Wonderful and Crazy Ideas in Computer Science and Math
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Probability Theory: Counting in Terms of Proportions

Counting ways to order balls

Suppose we have n different balls.
How many ways can we order them?

n=3

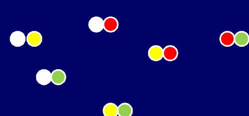
n ways to pick ball 1, n-1 to pick ball 2,
n-2 to pick ball 2, …, 1 way to pick last ball

n*(n-1)*(n-2)*…*2*1 = n!

Counting balls

Suppose we have **n** balls.
How many ways are there to pick i balls?
Here we don't care about the order.

6 ways to pick 2 balls out of 4:



Counting balls

Suppose we have n balls.
How many ways are there to pick i balls when we don't care about the order?

First count the ways to pick i balls when the **order** matters.

n ways to pick ball 1,
n-1 ways to pick ball 2,
…
n-i+1 ways to pick ball i.

n(n-1)(n-2)…(n-i+1) = n! / (n-i)!

Counting balls

First count the ways to pick i balls in some order:

n(n-1)(n-2)…(n-i+1) = n! / (n-i)!

Claim: Each set of i balls is counted i! times in the above expression.

Proof: Fix i balls {B1, B2, …, Bi}. There are **i!** possible ways to order them.

Hence {B1, B2, …, Bi} appears i! times in the list of ways to pick i balls from n balls when the order matters.

The choose function

First count the ways to pick i ordered balls:

n(n-1)(n-2)…(n-i+1) = n! / (n-i)!

Each set of i balls is counted i! times.

Hence the number of ways to pick i balls is:

$$\binom{n}{i} = \frac{n!}{(n-i)!\, i!}$$

The number of ways to pick i items out of n.

"n choose i" or i-th binomial coefficient

Another proof: Picking i balls out of n

$$\left[\begin{matrix} n \\ i \end{matrix}\right] = \frac{n!}{i!\,(n-i)!}$$

The number of ways to pick i items out of n.

Proof by induction:

Base case: n=1, i=1, only one way to choose: $\left[\begin{matrix} 1 \\ 1 \end{matrix}\right] = 1$.

Suppose $\left[\begin{matrix} n-1 \\ i-1 \end{matrix}\right] = \frac{(n-1)!}{(i-1)!(n-i)!}$ $\left[\begin{matrix} n-1 \\ i \end{matrix}\right] = \frac{(n-1)!}{i!\,(n-i-1)!}$

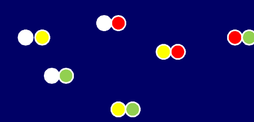
Count the ways to pick i balls which contain ball 1, and then the ways to pick i balls that don't contain ball 1.

$$\left[\begin{matrix} n \\ i \end{matrix}\right] = \left[\begin{matrix} n-1 \\ i-1 \end{matrix}\right] + \left[\begin{matrix} n-1 \\ i \end{matrix}\right] = \frac{(n-1)!}{(i-1)!(n-i)!} + \frac{(n-1)!}{i!\,(n-i-1)!} = \frac{n!}{i!\,(n-i)!}$$

Choose function: example

$$\left[\begin{matrix} 4 \\ 2 \end{matrix}\right] = \frac{4!}{(4-2)!\,2!} = \frac{4*3}{2} = 6$$

6 ways to pick 2 balls out of 4:



The area of mathematics devoted to counting is called combinatorics.

Proportions

Suppose we have 20 people:

- 7 born in the summer
- 9 born in the fall
- 4 born in the winter

The proportion of people born in the summer is 7/20 = 35%.

The proportion of people born in the fall is 9/20=45%, and of those born in the winter is 4/20 = 20%.

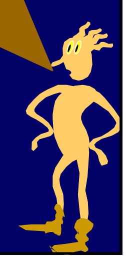
Probability

Suppose we keep rolling a fair 20 sided die: a side for each person.

What fraction of the time will it name a person born in the fall?

9/20 **This is the probability that a person is born in the fall.**

Probabilities and Combinatorics are intimately related.



