Mutilated Chessboard problem is exponentially hard for Resolution

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Abstract

Mutilated Chessboard principle CB_n says that it is impossible to cover by domino tiles the chessboard $2n \times 2n$ with two diagonally opposite corners removed. We prove $2^{\Omega(\sqrt{n})}$ lower bound on the size of minimal Resolution refutation of CB_n.

1 Introduction

Mutilated Chessboard problem corresponds to the well-known puzzle that often appears in mathematical popular literature. It asks to cover by dominos a $2n \times 2n$-chessboard with two white squares removed from the opposite corners. The goal is well known to be unachievable because every domino tile covers exactly one white and one black square while the mutilated chessboard contains two white squares less.

This principle is probably one of the earliest proposed hard problems for theorem provers. Expressed as a tautology in first order logic, it was proposed as a "tough nut for proof procedures" by McCarthy in 1964 [M64]. It was considered again by Krishnamurthy in 1985 [K85] who showed a short proof in Resolution augmented with auxiliary variables. Dantchev and Riis proved that any tree-like resolution of CB_n requires $2^{\Omega(n)}$ clauses ([DR00]).

If one considers the graph of the Mutilated Chessboard then the problem can be naturally stated in terms of matchings in a bipartite graph. In his survey of matching principles for Resolution [U00], Urquhart distinguished Mutilated Chessboard along with the weak Pigeon Hole principle as the one "where current techniques appear impotent". Recently there was made a great breakthrough in proving lower bounds for $WPHP$ in [PR01] and [R01], thus CB_n can have historical interest as one of the first suggested.

In this paper we show how to fit CB_n into the general framework of random restrictions in the spirit of Beame and Pitassi [BP96] and show that any Resolution refutation of this principle requires size $2^{\Omega(\sqrt{n})}$. In order to do it we use somewhat unusual critical assignments based on the idea of Road game from [DR00].

A similar result was independently achieved by Stefan Dantchev and Søren Riis [DR01].

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2 Preliminaries

Let x be a Boolean variable, i.e. a variable that ranges over the set $\{0, 1\}$. A *literal* of x is either x (denoted sometimes as x^1) or $\bar{x}$ (denoted sometimes as x^0). A *clause* is a disjunction of literals.

For any Boolean function $f : \{0, 1\}^n \to \{0, 1\}$, $Vars(f)$ will denote the set of its essential variables. An *assignment to* f is a mapping $\alpha : Vars(f) \to \{0, 1\}$. A *restriction of* f is a mapping $\rho : Vars(f) \to \{0, 1, \star\}$. We denote by $|\rho|$ the number of assigned variables, $|\rho| \stackrel{\mathrm{def}}{=} |\rho^{-1}(\{0, 1\})|$.

The *restriction of* f *by* ρ, denoted $f|_\rho$, is the Boolean function obtained from f by setting the value of each $x \in \rho^{-1}(\{0, 1\})$ to $\rho(x)$, and leaving each $x \in \rho^{-1}(\star)$ as a variable.

We say that an assignment α *satisfies* f if $f(\alpha) = 1$. For Boolean functions $f_1, \ldots, f_k, g$ we say that $f_1, \ldots, f_k$ *semantically imply* g, (denoted $f_1, \ldots, f_k \models g$), if every assignment to $V \stackrel{\mathrm{def}}{=} Vars(f_1) \cup \ldots \cup Vars(f_k) \cup Vars(g)$ satisfying $f_1, \ldots, f_k$, satisfies g as well (i.e. $\forall \alpha \in \{0, 1\}^V (f_1(\alpha) = \cdots = f_k(\alpha) = 1 \Rightarrow g(\alpha) = 1)$).

In this paper we consider a *chessboard* of size $n \times n$. It corresponds naturally to the $n \times n$ lattice. Also it can be represented as a *graph* G_n whose vertices are *squares* (sometimes we call them *points*) of the chessboard and two vertices are connected with an edge iff they are adjacent. Let us introduce a system of coordinates with the center in the bottommost left corner of the chessboard (its coordinates are $(1, 1)$).

