mathbb{R}} f$ is a limit of the sequence of real numbers $\left\{\int_{-n}^{n} f\right\}_{n\geq 0}$. Here $\int_{-n}^{n} f$ denotes the integral $\int_{[-n,n]} f\mid_{[-n,n]}$.

Proof. Define the function

$$f_n = \begin{cases} f(x) & x \in [-n, n], \\ 0 & \text{else.} \end{cases}$$

Then we write

$$\int_{-n}^{n} f = \int_{\mathbb{R}} f_n.$$

We have that the $|f_n|$ are bounded above by the integrable function $|f|$, which is integrable because f itself is. Moreover, $\{f_n\}$ clearly converges pointwise to f, so by the Dominated Convergence Theorem,

$$\int_{\mathbb{R}} f = \lim_{n\to\infty} \int_{\mathbb{R}} f_n = \lim_{n\to\infty} \int_{-n}^{n} f.$$

□