Hardy Functions

Junior Paper

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1 Introduction

In this paper we use techniques from both Fourier and complex analysis to prove six nice theorems. We start with the Phragmén-Lindelöf Theorem, which then allows us to prove a beautiful theorem by G.H. Hardy. We then further utilize the Phragmén-Lindelöf Theorem to prove the Paley-Wiener Theorem, which motivates the remainder of the paper. The Paley-Wiener Theorem states that precisely those functions f which may be extended to entire functions have Fourier transforms $\hat{f}$ with compact support. A function f with this property when regarded as a function of time describes a type of communications signal. We spend the rest of the paper studying functions which can act as filters applied to such signals. Specifically we study a class of functions living in the upper half plane called Hardy functions. We prove several theorems about the behavior of Hardy functions and then discuss their utility as filters.

2 Hardy's Theorem

We begin with the result which will play a key role in the proofs of the Hardy and Paley-Wiener theorems.

Theorem 1 (Phragmén-Lindelöf) *Suppose f is holomorphic in a sector D of the complex plane with angular opening $< \pi$ and that f is continuous on the closure of D. If $|f(z)| \leq Ce^{c|z|}$ throughout D for constants $C, c > 0$ and if $|f(z)| \leq M$ on the boundary of D, then $|f(z)| \leq M$ for all $z \in D$.*

Proof: Without loss of generality, let D be symmetrically placed about the positive real axis with angle ψ on each side of the axis. Then $\psi < \frac{\pi}{2}$ so there exists $a > 1$ s.t. $a\psi < \frac{\pi}{2}$. For any fixed $A > 0$ define $F(z) = f(z)e^{-Az^a}$. Then F is holomorphic in D and continuous on the closure of D. For $z = Re^{i\theta}$ in D with $|\theta| < \psi$, we have

$$|F(z)| = |f(z)||e^{-Az^a}| \leq Ce^{c|z|}|e^{-A(Re^{i\theta})^a}| \leq Ce^{cR}e^{-AR^a \cos(a\psi)}$$

Thus F decreases rapidly in the sector as $R \to \infty$, since $a > 1$. Let $B = \sup_{z\in D} |F(z)|$. We claim $B \leq M$. If $B = 0$ this is clearly true, so we may suppose $B \neq 0$. Let z_i be a sequence of points in D s.t. $|F(z_i)|$ converges to B. $F(z)$ vanishes as $|z| = R$ increases to infinity, but $B \neq 0$, thus the sequence z_i cannot travel to infinity, but must converge at an accumulation point in the closure of D. By the maximum principle, the maximum of the holomorphic function F must occur on the boundary of D. But by assumption, for z on the boundary of D,

$$|F(z)| = |f(z)||e^{-Az^a}| \leq Me^{-AR^a \cos(a\psi)} \leq M$$

hence we may conclude $\sup_{z\in D} |F(z)| \leq M$. Then letting $A \to 0$, $|f(z)| \leq M$ throughout the sector D as desired.

We are now ready to prove our first main theorem. Although this theorem does not connect directly to the subject of Hardy functions (other than that it is also the work of G.H. Hardy), its result is so nice that it is impossible to resist including it. The idea of the theorem is similar to that of the Paley-Wiener Theorem in that a bound on the growth of the function gives specific information about the function and its transform.

Theorem 2 (Hardy) *Suppose f is a continuous function of moderate decrease with*

$$\begin{aligned} |f(x)| &\leq Ce^{-\alpha x^2} \\ |\hat{f}(\xi)| &\leq C'e^{-\beta \xi^2} \end{aligned}$$

on the real line, for $\alpha, \beta > 0$.

If $\alpha\beta = \pi^2$, the function f is of the type $f(x) = ce^{-\alpha x^2}$. If $\alpha\beta > \pi^2$ then $f = 0$. If $\alpha\beta < \pi^2$, then there are infinitely many such functions f, the Hermite functions being one such family.

Proof: We first prove the result for the case $\alpha\beta = \pi^2$. With appropriate scaling we may assume $\alpha = \beta = \pi$. Complexifying the variable $\xi = a+ib$ we see that $\hat{f}(\xi)$ is an entire function since the assumed bound on f provides sufficient decrease in the definition of the Fourier transform:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i\xi x}\,dx = \int_{-\infty}^{\infty} f(x)e^{-2\pi i(a+ib)x}\,dx$$

The integral of a holomorphic function on a compact interval is holomorphic, and the tails of the integral converge to zero nicely for sufficiently large M,

$$\int_{|x|>M} |f(x)e^{-2\pi i\xi x}|\,dx \leq C\int_{|x|>M} e^{-\pi x^2}e^{2\pi bx}\,dx$$

Thus $\hat{f}$ is holomorphic in the complex plane, with the following bound,

$$\begin{aligned} |\hat{f}(\xi)| &= \left|\int_{-\infty}^{\infty} f(x)e^{-2\pi i\xi x}\,dx\right| \\ &\leq C\int_{-\infty}^{\infty} e^{-\pi x^2}e^{2\pi bx}\,dx \\ &\leq Ce^{\pi b^2} \end{aligned}$$

We will first prove the result for f an even function. If f is even then $\hat{f}$ is also even, as can be seen from the definition of the Fourier transform. Then the power series expansion for $\hat{f}$ contains only even powers and we can define an entire function $h(\xi) = \hat{f}(\sqrt{\xi})$. We will prove that $e^{\pi\xi}h(\xi)$ is a constant function on the complex plane, which will then give us the desired result for f an even function.

To prove that $e^{\pi\xi}h$ is constant we first find a bound on h by using the bound on $\hat{f}$ found above. For $\xi = Re^{i\theta}$ we have

$$|h(\xi)| = |\hat{f}(\xi)| \leq Ce^{\pi(Im\sqrt{\xi})^2} = Ce^{\pi R\sin^2(\frac{\theta}{2})} \leq Ce^{c|\xi|}$$

for some $c > 0$. This bound on h leads us to think of the Phragmén-Lindelöf Theorem. We will apply this theorem to the function $g(\xi)h(\xi)$ where the complex function g is defined

$$g(\xi) = e^{(i\pi\xi e^{-i\delta/2})/\sin(\delta/2)}$$

Specifically we will apply the Phragmén-Lindelöf Theorem to gh in a sector D of the first quadrant of the complex plane with angular opening $\delta < \pi$. First we find a bound for gh on the boundary lines of the sector.

For $\xi > R$ on the positive real line we have a bound inherited from the bound on $\hat{f}$ assumed in the statement of the theorem, namely

$$|g(\xi)h(\xi)| = |h(\xi)| \leq C'e^{-\pi R}$$

In general for $\xi = Re^{i\theta}$ we have, by trigonometric manipulations,

$$|g(\xi)h(\xi)| = e^{(-\pi R \sin(\theta - \frac{\delta}{2}))/\sin(\frac{\delta}{2})}|h(Re^{i\theta})| \leq C'' e^{c|\xi|}$$

so gh satisfies an exponential bound within the sector. For the values $\theta = 0$ and $\theta = \delta < \pi$, i.e. on the boundary lines of the sector D, we then have $|g(\xi)h(\xi)| \leq C'$. We may now apply the Phragmén-Lindelöf Theorem to conclude that

$$|g(\xi)h(\xi)| = e^{(-\pi R \sin(\theta - \frac{\delta}{2}))/\sin(\frac{\delta}{2})}|h(Re^{i\theta})| \leq C'$$

throughout the sector D. Letting $\delta \to \pi$ we see

$$e^{\pi R \cos\theta}|h(\xi)| \leq C'$$

Similarly we can apply this argument to any sector in the complex plane with angular opening strictly less than π. Conclude that

$$e^{\pi R \cos\theta}|h(\xi)| \leq C'$$

for all $\xi = Re^{i\theta}$ in the complex plane. Noting that

$$|e^{\pi\xi}h(\xi)| = e^{\pi R \cos\theta}|h(\xi)|$$

we then see that the entire function $e^{\pi\xi}h$ is bounded by C' in the complex plane. Hence by Liouville's Theorem $e^{\pi\xi}h$ is a constant function. Thus for some constant c, $\hat{f}(\sqrt{\xi}) = h(\xi) = ce^{-\pi\xi}$, hence $\hat{f}(\xi) = ce^{-\pi\xi^2}$. With a handy application of the fact that the Gaussian is its own transform we then have the desired result for f an even function:

$$f(x) = ce^{-\pi x^2}$$

If f is odd, then $\hat{f}$ is also odd and $\hat{f}(0) = 0$. Examining the function $\xi^{-1}\hat{f}(\xi)$ we see that this is an entire function since the zero of $\hat{f}$ cancels the pole of ξ^{-1} at the origin. Moreover, this function is even, so we may apply the above argument to $\xi^{-1}\hat{f}(\xi)$ to conclude that $\xi^{-1}\hat{f}(\xi) = ce^{-\pi\xi^2}$ and hence

$$\hat{f}(\xi) = c\xi e^{-\pi\xi^2}$$

But since we assumed the bound $|\hat{f}(\xi)| \leq C'e^{-\pi\xi^2}$, we must conclude that in fact $c = 0$ and hence $\hat{f} = f = 0$. Finally, for the general function f we split f into its even and odd parts. Applying the arguments given above to the even and odd parts of f we may conclude that f is a constant multiple of $e^{-\pi x^2}$ as desired. This proves the case $\alpha = \beta = \pi$.