Probabilities and counting

Say we want to count
the number of X's with property P

One way to do it is to ask
"if we pick an X at random,
what is the probability that X has property P?"
and then multiply by the number of X's.

$$\begin{bmatrix}\text{Probability of X} \\ \text{with property P}\end{bmatrix} = \frac{\text{(\# of X with property P)}}{\text{(total \# of X)}}$$

Probabilities and counting

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Probabilities and counting

Say we want to count
the number of people born in the fall

One way to do it is to ask
"if we pick a person at random, what is the probability that he/she was born in the fall?"
and then multiply by the number of people.

$$\begin{bmatrix}\text{Probability of X} \\ \text{born in the fall}\end{bmatrix} = \frac{\text{(\# of X born in the fall)}}{\text{(total \# of X)}}$$

Probability

When we talk about probabilities we have a set of possible events in mind:
For example: the event of picking a person born in the fall.

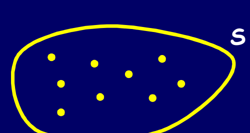
The set of events is the sample space.

The probability of an event E is a real number between 0 and 1: this is the fraction of time we expect E to occur if we select events at random.

Sample space

A (finite) <u>probability distribution</u> is a finite set S of elements, where each element x in S has a positive real number p(x) called the <u>probability</u> of x.

Sample space



Probability

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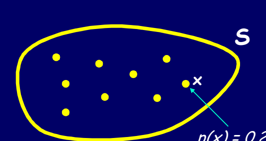
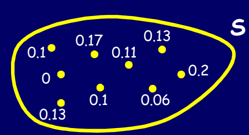
<u>Probability</u> of x



Probability Distribution

A (finite) probability distribution is a finite set S of elements, where each element x in S has a positive real number p(x) called the probability of x.

probabilities must sum to 1



Event 1 or Event 2

What is the probability that we pick a person who was either born in the summer or in the fall?

(7+9)/20 = 16/20=0.8

Probability(Event1 OR Event2) = Probability(Event1)+Probability(Event2)

Sum of probabilities of all events = 1

disjoint

Events

A (finite) probability distribution is a finite set S of elements, where each element x in S has a positive real number p(x) called the probability of x.

Any subset E of S is called an event.
The probability of event E is

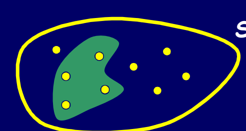
Sum of p(x) for all x in E.

If E={x_1, x_2, x_3} then this means that E is the event that **either x_1 or x_2 or x_3 occurs.**

Events

A (finite) probability distribution is a finite set S of elements/events, where each element x in S has a positive real number p(x) called the probability of x.

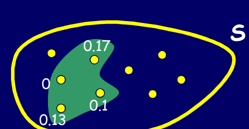
Event E



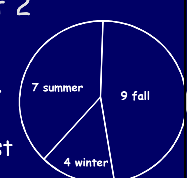
Events

A (finite) probability distribution is a finite set S of elements/events, where each element x in S has a positive real number p(x) called the probability of x.

Pr[E] = 0.4



Event 1 and Event 2



Suppose we pick a person at random, and then put them back.
Then we pick another person.
What is the probability we'll first get one born in the fall and then one born in the winter?

Multiply probabilities: (9/20)*(4/20)

Answer: (9*4)/(20*20) because there are 9*4 pairs of people (A, B) where A is born in the fall and B is born in the winter, and because there are 20*20 ways to pick twice from the set of all 20 people.

Wait!

What if after we pick a person, we **don't** put them back?
(picking without replacement)

Then the probabilities change!
After we have removed a person born in the fall, the probability that we pick a person born in the winter is 4/19, not 4/20!

dependence

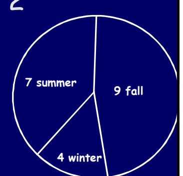
Independence

Two events are independent if knowing whether one of them occurred does not change the probability of the other occurring.

