

Problem One.

Let $f : [0, \pi] \rightarrow \mathbb{R}$ be a continuous function such that $f(0) = f(\pi) = 0$, and define real numbers a_1, a_2, \dots by the formula

$$a_n = \frac{2}{\pi} \int_0^\pi \sin(nx) f(x) dx.$$

Show that the sum $\sum_{n>0} (a_n)^2$ converges (hint: compare the sum with the integral $\int_0^\pi f(x)^2 dx$).

Proof. By way of motivation, we would somehow like to use the decomposition of $f(x)$ as a Fourier series $\sum_{n=1}^\infty a_n \sin nx$. Even though we haven't proven anything about the convergence of the Fourier series, we might nevertheless guess that $f(x)$ is well-approximated by the partial sums $\sum_{n=1}^N a_n \sin nx$ for large N , at least as far as integrals are concerned (though perhaps not pointwise). This means that the inequality

$$\int_0^\pi \left(f(x) - \sum_{n=1}^N a_n \sin nx \right)^2 dx \geq 0$$

should be pretty sharp (at least when N is large). It may therefore be useful to us.

Note that this integral is finite-valued: f is continuous and therefore achieves some maximum M on a compact domain, so that $M^2 + \sum_{n=1}^N a_n^2$ is a bound on the absolute value of the integrand. Similar arguments will ensure that all subsequent integrals are finite.

Anyway, expanding the brackets, the above inequality is equivalent to

$$\int_0^\pi f(x)^2 dx - 2 \int_0^\pi f(x) \sum_{n=1}^N a_n \sin nx dx + \int_0^\pi \left(\sum_{n=1}^N a_n \sin nx \right)^2 dx \geq 0$$

or, using that Riemann integration commutes with finite summation of integrable functions,

$$2 \sum_{n=1}^N a_n \int_0^\pi f(x) \sin nx dx - \sum_{m,n=1}^N a_m a_n \int_0^\pi \sin mx \sin nx dx \leq \int_0^\pi f(x)^2 dx.$$

But $a_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx dx$, so we can rewrite the first term on the LHS as

$$\pi \sum_{n=1}^N a_n^2$$

and it was shown in class that

$$\int_0^\pi \sin mx \sin nx dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{\pi}{2} & \text{if } m = n. \end{cases}$$

Therefore, we have

$$\begin{aligned} \pi \sum_{n=1}^N a_n^2 - \frac{\pi}{2} \sum_{n=1}^N a_n^2 &\leq \int_0^\pi f(x)^2 dx \\ \sum_{n=1}^N a_n^2 &\leq \frac{2}{\pi} \int_0^\pi f(x)^2 dx \end{aligned}$$

for any N . Thus $\sum_{n=1}^{\infty} a_n^2 \leq \frac{2}{\pi} \int_0^{\pi} f(x)^2 dx \leq 2M^2$, where $|f(x)| \leq M$ on $x \in [0, \pi]$. So by the monotone convergence theorem in \mathbb{R} , it follows that $\sum_{n=1}^{\infty} a_n^2$ converges. \square

Remark. This problem is a special case of *Parseval's identity*, which we'll cover later in this class. This proof imitates the proof of *Bessel's inequality*. Those of you who defined and used an inner product on the space of continuous functions $[0, \pi] \rightarrow \mathbb{R}$ having $f(0) = f(\pi) = 0$ accomplished essentially the same proof, but more sophisticatedly.

Remark. We can't assume results about the convergence of Fourier series at this point in the course. It's also not clear whether these integrals commute with infinite sums.

Problem Two.

By brute force:

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_{\pi/4}^{3\pi/4} \sin nx dx \\ &= \frac{2}{\pi n} \left(\cos \frac{\pi n}{4} - \cos \frac{3\pi n}{4} \right) \\ &= \frac{2}{\pi n} \left(\cos \left(\frac{\pi n}{2} - \frac{\pi n}{4} \right) - \cos \left(\frac{\pi n}{2} + \frac{\pi n}{4} \right) \right) \\ &= \frac{4}{\pi n} \sin \frac{\pi n}{2} \sin \frac{\pi n}{4}. \end{aligned}$$

The usefulness of expressing the answer in this form becomes apparent when we try to calculate $g(\frac{\pi}{4})$ and $g(\frac{3\pi}{4})$.

$$\begin{aligned} g(x) &= \sum_{n>0} \frac{4}{\pi n} \sin \frac{\pi n}{2} \sin \frac{\pi n}{4} \sin nx \\ \implies g\left(\frac{1}{4}\right) &= \sum_{n>0} \frac{4}{\pi n} \sin \frac{\pi n}{2} \sin^2 \frac{\pi n}{4} \\ &\text{and} \\ g\left(\frac{3}{4}\right) &= \sum_{n>0} \frac{4}{\pi n} \sin \frac{\pi n}{2} \sin \frac{\pi n}{4} \sin \frac{3\pi n}{4}. \end{aligned}$$