For the case $\alpha\beta > \pi^2$, with appropriate scaling we may assume $\alpha = \beta > \pi$. The bounds

$$\begin{aligned} |f(x)| &\leq Ce^{-\alpha x^2} \\ |\hat{f}(\xi)| &\leq C'e^{-\beta\xi^2} \end{aligned}$$

then give for some other constants C, C'

$$\begin{aligned} |f(x)| &\le Ce^{-\pi x^2} \\ |\hat{f}(\xi)| &\le C'e^{-\pi\xi^2} \end{aligned}$$

so we may apply the previous case to obtain the result $f(x) = ce^{-\pi x^2}$. But this contradicts the bound $|f(x)| \le Ce^{-\alpha x^2}$ unless $f = 0$.

For the final case $\alpha\beta < \pi^2$ we provide an infinite number of functions f satisfying the hypotheses of the theorem, namely the family of Hermite functions h_n defined for $n \ge 0$ by

$$h_n(x) = \frac{(-1)^n}{n!} e^{\pi x^2} \frac{d^n}{dx^n}\left(e^{-2\pi x^2}\right)$$

We again assume $\alpha = \beta < \pi$. It is simple to see that $|h_n(x)| \le Ce^{-\alpha x^2}$ for $\alpha < \pi$ by using induction to show that h_n is the product of $e^{-\pi x^2}$ and a polynomial of degree n. For $n = 0$, clearly $h_0(x) = e^{-\pi x^2}$. Assume that

$$h_n(x) = \frac{(-1)^n}{n!} e^{-\pi x^2} p_n(x)$$

where $p_n(x)$ is a polynomial of degree n. Then comparing this with the definition of h_n we see that

$$\frac{d^n}{dx^n}\left(e^{-2\pi x^2}\right) = e^{-2\pi x^2} p_n(x)$$

Then

$$\begin{aligned} h_{n+1}(x) &= \frac{(-1)^{n+1}}{(n+1)!} e^{\pi x^2} \frac{d^{n+1}}{dx^{n+1}}\left(e^{-2\pi x^2}\right) \\ &= \frac{(-1)^{n+1}}{(n+1)!} e^{\pi x^2} \frac{d}{dx}\left(e^{-2\pi x^2} p_n(x)\right) \\ &= \frac{(-1)^{n+1}}{(n+1)!} e^{\pi x^2} \left((-2\pi)(2x)e^{-2\pi x^2} p_n(x) + e^{-2\pi x^2} p_n'(x)\right) \\ &= \frac{(-1)^{n+1}}{(n+1)!} e^{-\pi x^2} p_{n+1}(x) \end{aligned}$$

Since any polynomial is bounded by a function of exponential growth $e^{\epsilon x^2}$ for any $\epsilon > 0$, we may conclude that

$$|h_n(x)| \le Ce^{-\alpha x^2}$$

for $\alpha < \pi$ as desired.

Now we need only show that $\hat{h}_n$ obeys a similar bound. To do so we show that $\hat{h}_n = (-i)^n h_n$ for each $n \ge 0$, which then clearly gives the desired bound on $\hat{h}_n$. To prove this relation between $\hat{h}_n$ and h_n we first show that h_n satisfies the recursion

$$h_n' - 2\pi x h_n = -(n+1)h_{n+1}$$

This follows plainly from the definition for h_n:

$$
\begin{aligned}
h_n' - 2\pi x h_n &= \frac{(-1)^n}{n!}(\pi 2x)e^{\pi x^2}\frac{d^{n+1}}{dx^{n+1}}\left(e^{-2\pi x^2}\right) + \frac{(-1)^n}{n!}e^{\pi x^2}\frac{d^{n+1}}{dx^{n+1}}\left(e^{-2\pi x^2}\right) \\
&\quad -2\pi x\frac{(-1)^n}{n!}e^{\pi x^2}\frac{d^n}{dx^n}\left(e^{-2\pi x^2}\right) \\
&= \frac{(-1)^n}{n!}e^{\pi x^2}\frac{d^{n+1}}{dx^{n+1}}\left(e^{-2\pi x^2}\right) \\
&= -(n+1)h_n
\end{aligned}
$$

From this recursion formula for h_n we then see that $(-i)^n h_n$ satisfies the recursion

$$-(-i)^{n+1}(n+1)h_{n+1} = -i(-i)^n h_n' + (-i)^n 2\pi i x h_n$$

Now using the relations $(-2\pi i x f)\hat{} = (\hat{f})'$ and $(f')\hat{} = 2\pi i\xi\hat{f}$ we can calculate the recursion for $\hat{h}_n$:

$$
\begin{aligned}
-(n+1)(h_{n+1})\hat{} &= (h_n')\hat{} - (2\pi x h_n)\hat{} \\
-(n+1)\hat{h}_{n+1} &= 2\pi i x h_n - i(-2\pi i x h_n)\hat{} \\
-(n+1)\hat{h}_{n+1} &= 2\pi i x h_n - i h_n'
\end{aligned}
$$

Thus we see that $\hat{h}_n$ and $(-i)^n h_n$ satisfy the same recursion relation. Furthermore, for $n = 0$ we see that $(-i)^n h_n = e^{-\pi x^2}$. The Gaussian is its own transform, hence $\hat{h}_0 = h_0 = (-i)^0 h_0$. This, with the recursion relation, implies $\hat{h}_n = (-i)^n h_n$ for all $n \geq 0$,and gives us the desired bound $|\hat{h}| \leq C' e^{-\beta\xi^2}$. This concludes our discussion of the theorem.

3 The Paley-Wiener Theorem

We now discuss a second theorem which uses the Phragmén-Lindelöf Theorem at a key point in the proof. The Paley-Wiener Theorem states that a continuous function f of moderate decrease may be extended to an entire function with a certain bounded growth if and only if $\hat{f}$ has compact support. This theorem is interesting not only for its striking result but also because of its application to signal processing. A signal can be regarded as a function of time $f(t)$, with power

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt = 1$$

Two characteristic quantities α and β of a signal f are defined for fixed $a, b > 0$ as

$$\alpha^2 = \int_{-a}^{a} |f(t)|^2 \, dt$$

$$\beta^2 = \int_{-b}^{b} |\hat{f}(\xi)|^2 \, d\xi$$

For the purposes of signals communications, α and β are desired to be as close as possible to 1. A signal with $\alpha = 1$ is termed time-limited, while a signal with $\beta = 1$ is band-limited. It is possible to show that both $\alpha = 1$ and $\beta = 1$ cannot be acheived for a nonzero signal f. For suppose $\alpha = 1$ and $\beta = 1$ with both f and $\hat{f}$ of moderate decrease. Then since the total power of the signal is 1,

$$\alpha^2 = \int_{-a}^{a} |f(t)|^2 dt = 1$$

implies $f(t) = 0$ for all $|t| > a$. Similarly $\beta^2 = 1$ implies $\hat{f}(\xi) = 0$ for all $|\xi| > b$. Then Fourier inversion gives

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi t} \, d\xi = \int_{-b}^{b} \hat{f}(\xi) e^{2\pi i \xi t} \, d\xi = 0$$

for all $|t| > a$. Differentiating under the integral sign yields

$$\int_{-b}^{b} \hat{f}(\xi) e^{2\pi i \xi a} \xi^n \, d\xi = 0 \quad \text{for } n \geq 0$$

Using the power series expansion for the exponential function we can write for any real t:

$$\begin{aligned} f(t) &= \int_{-b}^{b} \hat{f}(\xi) e^{2\pi i \xi t}\, d\xi \\ &= \int_{-b}^{b} \hat{f}(\xi) e^{2\pi i \xi (t-a)} e^{2\pi i \xi a}\, d\xi \\ &= \int_{-b}^{b} \hat{f}(\xi) e^{2\pi i \xi a} \left[\sum_{n=0}^{\infty} \frac{(2\pi i \xi (t-a))^n}{n!} \right] d\xi \\ &= \sum_{n=0}^{\infty} \left[\frac{(2\pi i (t-a))^n}{n!} \int_{-b}^{b} \hat{f}(\xi) e^{2\pi i \xi a} \xi^n \, d\xi \right] \\ &= 0 \end{aligned}$$

Hence $h = 0$.