If E1 and E2 are independent,
**Probability[E1 and E2] =
Probability[E1]*Probability[E2]**

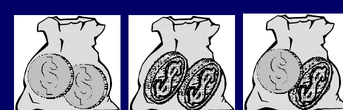
Event 1 and Event 2

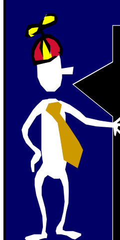
If we pick two people at random *with replacement*, what is the probability we'll first get one born in the fall and then one born in the winter?



Independent events:
9/20 * 4/20 = 36/400 = 0.09

Silver and Gold

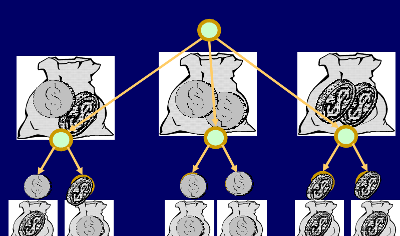


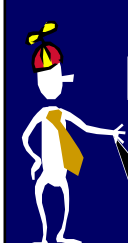


One bag has two silver coins, another has two gold coins, and the third has one of each.

One of the three bags is selected at random. Then one coin is selected at random from the two in the bag. It turns out to be gold.

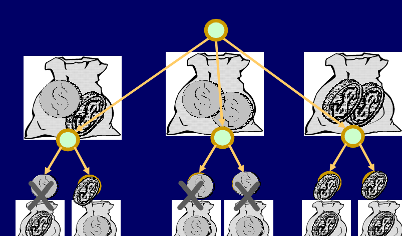
What is the probability that the other coin is gold?

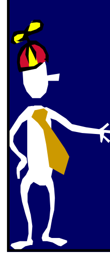




3 choices of bag
2 ways to order bag contents

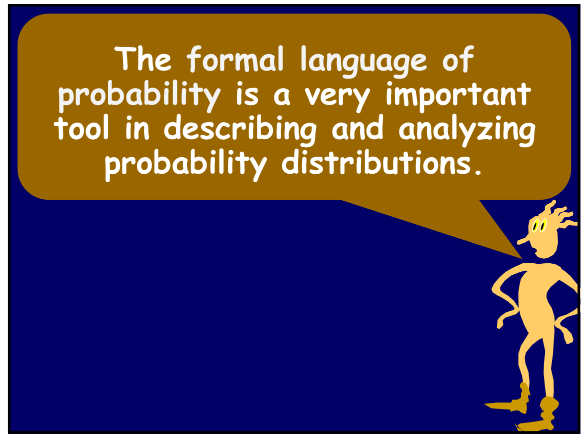
6 equally likely paths.

