A quasicategorical theory of $E_\infty$ bimonoids

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- Constructing symmetric bimonoidal $\infty$-categories
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Let $C$ be a category with finite products.
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**Definition**

A *commutative monoid* in $\mathcal{C}$ is a functor $M : \mathcal{F} \to \mathcal{C}$ such that for every $n$, the $n$ inert maps $\langle n \rangle \to \langle 1 \rangle$ express $M(\langle n \rangle)$ as the product of $n$ copies of $M(\langle 1 \rangle)$.
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For example, an ordinary commutative monoid is a commutative monoid in $\text{Set}$. A monoid in an abelian category $\mathcal{A}$ is nothing more than an object of $\mathcal{A}$. A monoid in $\text{Cat}$ is a **strict symmetric monoidal category**.
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We don’t expect to see this structure very often in nature. Clearly the definition above is not the correct one for a 2-category such as Cat.
The Grothendieck construction and cocartesian fibrations

Let \( \mathcal{D} \) be a small category and let \( F : \mathcal{D} \to \text{Cat} \) be a functor. We can define a new category \( \mathcal{G} \) with a functor \( p : \mathcal{G} \to \mathcal{D} \) in such a way that for all objects \( d \in \mathcal{D} \),

\[
p^{-1}(d) \cong F(d).
\]

This is known as the Grothendieck construction.

It's possible to axiomatize this situation.

Definition

Let \( \mathcal{G} \) and \( \mathcal{D} \) be categories. A functor \( p : \mathcal{G} \to \mathcal{D} \) is called a cocartesian fibration if it looks like it came from the Grothendieck construction.
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Cocartesian fibrations, continued

In general, cocartesian fibrations correspond to *functors up to isomorphism*. 
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The data of a (fully weak) symmetric monoidal category is encapsulated by a cocartesian fibration $p : C^\otimes \to \mathcal{F}$ satisfying a Segal condition:

$$p^{-1}(\langle n \rangle) = \prod_n p^{-1}(\langle 1 \rangle).$$
Review of ∞-categories

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\[ \text{Definition} \]
A symmetric monoidal $\infty$-category is a functor $M : N(F) \to \text{Cat}_\infty$ satisfying the Segal condition
\[ M(\langle n \rangle) = \prod_n M(\langle 1 \rangle), \]
or equivalently, a cocartesian fibration $p : C \otimes N(F) \to \text{Cat}_\infty$ satisfying the Segal condition
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Products induce a canonical symmetric monoidal structure on any $\mathbf{C}$ in which they exist.
The category $\mathcal{F} \int^{\wedge} \mathcal{F}$

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- An object of $\mathcal{F} \int^\wedge \mathcal{F}$ is a finite pointed set $S$ together with a finite pointed set $T_s$ for each legit $s \in S$. We’ll denote this object $(S, [T_s])$. 
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- A morphism $f : (S, [T_s]) \to (U, [V_u])$ in $\mathcal{F} \int^\wedge \mathcal{F}$ consists of
  - A map $f : S \to U$;
  - For each legit $u \in U$, a map

$$f_u : \left( \bigwedge_{s \in f^{-1}(u)} T_s \right) \to V_u.$$
Bilinear maps and tensor products

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general a context as possible.

Let $\mathcal{C}$ be a category with finite products. A commutative monoid $Z$ in $\mathcal{C}$ is
classified by a functor $\mathcal{F} \to \mathcal{C}$, and a pair of commutative monoids $(X, Y)$
in $\mathcal{C}$ is classified by a functor $\mathcal{F} \times \mathcal{F} \to \mathcal{C}$.
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By a bilinear map from $(X, Y)$ to $Z$, we mean a map

$$\left( \mathcal{F} \int ^{\wedge} \mathcal{F} \right)_{(\mu: \langle 2 \rangle \to \langle 1 \rangle)} \to \mathcal{C}$$

which pulls back to $(X, Y): \mathcal{F} \times \mathcal{F} = \left( \mathcal{F} \int ^{\wedge} \mathcal{F} \right)_{\langle 2 \rangle} \to \mathcal{C}$ and $Z: \mathcal{F} = \left( \mathcal{F} \int ^{\wedge} \mathcal{F} \right)_{\langle 1 \rangle} \to \mathcal{C}$. 
Bilinear maps and tensor products, continued

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This all works for $\infty$-categories.
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**Definition**

A *commutative bimonoid* in $\mathbf{C}$ is a functor $B : N(\mathcal{F} \int^\wedge \mathcal{F}) \to \mathbf{C}$ satisfying an enhanced Segal condition:

$$B(S, [T_s]) = \prod \left( \bigvee s \cdot T_s \right) \circ B(e = (\langle 1 \rangle, \langle 1 \rangle)).$$

A commutative bimonoid in $\text{Cat}_\infty$ is called a *symmetric bimonoidal $\infty$-category*.