It is precisely the band-limited functions which the Paley-Wiener Theorem describes. Now we review the proof of the Paley-Wiener Theorem.

Theorem 3 (Paley-Wiener) *Suppose f is a continuous function with moderate decrease on the real line. Then f is extendable to an entire function of growth $|f(z)| \leq Ce^{2\pi M|z|}$ for a constant $C > 0$ if and only if $\hat{f}$ has compact support in the interval $[-M, M]$.*

Proof: First suppose that $\hat{f}$ has compact support in $[-M, M]$. Then $\hat{f}$ has moderate decrease and by assumption f also has moderate decrease, so we may apply the Fourier inversion formula to obtain

$$f(x) = \int_{-M}^{M} \hat{f}(\xi) e^{2\pi i \xi x}\, d\xi$$

Since the range of integration is finite, complexifying the variable only introduces a constant term, so we may extend f to a complex function of z,

$$f(z) = \int_{-M}^{M} \hat{f}(\xi) e^{2\pi i \xi z}\, d\xi$$

Differentiation under the integral sign shows that $f(z)$ is holomorphic on the entire complex plane, so $f(z)$ is the desired entire extension of the real function $f(x)$. We also obtain the desired bound on $f(z)$: if $z = x + iy$ then

$$\begin{aligned} |f(z)| &\leq \int_{-M}^{M} |\hat{f}(\xi)| e^{-2\pi \xi y}\, d\xi \\ &\leq \sup_{y \in [-M,M]} |\hat{f}(\xi)| \int_{-M}^{M} e^{-2\pi \xi y}\, d\xi \\ &\leq Ce^{2\pi M|z|} \end{aligned}$$

This proves one direction of the theorem.

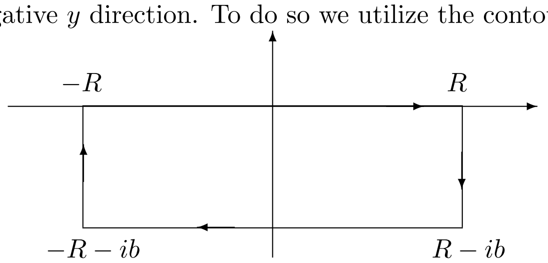
In order to prove the converse, we first assume a boundedness condition for the extension of f that is stronger than $|f(z)| \leq Ce^{2\pi M|z|}$. Specifically, we assume that for some constant A,

$$|f(x+iy)| \leq A\frac{e^{2\pi M|y|}}{1+x^2}$$

We will use contour integration to prove that $\hat{f}(\xi) = 0$ for $|\xi| > M$. For suppose $\xi > M$. Taking the Fourier transform of f gives

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i\xi x}\,dx$$

We wish to shift the line of integration down below the real axis by an amount $b > 0$ in the negative y direction. To do so we utilize the contour shown below.



Define the complex function $g(z) = f(z)e^{-2\pi i\xi z}$. Then the integral of g over the vertical sides of the contour vanishes as $R \to \infty$. Specifically, the integral along the right vertical side is bounded by

$$\begin{aligned}
&\leq \int_0^{-b} |f(R+it)||e^{-2\pi i\xi(R+it)}|\,dt \\
&\leq \int_0^{-b} A\frac{e^{2\pi Mb}}{1+R^2}e^{2\pi t\xi}\,dt \\
&\leq \frac{c}{1+R^2}
\end{aligned}$$

which vanishes as $R \to \infty$. Similarly the integral along the left vertical side vanishes. Since f and $e^{-2\pi i\xi z}$ are entire functions then g is entire and hence has no poles in the contour. Cauchy's theorem then implies that the integral of g around the contour is zero, hence by orientation considerations we see that

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x-ib)e^{-2\pi i\xi(x-ib)}\,dx$$

Using the strong assumed bound on f we can now bound $\hat{f}(\xi)$ by

$$\begin{aligned}
|\hat{f}(\xi)| &\leq \int_{-\infty}^{\infty} A\frac{e^{2\pi Mb}}{1+x^2}e^{-2\pi\xi b}\,dx \\
&\leq Ae^{-2\pi b(\xi-M)}\int_{-\infty}^{\infty}\frac{dx}{1+x^2} \\
&\leq Ce^{-2\pi b(\xi-M)}
\end{aligned}$$

By assumption $\xi > M$ hence as $b \to \infty$, we see that $\hat{f}(\xi) = 0$. Similarly we can shift the contour above the real line for $\xi < -M$, and we have proved that $\hat{f}$ has the desired compact support under our stronger assumption on the boundedness of f.

Next we assume that f has the somewhat weaker boundedness condition that for some A',

$$|f(x+iy)| \le A' e^{2\pi M|y|}$$

From this we will define functions f_ϵ which satisfy the stronger assumption used in the argument above, and then show that these functions give the desired result of compact support for the transform of f itself. Suppose $\xi > M$. Fix $\epsilon > 0$ and define

$$f_\epsilon(z) = \frac{f(z)}{(1+i\epsilon z)^2}$$

Since

$$|1 + i\epsilon(x+iy)|^2 = (1-\epsilon y)^2 + (\epsilon x)^2$$

then for $x + iy$ in the closed lower half plane we see that $1/|1 + i\epsilon(x+iy)|^2 \le 1$ and that this factor converges to 1 as $\epsilon \to 0$. Then we also see that $\hat{f}_\epsilon(\xi) \to \hat{f}(\xi)$ as $\epsilon \to 0$ since

$$|\hat{f}_\epsilon(\xi) - \hat{f}(\xi)| \le \int_{-\infty}^{\infty} |f(x)| \left| \frac{1}{(1+i\epsilon x)^2} - 1 \right| dx$$

which converges to zero since f has moderate decrease, hence is bounded. Next note that for each fixed $\epsilon > 0$, f_ϵ satisfies the stronger boundedness assumption of our previous argument. For $x + iy$ in the lower half plane we have

$$|f_\epsilon(x+iy)| \le \frac{A' e^{2\pi M|y|}}{|1 + i\epsilon x - \epsilon y|^2} \le \frac{A' e^{2\pi M|y|}}{|1 + i\epsilon x|^2} \le A'' \frac{e^{2\pi M|y|}}{1+x^2}$$

Thus we may apply our previous argument to conclude that for each fixed $\epsilon > 0$, $\hat{f}_\epsilon(\xi) = 0$ for $\xi > M$. Letting $\epsilon \to 0$ we then see that $\hat{f}(\xi) = 0$ for $\xi > M$. We can repeat the same argument for $\xi < -M$, defining f_ϵ by altering f by a factor of $1/(1 - i\epsilon z)^2$ and working in the closed upper half plane.

Now in order to complete the proof of the theorem we need only conclude that for an entire function f with $|f(z)| \le Ce^{2\pi M|z|}$ for all z in the complex plane and with f of moderate decrease on the real line, we do indeed have the bound $|f(x+iy)| \le A' e^{2\pi M|y|}$. But this follows directly from the Phragmén-Lindelöf Theorem. Let D be a sector in the first quadrant with angular opening $\delta < \pi$. Defining

$$F(z) = f(z) e^{2\pi i M z}$$

we apply the Phragmén-Lindelöf Theorem to F in the sector D, as $\delta \to \pi$. f has moderate decrease on the real line, hence f is bounded, $|f(x)| \le A$ for all real x and some constant A. Then $|F(x)| = |f(x)| \le A$ on the positive real line and $|F(iy)| \le Ce^{2\pi My} e^{-2\pi My} = C$ on the positive imaginary axis. Defining $A' = \max(A, C)$ we then have $|F(z)| \le A'$ on the boundary of D. Also, in the sector D, $|F(z)| \leq Ce^{2\pi M|z|}e^{-2\pi My} \leq C'e^{c|z|}$. Thus by the Phragmén-Lindelöf Theorem we can conclude that $|F(z)| \leq A'$ throughout the first quadrant. A similar argument applied to the other quadrants then gives the desired bound $|f(x+iy)| \leq A'e^{2\pi M|y|}$ for all $x+iy$ in the complex plane. This completes the proof of the Paley-Wiener Theorem.

4 Hardy Functions

We now begin our discussion of Hardy functions. A function h which is holomorphic on the open upper half plane $\mathbb{H}$ and satisfies

$$\sup_{b>0} \|h_b\|^2 = \sup_{b>0} \int_{-\infty}^{\infty} |h(a+ib)|^2 da < \infty$$

is said to be a Hardy function of class $\mathcal{H}^{2+}$.