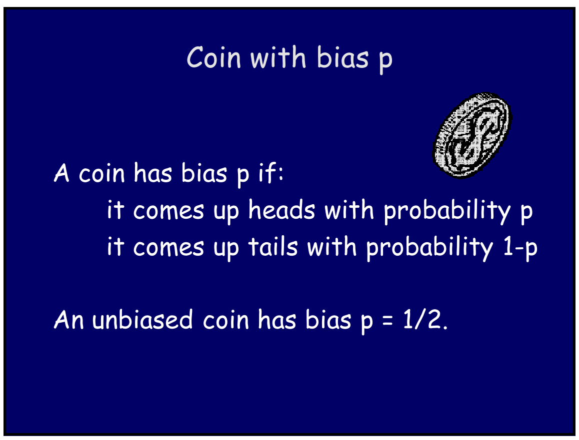


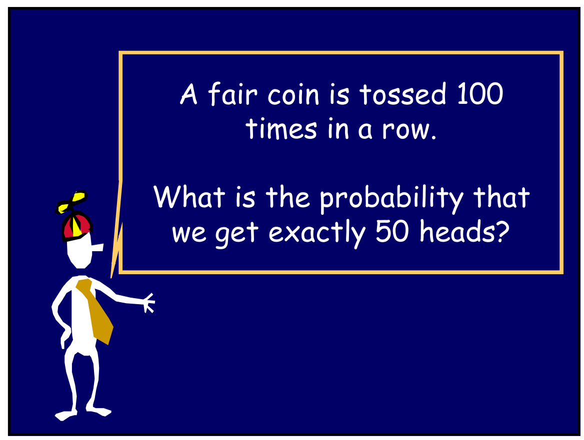


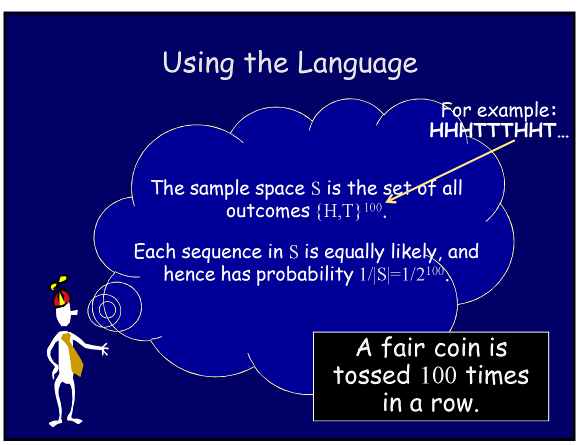
Given you see a , 2/3 of the remaining paths have a second .

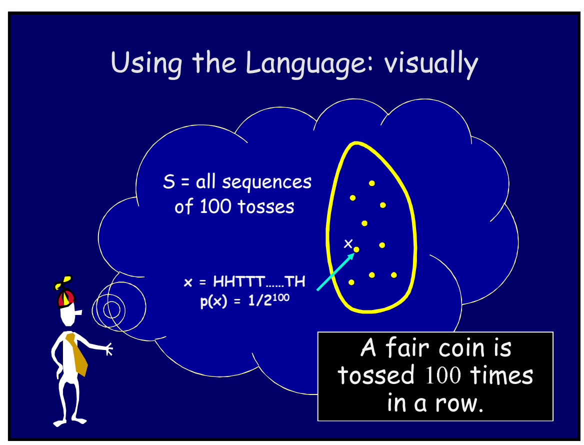

So, sometimes, probabilities can be counter-intuitive.

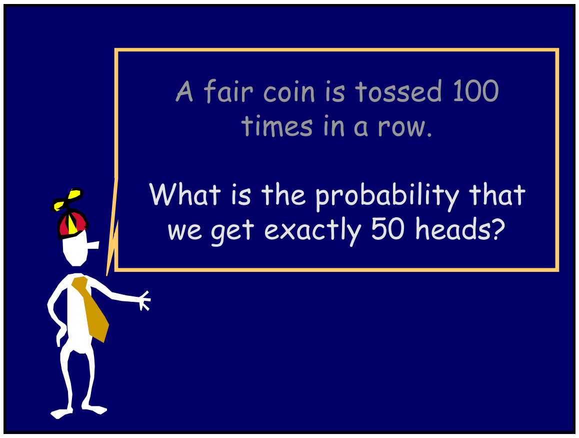

The formal language of probability is a very important tool in describing and analyzing probability distributions.


Coin with bias p
A coin has bias p if:
it comes up heads with probability p
it comes up tails with probability 1-p
An unbiased coin has bias p = 1/2.


A fair coin is tossed 100 times in a row.
What is the probability that we get exactly 50 heads?


Using the Language
For example:
HHHTTTHHT…
The sample space S is the set of all outcomes {H,T}^{100}.
Each sequence in S is equally likely, and hence has probability 1/|S|=1/2^{100}.
A fair coin is tossed 100 times in a row.