In each case, $\sin \frac{\pi n}{2} = 0$ whenever n is even. Considering only the terms where $n = 2k - 1$ is odd, the first sum yields

$$\begin{aligned} g\left(\frac{1}{4}\right) &= \sum_{k>0} \frac{4}{\pi(2k-1)} (-1)^k \sin^2 \frac{\pi(2k-1)}{4} \\ &= \sum_{k>0} \frac{4}{\pi(2k-1)} \frac{(-1)^k}{2} \\ &= \frac{2}{\pi} \sum_{k>0} \frac{(-1)^k}{2k-1} \\ &= \frac{1}{2} \end{aligned}$$

and likewise the second sum yields

$$\begin{aligned} g\left(\frac{3}{4}\right) &= \sum_{k>0} \frac{4}{\pi(2k-1)} (-1)^k \sin \frac{\pi(2k-1)}{4} \sin \frac{3\pi(2k-1)}{4} \\ &= \dots \\ &= \frac{1}{2}. \end{aligned}$$

Remark. The formulae that convert between trigonometric sums and products are properly called the *prosthaphaeretic formulae*. Historically they were very important in navigation before the invention of logarithms. They were used to multiply large numbers quickly, much like the formula $\log ab = \log a + \log b$.

Problem Three.

I will generalise Prof Lurie's proof from class that the line $y = x$ has outer measure zero in \mathbb{R}^2 . The idea of that proof was to chop up the line segment from $(0, 0)$ to $(1, 1)$ into m pieces of equal length, and put a box of sidelength $\frac{1}{m}$ at the endpoints of each piece. We will do something similar.

Let $\{v_1, v_2, \dots, v_k\}$ be an orthonormal basis for the k -dimensional linear subspace $V \subset \mathbb{R}^n$.

It suffices to show that the k -dimensional parallelepiped P spanned by $\{v_1, v_2, \dots, v_k\}$ has outer measure $m^*(P) = 0$. V can then be expressed as a countable union of translates of P . Since outer measure is translation-invariant, we will then have by countable subadditivity that $m^*(V) = 0$ as well.

Here are some collections of boxes that cover P .

Collection 0.

$$\mathcal{C}_0 = \{B_{i_1 i_2 \dots i_k}^{(0)} : 0 \leq i_1, i_2, \dots, i_k \leq 1\}$$

where $B_{i_1 i_2 \dots i_k}^{(0)}$ is an open n -box of sidelength 2 centred at the vertex of the parallelepiped given by $\sum_{\alpha=1}^k i_\alpha v_\alpha$.

This is a cover of P , because the distance between any point in P and the centre of some box is no more than $\|\frac{1}{2} \sum_{\alpha=1}^k v_\alpha\| \leq \frac{1}{2^k} \sum_{\alpha=1}^k \|v_\alpha\| \leq \frac{k}{2^k} < 1$, and the open box of sidelength 2 contains the open ball of radius 1.

The total volume of \mathcal{C}_0 is the volume of each box times the number of boxes, or $2^n \cdot 2^k = 2^{n+k}$.

Collection 1.

$$\mathcal{C}_1 = \{B_{i_1 i_2 \dots i_k}^{(1)} : 0 \leq i_1, i_2, \dots, i_k \leq 2\}$$

where $B_{i_1 i_2 \dots i_k}^{(1)}$ is an open n -box of sidelength 1 centred at $\sum_{\alpha=1}^k \frac{1}{2} i_\alpha v_\alpha$. In other words we have put in all the midpoints of the v 's, together with sums of midpoints. This is a cover by the same argument as above.

The total volume of \mathcal{C}_1 is $1^n \cdot 3^k = 3^k$.

Collection 2.

$$\mathcal{C}_2 = \{B_{i_1 i_2 \dots i_k}^{(2)} : 0 \leq i_1, i_2, \dots, i_k \leq 3\}$$

where $B_{i_1 i_2 \dots i_k}^{(2)}$ is an open n -box of sidelength $\frac{2}{3}$ centred at $\sum_{\alpha=1}^k \frac{1}{3} i_\alpha v_\alpha$.

The total volume of \mathcal{C}_2 is $(\frac{2}{3})^n \cdot 4^k = \frac{2^{n+2k}}{3^n}$.

⋮

Collection m .

To cut a long and exceedingly painful story short,

$$\mathcal{C}_m = \{B_{i_1 i_2 \dots i_k}^{(m)} : 0 \leq i_1, i_2, \dots, i_k \leq m+1\}$$

where $B_{i_1 i_2 \dots i_k}^{(m)}$ is an open n -box of sidelength $\frac{2}{m+1}$ centred at $\sum_{\alpha=1}^k \frac{1}{m+1} i_\alpha v_\alpha$.

The total volume of \mathcal{C}_m is $(\frac{2}{m+1})^n \cdot (m+2)^k = 2^n \frac{(m+2)^k}{(m+1)^n}$ which can be made arbitrarily small as $m \rightarrow \infty$.

Therefore $m^*(P) = 0$ and so $m^*(V) = 0$. \square

Remark. Sorry this proof was so messy. Hopefully you get the idea.