Such functions are usually discussed in the context of the L^2 norm, but for the purposes of this paper we will only speak of square integrability. Also, in the following discussion we will assume that all functions and their transforms have sufficient decrease for the Fourier transform and inversion formulae to hold. Note that for convenient scaling in the following discussion of Hardy functions we will use these alternative forms of the Fourier transform and inversion formulae:

$$\begin{aligned} \hat{f}(\xi) &= \int_{-\infty}^{\infty} f(x) e^{i\xi x}\, dx \\ f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{-i\xi x}\, d\xi \end{aligned}$$

This then gives the modified Plancherel identity

$$\|\hat{f}\| = \sqrt{2\pi}\|f\|$$

We now prove a nice theorem which gives a necessary and sufficient condition for a function to be Hardy.

Theorem 4 *A function h is of class $\mathcal{H}^{2+}$ if and only if*

$$h(\omega) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\hat{f}(\xi)}{\xi - \omega}\, d\xi = \int_0^{\infty} f(x) e^{i\omega x}\, dx$$

in the upper half plane for some square integrable function f which vanishes on the left half real line, $f(x) = 0$ for $x < 0$.

Proof: First note that the equality of the two integrals in the theorem statement makes sense. Define for $\omega \in \mathbb{H}$,

$$g(\xi) = \frac{1}{2\pi i(\xi - \omega)}$$

Then g is square integrable on the real line and by contour integration we can show

$$2\pi g^{*\vee *}(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{i\xi x}}{\xi - \omega} d\xi = \begin{cases} e^{i\omega x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

Then for f a square integrable function with sufficient decrease,

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\hat{f}(\xi)}{\xi - \omega}\, d\xi = (\hat{f}, g^*) = 2\pi(f, g^{*\vee}) = \int_0^{\infty} f(x) e^{i\omega x}\, dx$$

Thus the statement of the theorem makes sense.

Now suppose

$$h(\omega) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\hat{f}(\xi)}{\xi - \omega}\, d\xi = \int_0^{\infty} f(x)e^{i\omega x}\, dx$$

for some such f. First we see that h is holomorphic in the upper half plane. Integrating the holomorphic complex function $f(x)e^{i\omega x}$ on any compact interval $[0, M]$ of the real line yields a holomorphic function. We need only check that the tail of the integral converges uniformly to zero to conclude that h is holomorphic in the upper half plane. But this follows because for $\omega = a + ib$ in the upper half plane,

$$\begin{aligned}
\int_M^{\infty} |f(x)e^{i\omega x}|\, dx &\leq \int_M^{\infty} |f(x)|e^{-bx}\, dx \\
&\leq \left(\int_M^{\infty} |f(x)|^2\, dx\right)^{\frac{1}{2}} \left(\int_M^{\infty} e^{-2bx}\, dx\right)^{\frac{1}{2}} \\
&\leq ce^{-bM} \to 0 \quad \text{as} \quad M \to \infty
\end{aligned}$$

since f is square integrable and $b > 0$. Also, h has the boundedness property required for a Hardy function: for $\omega = a + ib$ with fixed $b > 0$ we define $h(\omega) = h(a + ib)$ as $h_b(a)$. Then for each fixed $b > 0$ we have

$$\begin{aligned}
h_b(a) &= \int_0^{\infty} f(x)e^{i\omega x}\, dx \\
&= \int_0^{\infty} f(x)e^{i(a+ib)x}\, dx \\
&= \int_0^{\infty} f(x)e^{-bx}e^{iax}\, dx \\
&= (f(x)e^{-bx})\check{}(a)
\end{aligned}$$

Then by the Plancherel identity,

$$\|h_b\|^2 = \frac{1}{2\pi}\|\hat{h}_b\|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |f(x)|^2 e^{-2bx}\, dx < \infty$$

since f is assumed to be square integrable. Thus we can conclude that h is indeed a Hardy function.

Conversely, suppose that h is a Hardy function, i.e. h is holomorphic on the upper half plane and $\sup_{b>0} \|h_b\|^2 < \infty$. For fixed $A > 0$ and $\omega \in \mathbb{H}$ define

$$k(\omega) = e^{iA\omega}A^{-1}\int_0^A h_b(\omega + y)\, dy$$

Since $h_b(\omega+y)$ is holomorphic in the upper half plane for each y, then the integral over the compact interval yields a holomorphic function, thus k is holomorphic in the upper half plane. Our goal is to show that

$$k(\omega) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{k(\xi)}{\xi - \omega}\, d\xi$$

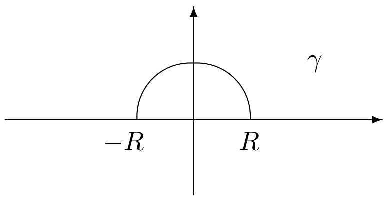
Once we have proved this formula, then as $A \to 0$ we obtain the formula

$$h_b(\omega) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{h_b(\xi)}{\xi - \omega}\, d\xi$$

This is because k approximates h_b locally uniformly and the tails of the integral formula for k are sufficiently small, for by the Cauchy-Schwarz inequality we have

$$\begin{aligned} \int_{|\xi|>M} \frac{|k(\xi)|}{|\xi - \omega|}\, d\xi &\leq \left(\int_{|\xi|>M} |k(\xi)|^2\, d\xi \right)^{\frac{1}{2}} \left(\int_{|\xi|>M} \left| \frac{1}{\xi - \omega} \right|^2 d\xi \right)^{\frac{1}{2}} \\ &\leq C\|k\|M^{-1} \\ &\leq C'\|h_b\|M^{-1} \end{aligned}$$

We will prove the Cauchy integral formula for $k(\omega)$ by contour integration of the complex function $k(z)/(z-\omega)$ on the semicircular contour γ in the upper half plane as shown below.



Since k is holomorphic on the upper half plane, the only pole of $k(z)/(z - \omega)$ occurs at ω. Thus by residue calculus,

$$2\pi i k(\omega) = \int_\gamma \frac{k(z)}{z - \omega}\, dz = \int_{-R}^{R} \frac{k(x)}{x - \omega}\, dx + \int_0^\pi \frac{k(Re^{i\theta}) i Re^{i\theta}}{Re^{i\theta} - \omega}\, d\theta$$

In order to conclude that we have the desired formula for $k(\omega)$ we must only show that as $R \to \infty$ the integral over the arc vanishes. Using the definition for $k(\omega)$ and the Cauchy-Schwarz inequality we have the following bound on k:

$$\begin{aligned} |k(\omega)| = |k(Re^{i\theta})| &\leq \frac{e^{-AR\sin\theta}}{A} \int_0^A |h_b(Re^{i\theta} + y)|\, dy \\ &\leq \frac{e^{-AR\sin\theta}}{A} \left(\int_0^A dy \right)^{\frac{1}{2}} \left(\int_0^A |h_b(Re^{i\theta} + y)|\, dy \right)^{\frac{1}{2}} \\ &\leq \frac{e^{-AR\sin\theta}}{\sqrt{A}} \|h_{b+R\sin\theta}\| \\ &\leq C \frac{e^{-AR\sin\theta}}{\sqrt{A}} \end{aligned}$$

since h_b is square integrable for all $b > 0$. Using this bound we see that the integral over the arc vanishes as $R \to \infty$,

$$\int_0^\pi \left| \frac{k(Re^{i\theta})iRe^{i\theta}}{Re^{i\theta} - \omega} \right| d\theta \leq \int_0^\pi C \frac{e^{-AR\sin\theta}}{\sqrt{A}} \, d\theta \leq \frac{C'}{R}$$

for each fixed $A > 0$. Thus we can conclude

$$k(\omega) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{k(x)}{x - \omega} \, dx$$

and consequently the equivalent formula for $h_b(\omega)$ also holds:

$$h_b(\omega) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{h_b(\xi)}{\xi - \omega} \, d\xi$$

We can now prove the result of the theorem. For suppose $\omega = a + ic$ is in the upper half plane. Since h is Hardy, h_b is square integrable with sufficient decrease so we may apply the result of the discussion at the beginning of this proof to write the Cauchy integral from of $h_b(\omega)$ in the alternate form

$$\begin{aligned} h_b(\omega) = h_b(a + ic) &= \int_0^\infty h_b^\vee(x) e^{i\omega x} \, dx \\ &= \int_0^\infty h_b^\vee(x) e^{iax} e^{-cx} \, dx \end{aligned}$$

But also

$$h_b(\omega) = h_b(a + ic) = h_{b+c}(a) = \int_0^\infty h_{b+c}^\vee(x) e^{iax} \, dx$$