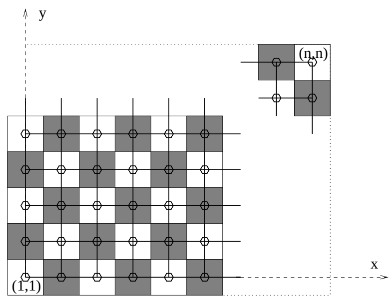

Figure 1: Chessboard - lattice - graph

We consider the discrete metrics d on G_n induced by the discrete l_∞-norm (the distance between two points is the minimum of their vertical and horizontal differences). For a subset S of points its *area* $|S|$ is the the cardinality of S, the *boundary* $\partial(S)$ is the set of points from S with a neighbor in $\bar{S}$, the *distance* between two sets $d(S_1, S_2)$ is the minimal distance $d(v_1, v_2)$ with $v_1 \in S_1, v_2 \in S_2$.

Mutilated Chessboard is the graph G'_{2n} that results from G_{2n} by removing vertices with coordinates $(1, 1)$ and $(2n, 2n)$.

2.1 Resolution

Resolution is the simplest and probably the most widely studied model. It operates with clauses and has one rule of inference called *resolution rule*:

$$\frac{A \vee x \qquad B \vee \bar{x}}{A \vee B}.$$

A *resolution refutation* of a CNF formula τ is a resolution proof of the empty clause from the clauses appearing in τ.

The *size* of a resolution proof is the number of different clauses in it. The *width* $w(C)$ *of a clause* C is the number of literals in C. The *width* $w(\tau)$ *of a set of clauses* τ (in particular, the width of a resolution proof) is the maximal width of a clause appearing in this set.

For an unsatisfiable set of clauses τ denote by $S(\tau)$ the size of minimal refutation in Resolution. Denote by $w(\tau \vdash \bot)$ the minimal refutation width over all possible refutations of τ.

3 Mutilated Chessboard and reduction from Infinite Road

Stated informally, CB_n principle says that one cannot cover the Mutilated Chessboard by domino tiles. To formalize it, for any pair of adjacent points v_1, v_2 of G'_{2n} we introduce a propositional variable $x_{v_1 v_2}$ saying that the corresponding edge participates in the matching.

Definition 3.1 *Let X be the set of variables $\{x_{v_1v_2} | (v_1, v_2) \in E(G'_{2n}\}$. CB_n is the following contradictory set of clauses over X:*

- $x_{v_1v_2} = x_{v_2v_1}$ $(v_1, v_2) \in E(G'_{2n})$
- $\bar{x}_{v_1v} \vee \bar{x}_{v_2v} \qquad v_1 \neq v_2$
- $\bigvee_{(v_1,v) \in E(G'_{2n})} x_{vv_1}$

In the rest of the section we reduce to CB_n the *Infinite Road* tautology. The latter principle uses the fact that any set of oriented curves on the finite plane has equal number of "sources" and "ends". To get a contradictory set of axioms we postulate the existence of a path (*Infinite Road*) that has exactly one "source" v_s. A similar idea was considered in [DR00].

Definition 3.2 *Let G be a graph with a fixed vertex v_s. Y be the following set of propositional variables: $\{y^v | v \in V(G)\} \cup \{y^v_{v_1v_2} | (v_1, v), (v_2, v) \in E(G), v_1 \neq v_2\}$. $IR(G, v_s)$ is the following contradictive set of axioms over Y:*

- *Axioms of the source v_s:*
  $\bigvee_{(v_s,v_1),(v_1,v_2) \in E(G)} y^{v_1}_{v_s v_2}$

  $\bar{y}^{v_1}_{v_s v_2} \vee \bar{y}^{v_3}_{v_s v_4}$ $v_1 \neq v_3$

- *Axioms for $v \neq v_s$:*
  $y^v = \bigwedge_{(v_1,v),(v,v_2)\in E(G)} \bar{y}^v_{v_1v_2}$

  $\bar{y}^v_{v_1v_2} \vee \bar{y}^v_{v_3v_4}$ $v_1 \neq v_3$ *or* $v_2 \neq v_4$

  $y^v_{v_1v_2} \to \bigvee_{(v_2,v_3)\in E(G)} y^{v_2}_{vv_3}$

  $y^v_{v_1v_2} \to \bigvee_{(v_0,v_1)\in E(G)} y^{v_1}_{v_0v}$ $v_1 \neq v_s$

For $v \neq v_s$ denote by A_v the conjunction of the axioms corresponding to this vertex. Let A_{v_s} be the conjunction of the source axioms (thus $IR(G, v_s) = \bigcup_{v\in V(G)} A_v$).