We could equally well define one of these as a cocartesian fibration, and indeed this is more practical for most purposes.
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Constructing symmetric bimonoidal $\infty$-categories

As often happens in this subject, we must bootstrap up from the discrete world.

Proposition

If $C$ is a symmetric bimonoidal category, then $N(C)$ naturally has the structure of a symmetric bimonoidal $\infty$-category.

For example, we have a symmetric bimonoidal $\infty$-category $N(F \vee, \wedge) \to N(F \int \wedge F)$: the category of finite pointed sets under wedge and smash. This is, in some sense, the archetypal example.

What can we do for an arbitrary $\infty$-category?

One might expect the category of commutative monoids to carry a symmetric bimonoidal structure, and this is sometimes true.
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What can we do for an arbitrary $\infty$-category? One might expect the category of commutative monoids to carry a symmetric bimonoidal structure, and this is sometimes true.
Constructing symmetric bimonoidal $\infty$-categories, continued

Given an $\infty$-category $\mathcal{C}$, we define a map of $\infty$-categories $p : \text{CMon}(\mathcal{C})^{\oplus, \otimes} \to \mathcal{N}(\mathcal{F} \int^\wedge \mathcal{F})$ as follows:

If $K \to \mathcal{N}(\mathcal{F} \int^\wedge \mathcal{F})$, there is a bijection, natural in $K$:

$$\text{Fun}_{\mathcal{N}}(\mathcal{F} \int^\wedge \mathcal{F})(K, \text{CMon}(\mathcal{C})^{\oplus, \otimes}) \cong \text{Fun}_{\text{Seg}}(K \times \mathcal{N}(\mathcal{F} \int^\wedge \mathcal{F}), \mathcal{C}).$$

The fiber of $p$ over $e$ is, by construction, equivalent to $\text{CMon}(\mathcal{C})$.

If $p$ is cocartesian, then this construction gives a symmetric bimonoidal $\infty$-category, and we say that $\mathcal{C}$ admits tensor products.

This can really only go wrong if the tensor product fails to be associative.
Constructing symmetric bimonoidal ∞-categories, continued

Given an ∞-category \( C \), we define a map of ∞-categories
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If $p$ is cocartesian, then this construction gives a symmetric bimonoidal $\infty$-category, and we say that $\mathbf{C}$ admits tensor products.

This can really only go wrong if the tensor product fails to be associative.
Constructing symmetric bimonoidal $\infty$-categories, continued, continued

One can obtain some nice sufficient conditions for $\mathcal{C}$ to admit tensor products.

Proposition

Suppose that $\mathcal{C}$ has internal homs (in the cartesian sense). Then $\mathcal{C}$ admits tensor products.

Examples include $\text{Top}$ and $\text{Cat}_\infty$. These hypotheses fail for $\text{Sp}$, and even $\text{Top}^\ast$. 
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The symmetric bimonoidal $\infty$-category of spectra

The symmetric bimonoidal $\infty$-category $\text{CMon}(\text{Top})^{\oplus, \otimes}$ is classified by a map

$$N(\mathcal{F} \int^\wedge \mathcal{F}) \to \text{Cat}_\infty$$
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We’d like to obtain $\text{Sp}$ as a symmetric bimonoidal $\infty$-category by postcomposition with some kind of ‘stabilization’ endofunctor.
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This gives $\text{Sp}^{\vee,\wedge}$ and a symmetric bimonoidal map

$$\Omega^\infty : \text{Sp}^{\vee,\wedge} \to \text{CMon}(\text{Top})^\oplus,\otimes.$$
An application to symmetric monoidal K-theory

Suppose $\mathbf{C}$ and $\mathbf{D}$ are $\infty$-categories which admit tensor products.

This recovers the delooping of the K-theory of a symmetric bimonoidal category to an $E_\infty$ ring spectrum.
An application to symmetric monoidal K-theory

Suppose $\mathbf{C}$ and $\mathbf{D}$ are $\infty$-categories which admit tensor products. Then any functor $F : \mathbf{C} \to \mathbf{D}$ induces a functor $\text{BiMon}(\mathbf{C}) \to \text{BiMon}(\mathbf{D})$. This recovers the delooping of the K-theory of a symmetric bimonoidal category to an $E_\infty$ ring spectrum.
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