Thus

$$h_{b+c}^\vee(x) = h_b^\vee(x) e^{-cx} \quad \text{for } c > 0,\ b > 0$$

Defining $f(x) = e^{bx} h_b^\vee(x)$ we see that this definition is independent of $b > 0$, since for any $\epsilon > 0$,

$$e^{(b+\epsilon)x} h_{b+\epsilon}^\vee(x) = e^{bx+\epsilon x} h_b^\vee(x) e^{-\epsilon x} = e^{bx} h_b^\vee(x)$$

Now we see that f as defined is the desired function, with

$$\begin{aligned} \int_0^\infty |f(x)|^2 \, dx &= \lim_{b \to 0} \int_0^\infty e^{-2bx} |f(x)|^2 \, dx \\ &= \lim_{b \to 0} \|h_b^\vee\|^2 \\ &= \lim_{b \to 0} \frac{1}{2\pi} \|h_b\|^2 < \infty \end{aligned}$$

Also, since f is defined $f(x) = e^{bx} h_b^\vee(x)$, we see

$$h_b(\omega) = h(\omega + ib) = \int_0^\infty h_b^\vee(x) e^{i\omega x} \, dx = \int_0^\infty e^{-bx} f(x) e^{i\omega x} \, dx$$

Letting $b \to 0$ for fixed ω we then have

$$h(\omega) = \int_0^\infty f(x)e^{i\omega x}\,dx = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\hat{f(\xi)}}{\xi - \omega}\,d\xi$$

and the proof of the theorem is complete.

One result of this theorem is that it gives us a convenient way to think of Hardy functions. Specifically, we can now think a Hardy function h in terms of the Fourier transform of its accompanying function f. In fact, from now on we will often use the alternative definition

$$\mathcal{H}^{2+} = \{\hat{f} \mid \hat{f} \text{ square integrable, } f(x) = 0 \quad \text{for } x < 0\}$$

Equivalently we can define a class of functions

$$\mathcal{H}^{2-} = \{\hat{f} \mid \hat{f} \text{ square integrable, } f(x) = 0 \quad \text{for } x > 0\}$$

Although we will focus primarily on the class of functions $\mathcal{H}^{2+}$, we will see some examples of functions of class $\mathcal{H}^{2-}$.

It is now convenient to exhibit an orthonormal basis for $\mathcal{H}^{2+}$, namely the family

$$e_n(\xi) = \frac{1}{\sqrt{\pi}}\left(\frac{1}{i\xi - 1}\right)\left(\frac{i\xi + 1}{i\xi - 1}\right)^n$$

for $n \geq 0$. First we prove the following proposition.

Proposition 1 *A rational function $\hat{f}$ of moderate decrease is Hardy of class $\mathcal{H}^{2+}$ if and only if all its poles lie in the open lower half plane $\mathbb{H}^-$.*

Proof: Suppose $\hat{f}$ is a function of moderate decrease with no poles in the closed upper half plane. We must show that the inverse Fourier transform

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{-i\xi x}\,d\xi$$

vanishes on the left half real line. Examine the complex function $\hat{f}(z)e^{-izx}$. Since $\hat{f}$ has no poles in the upper half plane, contour integration on the semi-circular contour γ of radius R in the upper half plane (as used in Theorem 4) then yields for any fixed real $a < 0$:

$$0 = \int_\gamma \hat{f}(z)e^{-iza}\,dz = \int_{-R}^{R} \hat{f}(\xi)e^{-i\xi a}\,d\xi + \int_0^\pi \hat{f}(Re^{i\theta})e^{-iaRe^{i\theta}}iR\,d\theta$$

As $R \to \infty$, the integral over the arc goes to zero:

$$\left|\int_0^\pi \hat{f}(Re^{i\theta})e^{-iaRe^{i\theta}}iR\,d\theta\right| \leq \int_0^\pi |\hat{f}(Re^{i\theta})|e^{aR\sin\theta}R\,d\theta \leq MR\int_0^\pi e^{aR\sin\theta}\,d\theta$$

since $\hat{f}$ has moderate decrease, hence is bounded. Noting that the integrand $e^{aR\sin\theta}$ has exponential decay for $a < 0$, except in neighborhoods near $\theta = 0$ and $\theta = \pi$ which we may make arbitrarily small, we see that the integral vanishes as $R \to \infty$. Thus we may conclude that for all real $a < 0$,

$$f(a) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{-i\xi a}\, d\xi = 0$$

Hence $\hat{f}$ is Hardy, as desired.

Conversely, suppose $\hat{f}$ is a rational Hardy function of moderate decrease. We will show that $\hat{f}$ is holomorphic in the upper half plane $\mathbb{H}$. Since $\hat{f}$ is Hardy, f vanishes on the left half line, so taking the Fourier transform yields

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{ix\xi}\, dx = \int_{0}^{\infty} f(x) e^{ix\xi}\, dx$$

Complexifying the variable ξ and regarding $\hat{f}$ as a function of $z = s + it$ in the upper half plane we then have

$$\hat{f}(z) = \int_{0}^{\infty} f(x) e^{ixz}\, dx = \int_{0}^{\infty} f(x) e^{isx} e^{-tx}\, dx$$

Examining the integrand $f(x)e^{isx}e^{-tx}$ we see that for $z \in \mathbb{H}$, $t > 0$ hence the exponential decay dominates the function f for $x > 0$. Thus the integrand is holomorphic in the upper half plane, for $x > 0$. Integrating on a compact interval yields a holomorphic function so we need only check that the tail of the integral converges uniformly to conclude that $\hat{f}$ is also holomorphic in the upper half plane. We have the bound

$$\begin{aligned} \left| \int_{M}^{\infty} f(x) e^{ixz} dx \right| &\leq \int_{M}^{\infty} |f(x)| e^{-tx}\, dx \\ &\leq \|f\| \left(\int_{M}^{\infty} e^{-2tx}\, dx \right)^{\frac{1}{2}} \\ &\leq c\|f\| e^{-tM} \end{aligned}$$

which clearly converges to zero as $M \to \infty$, for $t > 0$. Thus we may conclude $\hat{f}$ is holomorphic in the upper half plane and the proof is complete.

Proposition 1 gives us an easy way to see that e_n belongs to $\mathcal{H}^{2+}$ for $n \geq 0$ and to $\mathcal{H}^{2-}$ for $n < 0$. For any integer n, e_n is square integrable:

$$\left(\int_{-\infty}^{\infty} |e_n(x)|^2\, dx \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{\pi}} \left(\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} \right)^{\frac{1}{2}} < \infty$$

For $n \geq 0$, e_n clearly has poles at $-i$ but nowhere else, so by Proposition 1, $e_n \in \mathcal{H}^{2+}$. The equivalent of Proposition 1 for the class $\mathcal{H}^{2-}$ (proved in a similar manner) implies that a rational function $\hat{f}$ of moderate decrease is Hardy of class $\mathcal{H}^{2-}$ if and only if all its poles lie in the open upper half plane. Thus we see that for $n < 0$, e_n, which has its only poles at $+i$, belongs to $\mathcal{H}^{2-}$.

Furthermore, the family e_n for all integers n is orthonormal,

$$\int_{-\infty}^{\infty} e_n(x)e_m^*(x)\,dx = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

To show this we use the conformal mapping $z \longmapsto (1+iz)/(1-iz)$ of the upper half plane onto the unit disc. Defining for real x,

$$e^{i\phi(x)} = \frac{1+ix}{1-ix}$$

and then calculating the accompanying Jacobian, $d\phi/dx = 2/(1+x^2)$, we may conclude

$$\begin{aligned} \int_{-\infty}^{\infty} e_n(x)e_m^*(x)\,dx &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+x^2} e^{in\phi(x)}e^{-im\phi(x)}\,dx \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{i(n-m)\phi}\,d\phi \\ &= \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases} \end{aligned}$$

It is also possible to show that the family e_n for $n \geq 0$ spans $\mathcal{H}^{2+}$, i.e. the only function $h \in \mathcal{H}^{2+}$ which is perpendicular to e_n for all $n \geq 0$ is $h = 0$. Similarly the family e_n for $n < 0$ spans $\mathcal{H}^{2-}$. We will use the orthonormal basis e_n for $\mathcal{H}^{2+}$ in our next theorem about Hardy functions. First we prove two short lemmas.