The intuitive sense of a variable $y^v_{v_1v_2}$ is that the point v belongs to some curve going through the edges (v_1, v) and (v, v_2). The variable y^v indicates that the point doesn't belong to any curve. Notice that the axioms of $IR(G, v_s)$ don't prohibit local cycles as long as they don't intersect the main path starting from v_s.

Let us now consider this tautology on the graph G_n corresponding to the $n \times n$ chessboard, and fix the source v_s in the point $(1, 1)$. Denote the resulting principle by IR_n. In this case the variables $\{y^v_{v_1v_2}\}, y^v$ naturally correspond to the following tiles:

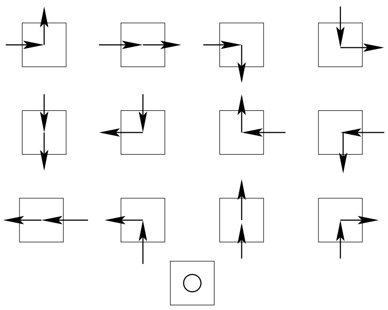

Figure 2: Possible assignments to $\{y^v_{v_1v_2}\}, y_v$ in case of G_n

Theorem 3.3 $S(IR_{\lfloor (n-2)/4 \rfloor}) = O(S(CB_n))$

Proof.

We need the following technical lemma about simulating of one tautology by another:

Lemma 3.4 *Let $\tau_1(X), \tau_2(Y)$ be two contradictive sets of clauses. Assume that each variable y_i is mapped to some restriction ρ_i on X so that the following holds:*

- *Any clause in $\tau_1(X)$ can be inferred from $\tau_2(Y) \cup \bigcup_{\rho_i(x_k)=\epsilon} \{y^i \to x^\epsilon_k\}$ in $O(1)$ resolution steps.*
- *For any variables y_i, y_j s.t. ρ_i and ρ_j are inconsistent on the value of some variable x_k the clause $\bar{y}_i \vee \bar{y}_j$ is provable from $\tau_2(Y)$ in $O(1)$ steps.*

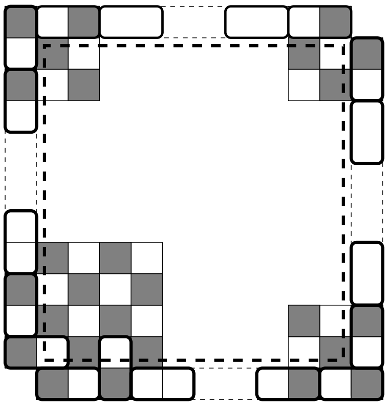

Figure 3: Reduction to $(2n-2)\times(2n-2)$ chessboard

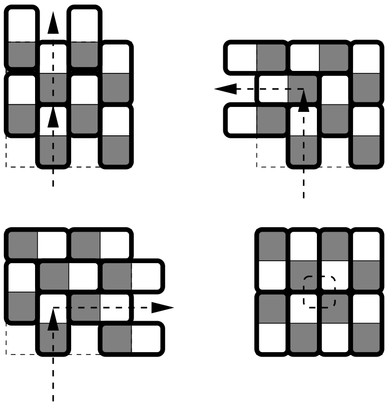

Figure 4: Coverings of the square 4×4

- *Any variable x_k is assigned by $O(1)$ restrictions ρ_i.*

Then $S(\tau_2) = O(S(\tau_1))$.

Proof.