Using the Language: visually
S = all sequences of 100 tosses
x
x = HHTTT……TH
p(x) = 1/2^{100}
A fair coin is tossed 100 times in a row.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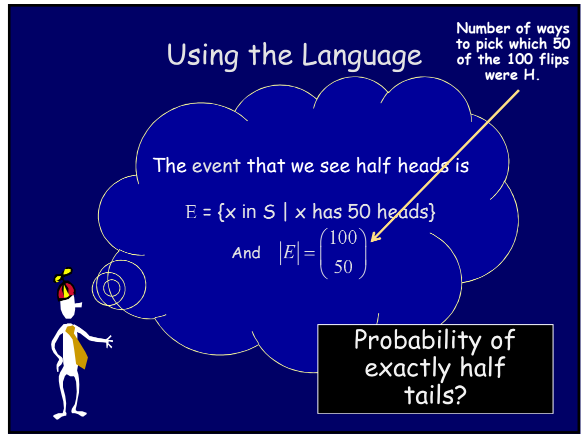
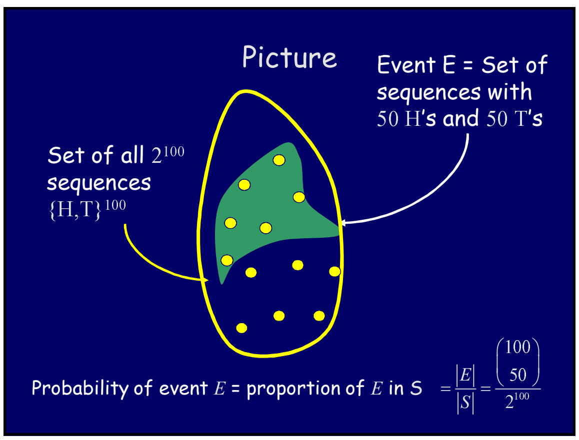
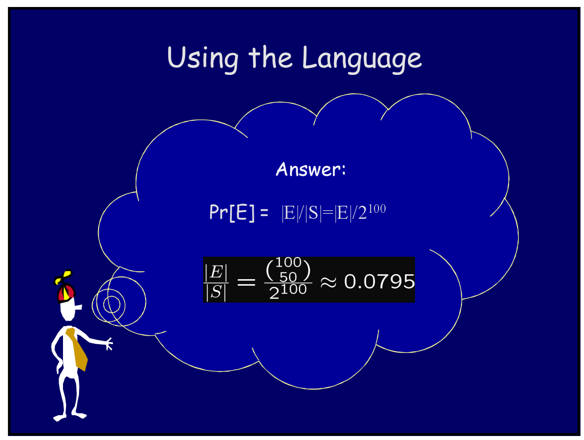
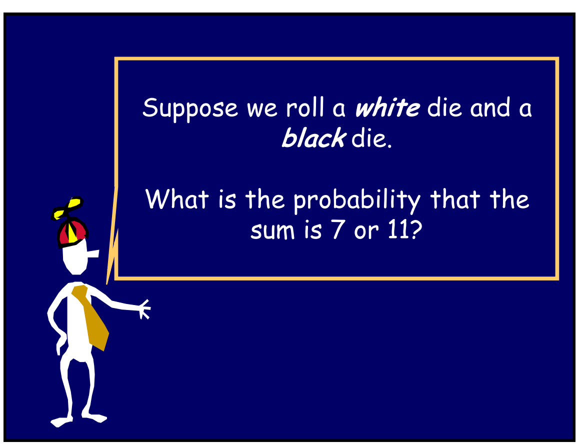
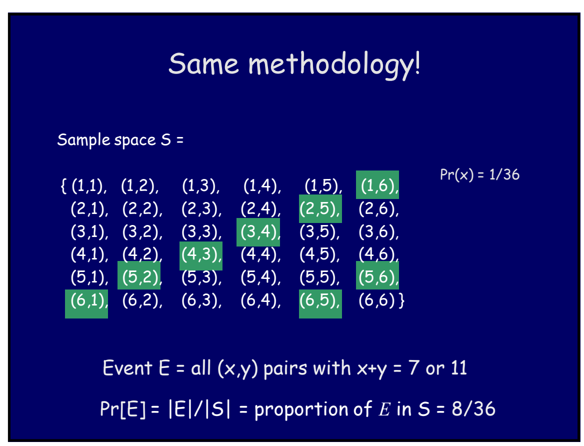
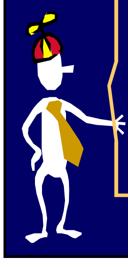

Using the Language
Number of ways to pick which 50 of the 100 flips were H.
The event that we see half heads is
E = {x in S | x has 50 heads}
And $|E| = \binom{100}{50}$
Probability of exactly half tails?