Lemma 1

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log|x-\omega|}{1+x^2}\,dx = \log|i-\omega| \quad \text{for } \omega \in \mathbb{H}^-$$

Proof: Define the complex function

$$f(z) = \frac{\log|z-\omega|}{1+z^2}$$

for ω in the lower half plane. The complex logarithm is defined as $\log(z) = \log(Re^{i\theta}) = \log R + i\theta$, using the principle branch cut. Note that $f(z)$ is well defined in the upper half plane, $z - \omega \neq 0$ since $\omega \in \mathbb{H}^-$. Then integrating over the same semicircular contour γ as in Theorem 4 we have

$$\int_\gamma f(z)\,dz = \int_{-R}^{R} \frac{\log(x-\omega)}{1+x^2}\,dx + \int_0^{\pi} \frac{\log(Re^{i\theta}-\omega)}{1+(Re^{i\theta})^2} iRe^{i\theta}\,d\theta$$

Note that as $R \to \infty$ the second integral goes to zero:

$$\begin{aligned}
\left| \int_0^\pi \frac{\log(Re^{i\theta} - \omega) iRe^{i\theta}}{1 + (Re^{i\theta})^2} \, d\theta \right| &\leq \int_0^\pi \frac{|\log(Re^{i\theta} - \omega)| R}{R^2 - 1} \, d\theta \\
&\leq \int_0^\pi \frac{|\log(R - \mathrm{Re}[\omega])| R}{R^2 - 1} \, d\theta \\
&\quad + \int_0^\pi \frac{|i\theta - \mathrm{Arg}[\omega]| R}{R^2 - 1} \, d\theta \\
&\leq C \frac{\log(R - c) R}{R^2 - 1} + C' \frac{R}{R^2 - 1}
\end{aligned}$$

which vanishes for fixed constants c, C, C' as $R \to \infty$ since the logarithm grows slower than any polynomial.

The poles of $f(z)$ occur at $+i$ and $-i$. Only the pole at $+i$ is included in the contour γ, with residue

$$\lim_{z \to i} (z - i) \frac{\log(z - \omega)}{(z - i)(z + i)} = \frac{\log(i - \omega)}{2i}$$

Then

$$\int_\gamma f(z) \, dz = (2\pi i) \frac{\log(i - \omega)}{2i} = \pi \log(i - \omega)$$

Thus we can conclude that for ω in the lower half plane

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log(x - \omega)}{1 + x^2} \, dx = \log(i - \omega)$$

Taking real parts we have the desired result:

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log |x - \omega|}{1 + x^2} \, dx = \log |i - \omega|$$

Lemma 2

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \pi$$

Proof: Define the complex function $f(z) = 1/(1 + z^2)$. Integrating on the same semicircular contour γ as before we have

$$\int_\gamma f(z) \, dz = \int_{-R}^{R} \frac{dx}{1 + x^2} + \int_0^\pi \frac{iRe^{i\theta}}{1 + (Re^{i\theta})^2} \, d\theta$$

Note that the second integral goes to zero as $R \to \infty$:

$$\left| \int_0^\pi \frac{iRe^{i\theta}}{1 + (Re^{i\theta})^2} \, d\theta \right| \leq c \frac{R}{R^2 - 1}$$

$f(z)$ has poles at $+i$ and $-i$, but the only pole included in γ is at $+i$, with residue $1/2i$. Then by residue calculus

$$\int_\gamma f(z)\,dz = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$$

Now we may prove our second theorem on Hardy functions.

Theorem 5 *For any Hardy function $h \neq 0$,*

$$\int_{-\infty}^{\infty} \frac{\log|h(x)|}{1+x^2}\,dx > -\infty$$

Proof: We first prove the result for h a nonzero rational Hardy function. Given such a rational function h we may shift it so that it has no poles on the real line. Then define $\tilde{h}$ to be the rational function with all the zeroes of h in the upper half plane $\mathbb{H}$ moved to their conjugates in the lower half plane $\mathbb{H}^-$. Then we have

$$\tilde{h}(z) = \frac{c\prod_i (z-\alpha_i)}{\prod_j (z-\beta_j)}$$

where the zeroes $\{\alpha_i\}$ of $\tilde{h}$ are in $\mathbb{H}^-$ by construction. Since $\tilde{h}$ has the same poles as h and h is a Hardy function, all the poles $\{\beta_j\}$ of $\tilde{h}$ are in $\mathbb{H}^-$. Note that for x on the real line, $|h(x)| = |\tilde{h}(x)|$. Furthermore, we also have

$$\log|\tilde{h}(z)| = \sum_i \log|z-\alpha_i| - \sum_j \log|z-\beta_j| + \log|c|$$

By Lemma 1, for any $\omega \in \mathbb{H}^-$,

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log|x-\omega|}{1+x^2}\,dx = \log|i-\omega|$$

Thus we obtain

$$\begin{aligned}
\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log|h(x)|}{1+x^2}\,dx &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log|\tilde{h}(x)|}{1+x^2}\,dx \\
&= \frac{1}{\pi}\sum_i \int_{-\infty}^{\infty} \frac{\log|x-\alpha_i|}{1+x^2}\,dx \\
&\quad -\frac{1}{\pi}\sum_j \int_{-\infty}^{\infty} \frac{\log|x-\beta_j|}{1+x^2}\,dx + \frac{\log(c)}{\pi}\int_{-\infty}^{\infty} \frac{dx}{1+x^2} \\
&= \sum_i \log|i-\alpha_i| - \sum_j \log|i-\beta_j| + log|c| \\
&= \log|\tilde{h}(i)| \\
&> -\infty
\end{aligned}$$

Now we use this result for rational Hardy functions with no roots on the real line to prove the result for any Hardy function h. We may approximate h arbitrarily closely by finite sums h_N of the basis elements e_n for $\mathcal{H}^{2+}$. The basis functions e_n have no roots on the real line so the finite sums h_N are rational Hardy functions with no roots on the real line. Thus we may apply the previous result to h_N for each N. We then find a lower bound for

$$\int_{-\infty}^{\infty} \frac{\log |h(x)|}{1+x^2}\, dx$$

as follows. By Cauchy's formula with C a circular contour around i,

$$\lim_{N\to\infty} h_N(i) = \lim_{N\to\infty} \frac{1}{2\pi i} \int_C \frac{h_N(\omega)}{\omega - i}\, d\omega = \frac{1}{2\pi i} \int_C \frac{h(\omega)}{\omega - i}\, d\omega = h(i)$$

Applying the previous result to h_N,

$$\begin{aligned} \log |h(i)| &= \log \left| \lim_{N\to\infty} h_N(i) \right| \\ &= \lim_{N\to\infty} \log |h_N(i)| \\ &\le \lim_{N\to\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log |h_N(x)|}{1+x^2}\, dx \end{aligned}$$

Define for real x

$$\log^+(x) = \begin{cases} \log x & \text{for } 1 \le x \le \infty \\ 0 & \text{for } 0 \le x < 1 \end{cases}$$

$$\log^-(x) = \begin{cases} -\log x & \text{for } 0 \le x \le 1 \\ 0 & \text{for } x > 1 \end{cases}$$

Then

$$\log |h(i)| \le \lim_{N\to\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log^+ |h_N(x)|}{1+x^2}\, dx - \lim_{N\to\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log^- |h_N(x)|}{1+x^2}\, dx$$

Applying the estimate $|\log^+ x - \log^+ y| \le |x-y|$ to the first integral we have for some constant $c \ge 2$

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log^+ |h_N(x)|}{1+x^2}\, dx &\le \lim_{N\to\infty} \frac{c}{\pi} \int_{-\infty}^{\infty} \frac{\log^+ |h(x)|}{1+x^2}\, dx \\ &\quad + \lim_{N\to\infty} \frac{c}{\pi} \int_{-\infty}^{\infty} \frac{\log^+ |h_N(x) - h(x)|}{1+x^2}\, dx \\ &\le \frac{c}{\pi} \int_{-\infty}^{\infty} \frac{\log^+ |h(x)|}{1+x^2}\, dx \\ &\quad + \limsup_{N\to\infty} \frac{c}{\pi} \int_{-\infty}^{\infty} \frac{|h(x) - h_N(x)|}{1+x^2}\, dx \end{aligned}$$

Thus in the expression for $\log|h(i)|$ we have

$$\begin{aligned}\log|h(i)| \quad &\leq \quad \frac{c}{\pi}\int_{-\infty}^{\infty}\frac{\log^{+}|h(x)|}{1+x^2}\,dx + \limsup_{N\to\infty}\frac{c}{\pi}\int_{-\infty}^{\infty}\frac{|h(x)-h_N(x)|}{1+x^2}\,dx \\ &\quad -\liminf_{N\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\log^{-}|h_N(x)|}{1+x^2}\,dx\end{aligned}$$

The middle integral vanishes as $N\to\infty$, since the h_N converge to h. Applying Fatou's Lemma to the third integral we obtain, up to a constant,

$$\log|h(i)| \leq \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\log^{+}|h(x)|}{1+x^2}\,dx - \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\log^{-}|h(x)|}{1+x^2}\,dx = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\log|h(x)|}{1+x^2}\,dx$$

To complete the proof of the theorem it is sufficient to prove that $\log|h(i)| > -\infty$, i.e. $h(i)\neq 0$. If $h(i)=0$, then since h is not identically zero h can only have a finite order zero at i, say of order n. Then defining

$$k(z) = \left(\frac{z+i}{z-i}\right)^n h(z)$$

we see that k is a Hardy function and $k(i)\neq 0$. Then we may apply the previous procedure to k to conclude

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\log|k(x)|}{1+x^2}\,dx \geq \log|k(i)| > -\infty$$

But $|k(x)| = |h(x)|$ on the real line so we may conclude

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\log|h(x)|}{1+x^2}\,dx > -\infty$$

and the proof is complete.