We give the sketch of the proof. Consider some refutation $\mathcal{P}$ of $\tau_1(X)$. We can simulate $\mathcal{P}$ by $\tau_2(Y)$ if we replace each literal x_k^ϵ with the clause $\hat{x_k^\epsilon} = \bigvee_{\rho_i(x_k)=\epsilon} y_i$. For any initial axiom $C \in \tau_1(X)$ the clause $\hat{C} = \bigvee_{x_k^\epsilon \in C} \hat{x}_k^\epsilon$ can be inferred from $\tau_2(Y)$ in $O(1)$ steps by the first two properties. By the second and the third properties, clauses $\hat{C} \vee \hat{x}_k$ and $\hat{D} \vee \hat{(\bar{x}_k)}$ imply $\hat{C} \vee \hat{D}$ in $O(1)$ resolution steps. The lemma follows.∎

Now we describe how to apply it in our case. First of all we reduce Mutilated Chessboard of size $2n \times 2n$ to another board of size $(2n-2)\times(2n-2)$ with two white squares removed from the corner as shown on Fig. 3.

Assume w.l.o.g. that $2n-2$ is divisible by four (if not we can cover two vertical and two horizontal lines and decrease n by 1). Divide our chessboard into 4×4 squares, the bottommost left one has two small white squares removed. Consider the set of domino assignments for each big square shown on Fig. 4.

Clearly there is a natural one-to-one correspondence between all possible rotations of these assignments and the variables of $IR_{(2n-2)/4}$ depicted on Fig 2. For each variable of $IR_{(2n-2)/4}$ these partial assignments define a restriction of domino tiles in the corresponding big square. The pattern in the square $(1,1)$ is determined by its neighbors in the obvious way. It is straightforward to check that the properties from Lemma 3.4 are satisfied. Theorem 3.3 follows.∎

4 Lower bound for Infinite Road on square lattice

In this section we prove that any Resolution refutation of IR_n requires size $2^{\Omega(\sqrt{n})}$. Our lower bound naturally falls into two parts. The first (Theorem 4.1) is the exponential relation between width and size via the method of random restrictions, and the second (Theorem 4.3) is the lower bound on the width in the spirit of [BW99].

Theorem 4.1 $2^{\Omega(w(IR_{n/20}\vdash\perp))} < S(IR_n)$.

Proof. Let us consider some proof $\mathcal{P}$ of IR_n. Assume w.l.o.g. that n is divisible by 20. We will apply a random restriction which kills every large clause in $\mathcal{P}$ with high probability. For that choose a random set of vertices $A \subseteq V(G_n)$ such that any point v is chosen independently at random with probability $1/2$. Let us choose only those points for which the distance from all other points is at least 15:

$$A' = \{v \in A | \forall v' \neq v, v' \in A \ \ d(v, v') \geq 15\}.$$

Clearly for any point the probability of being in A' is $\Omega(1)$ (that is the probability that the point is chosen and no other point in the neighborhood is) and if $d(v_1, v_2) > 30$ then the events $\{v_1 \in A'\}$ and $\{v_2 \in A'\}$ are independent.

Let us also throw away all points $v \in A'$ such that $d(v_s, v) < 15$. For each of the remaining points choose a variable from the set $\{y^v_{v_1 v_2}\} \cup \{y^v\}$ uniformly at random, set it to 1 and the rest to 0. Denote the resulting restriction by ρ.

Claim 4.2 *For any clause C, the probability $P\{C|_\rho \neq 1\}$ is less than $2^{-\Omega(w(C))}$.*

Proof. Let us fix a clause C. Clearly, we can extract a subclause C' of C such that $w(C') = \Omega(C)$ and any two literals of C' correspond to the vertices v_1, v_2 with $d(v_1, v_2) > 30$. Then all the literals in C' are satisfied independently with probability $\Omega(1)$. The Claim follows.■

By the Claim, there exists a constant ε for which the probability that a clause C is not killed is less than $2^{-\varepsilon w(C)}$. Assume for the sake of contradiction that $\mathcal{P}$ contains less than $2^{\varepsilon w(IR_{n/20}\vdash\perp)}$ clauses. Then by the union bound there exists a restriction ρ generated as above, which kills all clauses in $\mathcal{P}$ with width greater or equal than $w(IR_{n/20} \vdash \perp)$. To get a contradiction, we will show that the tautology $(IR_n)|_\rho$ "contains" the smaller principle $IR_{n/20}$, and hence $\mathcal{P}|_\rho$ should contain a clause of width $w(IR_{n/20} \vdash \perp)$.