Picture
Set of all 2^{100} sequences $\{H,T\}^{100}$
Event E = Set of sequences with 50 H's and 50 T's
Probability of event E = proportion of E in S $= \frac{|E|}{|S|} = \frac{\binom{100}{50}}{2^{100}}$


Using the Language
Answer:
Pr[E] = $|E|/|S| = |E|/2^{100}$
$\frac{|E|}{|S|} = \frac{\binom{100}{50}}{2^{100}} \approx 0.0795$


Suppose we roll a ***white*** die and a ***black*** die.
What is the probability that the sum is 7 or 11?


Same methodology!
Sample space S =
{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6),
(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),
(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),
(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),
(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),
(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) }
Pr(x) = 1/36
Event E = all (x,y) pairs with x+y = 7 or 11
Pr[E] = |E|/|S| = proportion of E in S = 8/36



23 people are in a room.

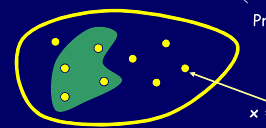
Suppose that all possible assignments of birthdays to the 23 people are equally likely.

What is the probability that at least two people will have the same birthday?

And the same methods again!

Sample space $\Omega = \{1, 2, 3, \ldots, 366\}^{23}$

Pretend it's always a leap year



x = (17,42,363,1,…, 224,177)

23 numbers from 1 to 366

Event E = {x in Ω | two numbers in x are the same}

What is |E|? (How big is E ? How many x have some day appear twice?)

Instead count $|\overline{E}|$ -- the number of lists with all 23 numbers different!

$\overline{E}$ = all sequences in Ω that have no repeated numbers

$$|\bar{E}| = 366 \cdot 365 \cdots 344$$

The probability that there are no repeated birthdays

$$\frac{|\bar{E}|}{|\Omega|} = \frac{366 \cdots 344}{366^{23}} \approx .494$$

$$\frac{|E|}{|\Omega|} \approx .51$$

Probability of some repeated birthday

Birthday "Paradox"

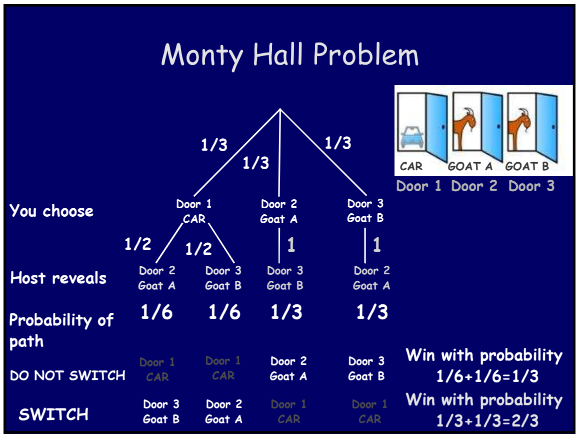
In a group of at least 23 people, the probability that there are two people with the same birthday is more than 50%!

Monty Hall Problem

Monty Hall Problem

- Announcer hides prize behind one of 3 doors at random.
- You select some door.
- From the remaining doors, Announcer picks a door with no prize at random and opens it.
- You can decide to keep or switch.

What to do?



Monty Hall Problem

- Sample space Ω =
  { prize behind door 1,
  prize behind door 2,
  prize behind door 3 }.

Each has probability 1/3.

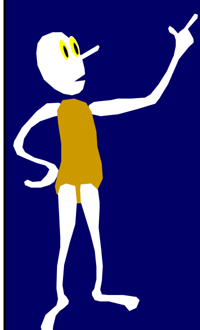
Staying
we win if we originally chose the correct door

Pr[choosing correct door] = 1/3.

Switching
we win if we originally chose the incorrect door

Pr[choosing incorrect door] = 2/3.

why was this tricky?



We are inclined to think:

"After one door is opened, others are equally likely…"

But his action is not independent of yours!