Although we will not prove it here, for completeness we note that the converse to this theorem is also true: any nonnegative square integrable function f with

$$\int_{-\infty}^{\infty}\frac{\log f(x)}{1+x^2}\,dx > -\infty$$

is the modulus of some Hardy function h, i.e. $f=|h|$ on the real line.

5 Hardy Functions as Filters

Hardy functions are important in signal processing because they may be used as signal filters. It is the vanishing of a Hardy function on the left half real line which makes it ideal for use as a filter. Suppose f is an input signal, K a filter. Applying K to f is equivalent to taking the convolution

$$Kf = \int_{-\infty}^{\infty} k(t-s)f(s)\,ds$$

for some square integrable function k. From this expression we can see that a filter K must have $k(t) = 0$ for $t < 0$, for otherwise the convolution depends on the future of the signal f. For k such a square integrable function with $k(t) = 0$ for all $t < 0$, $\hat{k}$ is in $\mathcal{H}^{2+}$ so we can now study Hardy functions in the context of signal filters. From now on we will refer to the function $\hat{k}$ associated with K interchangeably with the filter K itself. We define the gain of a filter as $G(\xi) = |\hat{k}(\xi)|$ and the phase shift of a filter as $\mathrm{Arg}[\hat{k}(\xi)]$. By Theorem 5 we thus see that for any filter $\hat{k}$ the gain satisfies

$$\int_{-\infty}^{\infty} \frac{\log G(\xi)}{1+\xi^2}\,d\xi = \int_{-\infty}^{\infty} \frac{\log|\hat{k}(\xi)|}{1+\xi^2}\,d\xi > -\infty$$

Also, by the converse to Theorem 5 we can see that any nonnegative square integrable function G with

$$\int_{-\infty}^{\infty} \frac{\log G(\xi)}{1+\xi^2}\,d\xi > -\infty$$

is the modulus of some Hardy function, hence the gain of some Hardy filter $\hat{k}$.

5.1 Circuit representation for a Hardy filter

In order to apply a filter K to a signal $f(t)$ we can construct an electrical circuit with input signal f and output Kf. Suppose $\hat{k}$ is a Hardy filter which can be expressed as a finite sum of the orthonormal basis functions e_n of $\mathcal{H}^{2+}$,

$$\hat{k}(\xi) = \sum_{n=0}^{N} c_n e_n(\xi)$$

where c_n are real constants. Applying this filter to a signal f we take the convolution to obtain the output signal

$$Kf(t) = \int_{-\infty}^{\infty} k(t-s)f(s)\,ds$$

Then taking the Fourier transform and using the identity $(f*g)\hat{} = \hat{f}*\hat{g}$ we obtain an equation which will aid us in designing a circuit for the filter:

$$\begin{aligned}(Kf)\hat{}(\xi) &= \hat{k}(\xi)\hat{f}(\xi) \\ &= \sum_{n=0}^{N} c_n e_n(\xi)\hat{f}(\xi)\end{aligned}$$

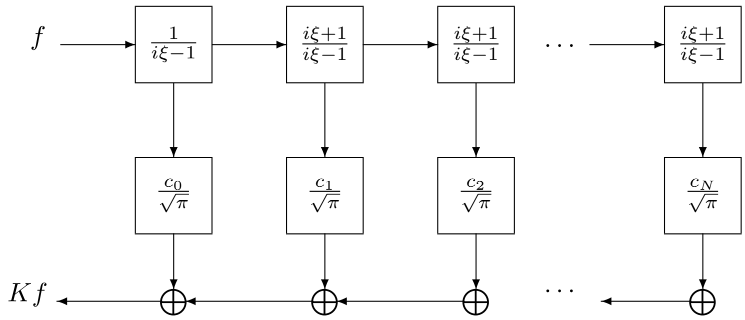
Recalling that

$$e_n(\xi) = \frac{1}{\sqrt{\pi}(i\xi - 1)} \left(\frac{i\xi + 1}{i\xi - 1} \right)^n$$

we see that

$$\hat{k}(\xi) = \frac{c_0}{\sqrt{\pi}(i\xi - 1)} + \frac{c_1}{\sqrt{\pi}(i\xi - 1)} \left(\frac{i\xi + 1}{i\xi - 1} \right) + \cdots + \frac{c_N}{\sqrt{\pi}(i\xi - 1)} \left(\frac{i\xi + 1}{i\xi - 1} \right)^N$$

Thus in order to construct a circuit for $\hat{k}$ it is sufficient to construct a circuit for the component $1/(i\xi - 1)$ and a circuit for the component $(i\xi + 1)/(i\xi - 1)$. Then we may apply the full filter $\hat{k}$ to the input signal f as shown in the diagram below.

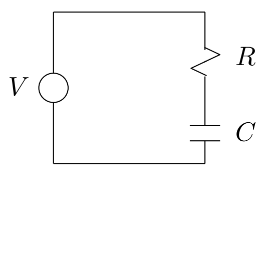


The box $1/(i\xi - 1)$ represents applying the factor $1/(i\xi - 1)$ to the signal, while the box $c_0/\sqrt{\pi}$ represents amplification of the signal by the factor $c_0/\sqrt{\pi}$. The symbol $\oplus$ represents the addition of signals. Thus we see that if the input signal passes through m upper boxes, then traverses downwards, being amplified by $c_m/\sqrt{\pi}$, we have calculated the application to the signal of the mth term

$$\frac{c_m}{\sqrt{\pi}(i\xi - 1)} \left(\frac{i\xi + 1}{i\xi - 1} \right)^m$$

in the expansion of $\hat{k}$. Summing on the bottom row finally yields the filtered signal.

We now examine a circuit representation for the factor $1/(i\xi - 1)$. For the circuit with unit capacitance and resistance shown below let $V(t)$ be the input voltage, $I(t)$ the current, and $Q(t)$ the charge stored on the capacitor.



We then have $V(t) = I(t) + Q(t)$ and hence

$$\frac{dV}{dt} = \frac{dI}{dt} + I$$

Applying the Fourier transform we then have $(-i\xi)\hat{V} = (1 - i\xi)\hat{I}$. Let $V_1(t)$ be the volage across the capacitor C. Then

$$V_1(t) = \int_{-\infty}^{t} I(t)\, dt$$

and hence

$$\frac{dV_1}{dt} = I$$

Again by the Fourier transform we then have $-i\xi\hat{V_1} = \hat{I}$. Thus we may conclude that $(-i\xi)\hat{V} = (1 - i\xi)(-i\xi\hat{V_1})$ and hence

$$-\hat{V_1} = \frac{\hat{V}}{i\xi - 1}$$

Thus if we input the signal $f(t)$ to the circuit and measure the negative of the voltage $V_1(t)$ across the capacitor, we obtain the signal after it has passed through the filter component e_0, except for amplification by the scaling factor $c_0/\sqrt{\pi}$. (In fact, this statement may be somewhat misleading since what we are actually doing is applying e_0 to the Fourier transform $\hat{f}$ of the signal, as indicated in the formula for $(Kf)\hat{}$ found above. Then when we measure $-V_1$ we regain the inverse transform of the filtered $\hat{f}$, i.e. the filtered signal itself. However, it is easier to talk of applying e_0 to the signal, even though we know we are actually transferring the signal to Fourier space first.)

Next we see that we can construct a circuit representation for the component $(i\xi + 1)/(i\xi - 1)$ using the same circuit but measuring the voltage $V_2(t)$ across the resistor. $V(t) = V_1(t) + V_2(t)$ and hence taking the Fourier transform of both sides we obtain

$$\hat{V_2} = \hat{V} - \hat{V_1} = \hat{V} + \frac{\hat{V}}{i\xi - 1} = \frac{i\xi\hat{V}}{i\xi - 1}$$

Thus we see that

$$\hat{V_2} - \hat{V_1} = \left(\frac{i\xi + 1}{i\xi - 1}\right)\hat{V}$$

Measuring the difference of the voltages across the resistor and the capacitor will give the signal after it has been filtered through the $(i\xi + 1)/(i\xi - 1)$ component. Thus we see that by assembling the component circuits as shown in the filter diagram and then taking the appropriate measurements we have a circuit representation for a Hardy filter.