Let us extend our restriction ρ to ρ' so that $(IR_n)|_{\rho'}$ is the Infinite Road tautology $IR(G, v_s)$ for a smaller graph G that results from G_n by removing rectangles of size 3×2 located sufficiently far from each other. To achieve this, for any assigned variable corresponding to the point $v \in A'$ we extend ρ to make a local cycle in the neighborhood of v (see Fig. 5).

Now it easy to see that $(IR_n)|_{\rho'}$ corresponds to some $IR(G, v_s)$, where the graph G was produced from the initial G_n by removing 2×3 rectangles D_i s.t. for any $i \neq j$ $d(D_i, D_j) > 10$.

To finish the proof, it is sufficient to find a minor subgraph of G isomorphic to $G_{n/20}$, clearly it would imply that $IR(G, v_s)$ is harder than $IR_{n/20}$. For that we define a set of $n/20$

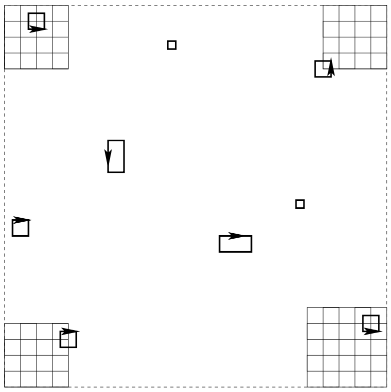

Figure 5: The extended restriction ρ'

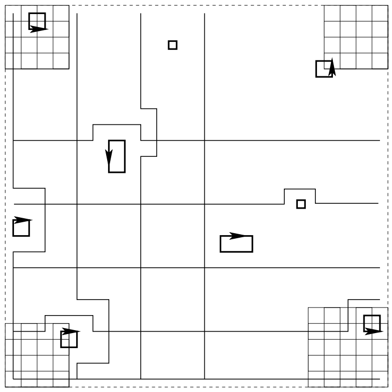

Figure 6: Extracting $G_{n/20}$ from G

horizontal lines h_i. Each h_i is drawn in parallel to $X - axis$ at the height $20i - 19$; whenever it meets an obstacle it "flows" around it and returns to the initial height. Similarly, we define $n/20$ vertical lines v_i so that each v_i intersects with each horizontal line exactly once (see Fig. 6). It is straightforward to check, that these lines define a minor isomorphic to $G_{n/20}$. Thus the minimal refutation width of $IR(G, v_s)$ is greater or equal than that of $IR_{n/20}$. By contradiction Theorem 4.1 is proved.■

Theorem 4.3 $w(IR_n \vdash \perp) \geq \sqrt{n}/4$.

Proof. Consider some refutation $\mathcal{P}$ of IR_n. Our strategy follows the idea of [BW99] to define a subadditive measure μ on the clauses and show that intermediate clauses have large width.

Recall that for v a vertex A_v is the conjunction of the corresponding axioms in Definition 3.2.

Definition 4.4 *We call an assignment* α critical *iff it satisfies all the axioms* A_v *except for some* $v_0 \neq v_s$.

It is easy to see that every critical assignment divides the chessboard with v_0 removed into isolated squares, non-intersecting cycles, and a path from v_s to v_0 (see Fig. 7).

Definition 4.5 *For* C *a clause,* $\mu(C)$ *is the minimal* k *s.t. there exists a set of* k *vertices* $I \subseteq V(G_n)$ *satisfying*

$$\bigcup_{v \in I} A_v \models C \text{ on the set of critical assignments.} \tag{1}$$

(this means that for any critical assignment α $(\forall v \in I\ A_v(\alpha) = 1) \rightarrow (C(\alpha) = 1))$.

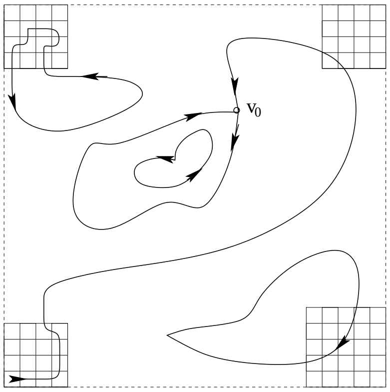

Figure 7: Critical assignments

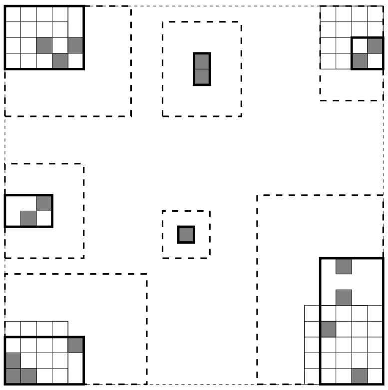

Figure 8: Covering of S by rectangles

Clearly μ is subadditive in the sense that if C resolves from C_1 and C_2 then $\mu(C) \leq \mu(C_1) + \mu(C_2)$. All initial clauses in the proof have μ equal to 1, and $\mu(\perp) = n^2 - 1$. These facts imply that there exists a clause C in the proof with $1/4n^2 < \mu(C) \leq 1/2n^2$. Fix this clause and assume for the sake of contradiction that $w(C) < \sqrt{n}/4$.

Consider the set S of all squares v such that C contains a variable $y^v_{v_1v_2}$ or y^v associated with v, let us also add the point v_s to S, thus $|S| < \sqrt{n}/4 + 1$.

Lemma 4.6 *There exists a covering of S by rectangles $D_1, ..., D_k$ with overall area $\sum_{i=1}^k |D_i| < 1/36n^2$ such that each rectangle D_i can be surrounded by a stripe of the width $|S \cap D_i|$ from each side not connected to the edge of the chessboard and these stripes don't either intersect or touch with each other (see Fig. 8).*

Proof. We will construct an admissible covering which is far from the optimal one, but its area is still small enough for our purposes. Since our metrics d is induced by l_∞-norm we can cover the projections of S onto X-axes and Y-axes independently. Denote by S_x the projection of S onto X-axes plus the elements 1 and n (corresponding to the left and right edges of the chessboard). Clearly $|S_x| < \sqrt{n}/4 + 2$. We need to cover S_x by intervals $[a_i, b_i]$ such that for any i the distance $|a_i - b_{i-1}|$ as well as $|a_{i+1} - b_i|$ is greater than $2|[a_i, b_i] \cap S_x| + 1$. Let us fix any covering with the minimal sum of the lengths.

Claim 4.7 *For any i the length $|b_i - a_i + 1| < (2|S_x| + 1) \cdot |[a_i, b_i] \cap S_x|$.*

Proof. The distance between any two adjacent points of S_x that belong to $[a_i, b_i]$ is less or equal $2|S_x| + 1$ (otherwise we could divide the interval $[a_i, b_i]$ into two smaller intervals). Thus the length of $[a_i, b_i]$ is upperbounded by the number of S_x-points multiplied by the maximal distance between two adjacent points.∎

By Claim 4.7 the sum of the lengths of our covering intervals is less than $(2|S_x| + 1) \cdot |S_x|$. In analogous way we can define the projection S_y and cover it by intervals $[c_j, d_j]$. Altogether these two coverings generate an admissible covering of S by rectangles $[a_i, b_i] \times [c_j, d_j]$ with area $\Sigma(b_i - a_i + 1) \cdot (d_j - c_j + 1) < (\Sigma(b_i - a_i + 1)) \cdot (\Sigma(d_j - c_j + 1)) < 4(|S_x| + 1)^2 \cdot (|S_y| + 1)^2 < n^2/36$ for sufficiently large n. Lemma is proved.■

Corollary 4.8 *The area of all D_i's with the corresponding stripes is less than $1/4n^2$.*

Let us now finish the proof of the Theorem. Consider the set of axioms I that imply the clause C in (1), $1/4n^2 < |I| \leq 1/2n^2$. Choose some $v_0 \in I$ and $v_0' \notin I$ that do not belong to D_i's or to their stripes. We can do it because of our bounds on $|I|$ and Corollary 4.8.