5.2 Outer Filters

We now further investigate the mathematical properties of Hardy functions in their application as filters. As we saw in the proof of Theorem 5, for any rational Hardy function $\hat{k}$ we may define a companion function $\tilde{k}$ which is obtained by moving all the zeroes of $\hat{k}$ in the open upper half plane to their conjugates in the open lower half plane. $\tilde{k}$ is then a Hardy function with the same gain as $\hat{k}$ since $|\hat{k}(\xi)| = |\tilde{k}(\xi)|$ on the real axis. Defining

$$j(\xi) = \frac{\hat{k}(\xi)}{\tilde{k}(\xi)}$$

we have $|j(\xi)| < 1$ on the open upper half plane and the gain $|j(\xi)| = 1$ on the real axis. We then make the definitions that $\tilde{k}$ is the outer part of the filter $\hat{k}$, while j is the inner part of the filter $\hat{k}$. Since the inner part of a filter has no gain it may be regarded as merely a phase shift. The important part of a filter is the outer part. Outer filters, i.e. Hardy filters with no zeroes in the upper open half plane, have many useful properties. For example, only outer filters transmit all nontrivial signals, whereas any other filter will block some nontrivial signal. Also, of all filters with a certain gain, outer filters have the biggest power and the smallest possible delay, where the delay ϕ is defined by $\hat{k}(\xi) = |\hat{k}(\xi)|e^{i\phi(\xi)}$.

We will now prove the useful fact that outer filters are amenable, i.e. inputting unit spikes to an outer filter can create any desired output signal. In fact, we will prove that outer filters are the only type of filter for which this is possible. Suppose we have a filter K with associated function k. Then applying K to a unit spike f at time T gives

$$\begin{aligned} Ff(t) &= \int_{-\infty}^{\infty} k(t-s)f(s)\,ds \\ &= k(t-T) \end{aligned}$$

Thus we can define a family $k_T = k(t-T)$ for all $T \geq 0$, obtained by inputting a unit spike at time T. Note that the Fourier transform of k_T is

$$\begin{aligned} \hat{k}_T(\xi) &= \int_{-\infty}^{\infty} k_T(x)e^{i\xi x}\,dx \\ &= \int_{-\infty}^{\infty} k(x-T)e^{i\xi x}\,dx \\ &= e^{i\xi T}\hat{k}(\xi) \end{aligned}$$

Thus we see that our claim that a filter is amenable if and only if it is an outer filter is equivalent to the following theorem.

Theorem 6 *The family $e^{i\xi T}\hat{k}$ for $T \geq 0$, $\hat{k}$ a rational Hardy function, spans $\mathcal{H}^{2+}$ if and only if $\hat{k}$ is outer.*

Proof: Suppose $h \in \mathcal{H}^{2+}$ is perpendicular to $e^{i\xi T}\hat{k}$ for all $T \geq 0$:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} h(\xi)\hat{k}^*(\xi)e^{-i\xi T}\,d\xi = 0 \quad \text{for } T \geq 0$$

Recall that $\mathcal{H}^{2-} = \{\hat{f} \mid \hat{f} \text{ square integrable, } f(x) = 0 \text{ for } x > 0\}$. Thus the above equation implies $h\hat{k}^* \in \mathcal{H}^{2-}$. Applying the equivalent of Theorem 4 for functions in $\mathcal{H}^{2-}$, we have for $\omega \in \mathbb{H}$,

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{h(\xi)\hat{k}^*(\xi)}{\xi - \omega}\,d\xi = 0$$

For $\hat{k}$ a rational Hardy function, we have up to some constant

$$\hat{k}(\xi) = \frac{\prod_{j=1}^{m}(\xi - \alpha_j)}{\prod_{j=1}^{n}(\xi - \beta_j)}$$

where all the poles $\{\beta_j\}$ are in open lower half plane by Proposition 1. We will first examine the case when $\hat{k}$ has only simple poles $\beta_1, \ldots, \beta_n \in \mathbb{H}^-$. Then partial fraction decomposition yields

$$\hat{k}(\xi) = \sum_{k=1}^{n} \frac{\prod_{j=1}^{m}(\beta_k - \alpha_j)}{\prod_{j\neq k}^{n}(\beta_k - \beta_j)} \frac{1}{(\xi - \beta_k)} = \sum_{k=1}^{n} c_k(\xi - \beta_k)^{-1}$$

for c_k complex constants. Then

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{h(\xi)\hat{k}^*(\xi)}{\xi - \omega}\,d\xi \\ &= \frac{1}{2\pi i}\int \frac{h(\xi)}{\xi - \omega}\left(\sum_{k=1}^{n} c_k(\xi - \beta_k)^{-1}\right)^* d\xi \\ &= \frac{1}{2\pi i}\sum_{k=1}^{n} \bar{c}_k \int \frac{h(\xi)}{(\xi - \omega)(\xi - \bar{\beta}_k)}\,d\xi \\ &= \sum_{k=1}^{n} \frac{\bar{c}_k}{\xi - \bar{\beta}_k}\frac{1}{2\pi i}\int h(\xi)\left[\frac{1}{\xi - \omega} - \frac{1}{\xi - \bar{\beta}_k}\right] d\xi \\ &= \sum_{k=1}^{n} \frac{\bar{c}_k}{\xi - \bar{\beta}_k}\left[h(\omega) - h(\bar{\beta}_k)\right] \end{aligned}$$

Thus

$$h(\omega)\sum_{k=1}^{n}\frac{\bar{c}_k}{\omega - \bar{\beta}_k} = \sum_{k=1}^{n}\frac{\bar{c}_k h(\bar{\beta}_k)}{\omega - \bar{\beta}_k}$$

Define functions f and g on the upper half plane by

$$\begin{aligned} f(\omega) &= \sum_{k=1}^{n}\frac{\bar{c}_k}{\omega - \bar{\beta}_k} \\ g(\omega) &= \sum_{k=1}^{n}\frac{\bar{c}_k h(\bar{\beta}_k)}{\omega - \bar{\beta}_k} \end{aligned}$$

Then for $\omega \in \mathbb{H}$,

$$h(\omega) = \frac{g(\omega)}{f(\omega)}$$

Thus h is a rational Hardy function, hence by Proposition 1 all the poles of h must lie in the open lower half plane. By inspection g has poles only at $\bar{\beta_k} \in \mathbb{H}$. Hence at each pole of h in the lower half plane, f must vanish. But $\hat{k}$ is outer, hence all its zeroes are in the lower half plane. The zeroes of f are the conjugates of the zeroes of $\hat{k}$, hence lie in the upper half plane. Thus h has no poles in the lower half plane, hence no poles in the complex plane. Then h is a polynomial. But since we assumed h is Hardy, thus square integrable on the real line, we must then have $h = 0$. If $\hat{k}$ has multiple poles then the partial fraction decomposition for $\hat{k}$ is more complicated, but a similar procedure works. Thus $h = 0$ is the only Hardy function perpendicular to $e^{i\xi T}\hat{k}$ for all $T \geq 0$. Thus we may conclude $e^{i\xi T}\hat{k}$ spans $\mathcal{H}^{2+}$, for $\hat{k}$ outer.

Conversely, suppose $\hat{k}$ is not outer, i.e. $\hat{k}$ has a root α in the upper half plane. Define the function

$$h(\xi) = \frac{1}{\xi - \bar{\alpha}}$$

Then clearly h is square integrable and has all its poles in the lower half plane, so by Proposition 1, $h \in \mathcal{H}^{2+}$. But also, h is perpendicular to $e^{i\xi T}\hat{k}$ for all $T \geq 0$.

$$\begin{aligned}
\frac{1}{2\pi i}\int_{-\infty}^{\infty} h^*(\xi) e^{i\xi T}\hat{k}(\xi)\, d\xi &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{i\xi T}\hat{k}(\xi)}{(\xi - \bar{\alpha})^*}\, d\xi \\
&= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{i\xi T}\hat{k}(\xi)}{(\xi - \alpha)}\, d\xi \\
&= e^{i\alpha T}\hat{k}(\alpha) \\
&= 0
\end{aligned}$$

since α is a root of $\hat{k}$. Thus we see that if $\hat{k}$ is not outer, $e^{i\xi T}\hat{k}$ does not span $\mathcal{H}^{2+}$ and the proof is complete.

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