Since I is the minimal set satisfying (1) if we remove A_{v_0} then $\bigcup_{v \in I \setminus \{v_0\}} A_v \not\models C$ on critical assignments. It means that there exists a critical assignment α_0, such that $A_{v_0}(\alpha_0) = 0$, $C(\alpha_0) = 0$ and $A_v(\alpha_0) = 1$ for any other v. Our idea is to switch α_0 to another assignment α_0' so that it satisfies all A_v's except the chosen $v_0' \notin I$ but agree with α_0 on the variables of C. The idea is to modify α_0 in every rectangle D_i independently using their stripes free of S-points.

So let us describe how to construct α_0' from α_0. Recall that each critical assignment divides the chessboard into non-intersecting cycles and a path (see Fig. 7). Consider some rectangle D_i. First assume that it doesn't contain v_s. Then according to α_0 it can be partitioned into free squares, cycles and transit paths ℓ with a starting point $\ell(0) \in \partial(D_i)$ and a leaving point $\ell(1) \in \partial(D_i)$. Consider the set of only those transit paths $\ell_1^i, ..., \ell_{k_i}^i$ that go through some vertex from $S \cap D_i$. Clearly $k_i \leq |S \cap D_i|$. If D_i contains v_s the only difference is that there exists a special path ℓ_0^i with $\ell_0^i(0) = v_s$.

Let us modify α_0 in the following way. Remove all paths outside $\bigcup D_i$. Remove all the transit paths that go through D_i but doesn't intersect S. There are some "ends" hanging around in the boundary of D_i left. In order to produce a valid critical assignment α_0' we connect them with each other in the corresponding stripe E_i around D_i.

We use the following algorithm. Consider the boundary of D_i. Choose two ends: a starting point and a leaving point of some transit (probably different) paths such that there is no other ends between them in the boundary of D_i. Connect these ends outside of D_i in E_i by the minimal path $\hat{\ell}_1^i$. Consider next two ends that don't have any others between them in the boundary besides already connected, and construct the corresponding minimal path $\hat{\ell}_2^i$ in $E_i \setminus \hat{\ell}_1^i$, and so on. Since there are at most $2k_i$ such ends we can finish the process without overflow of E_i (see Fig. 9).

Finally there is just one end v_1 left, that corresponds to the path from v_s. Since the stripes around D_i's don't touch each other there is enough free space to connect v_1 with v_0' by a path that doesn't intersect the paths constructed on previous steps.

We have built a new critical assignment α_0' (see Fig. 10) which violates $A_{v_0'}$ without change of the variables from C. Thus for any $v \in I$ $A_v(\alpha_0') = 1$ and $C(\alpha_0') = 0$. This contradicts to the assumption that $\bigcup_I A_v \models C$ on critical assignments. Theorem 4.3 is proved.■

Altogether Theorems 3.3, 4.1 and 4.3 imply the following

Corollary 4.9 $S(CB_n) > 2^{\Omega(\sqrt{n})}$

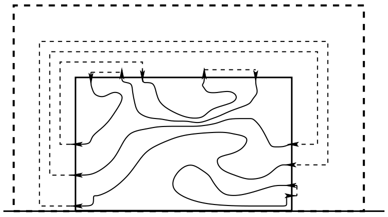

Figure 9: Switching critical assignment α_0 in D_i

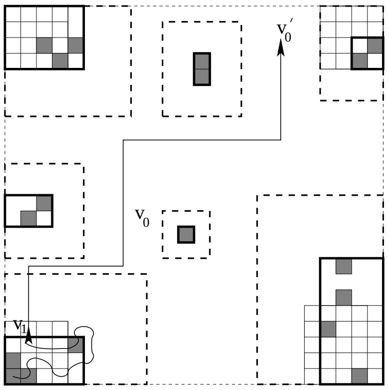

Figure 10: Building α_0'

5 Acknowledgements

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