

# ON THE NUMBER OF LINEAR FORMS IN LOGARITHMS

YOUNESS LAMZOURI

ABSTRACT. Let  $n$  be a positive integer. In this paper we estimate the size of the set of linear forms  $b_1 \log a_1 + b_2 \log a_2 + \dots + b_n \log a_n$ , where  $|b_i| \leq B_i$  and  $1 \leq a_i \leq A_i$  are integers, as  $A_i, B_i \rightarrow \infty$ .

## 1. INTRODUCTION

The theory of linear forms in logarithms, developed by A. Baker ([1] and [2]) in the 60's, is a powerful method in the transcendental number theory. It consists of finding lower bounds for  $|b_1 \log a_1 + b_2 \log a_2 + \dots + b_n \log a_n|$ , where the  $b_i$  are integers and the  $a_i$  are algebraic numbers for which  $\log a_i$  are linearly independent over  $\mathbb{Q}$ . We consider the simpler case where the  $a_i > 0$  are integers, and we let  $B_j = \max\{|b_j|, 1\}$ , and  $B = \max_{1 \leq j \leq n} B_j$ . Lang and Waldschmidt [4] conjectured the following

**Conjecture.** *Let  $\epsilon > 0$ . There exists  $C(\epsilon) > 0$  depending only on  $\epsilon$ , such that*

$$|b_1 \log a_1 + b_2 \log a_2 + \dots + b_n \log a_n| > \frac{C(\epsilon)^n B}{(B_1 \dots B_n a_1 \dots a_n)^{1+\epsilon}}.$$

One part of the argument they used to motivate the Conjecture, is that the number of distinct linear forms  $b_1 \log a_1 + b_2 \log a_2 + \dots + b_n \log a_n$ , where  $|b_j| \leq B_j$  and  $0 < a_j \leq A_j$ , is  $\asymp B_1 \dots B_n A_1 \dots A_n$ , if the  $A_i$  and the  $B_i$  are sufficiently large.

In this paper we estimate the number of these linear forms as  $A_i, B_i \rightarrow \infty$ .

An equivalent formulation of the problem is to estimate the size of the following set

$$R = R(A_1, \dots, A_n, B_1, \dots, B_n) := \{r \in \mathbb{Q} : r = a_1^{b_1} a_2^{b_2} \dots a_n^{b_n}, 1 \leq a_i \leq A_i, |b_i| \leq B_i\},$$

as  $A_i, B_i \rightarrow \infty$ .

For the easier case  $A_i = A$  and  $B_i = B$  for all  $i$ , a trivial upper bound on  $|R|$  is  $2^n A^n B^n / n! + o(A^n B^n)$ , since permuting the numbers  $a_i^{b_i}$  gives rise to the same number  $r$ . We prove that this bound is attained asymptotically as  $A, B \rightarrow \infty$ . Also we deal with the general case, which is harder since not every permutation is allowed for all the ranges. Indeed the size of  $R$  depends on the ranges of the  $A_i$  and the  $B_i$ , as we shall see in Corollaries 1 and 2.

Let  $E \subset \{(a_1, \dots, a_n, b_1, \dots, b_n), 1 \leq a_i \leq A_i, |b_i| \leq B_i\}$ . We say that  $r \in \mathbb{Q}$  has a representation in  $E$ , if  $r = a_1^{b_1} a_2^{b_2} \dots a_n^{b_n}$ , for some  $(a_1, \dots, a_n, b_1, \dots, b_n) \in E$ .

For  $r \in R$ , if  $\sigma \in S_n$  satisfies  $1 \leq a_{\sigma(i)} \leq A_i$ , and  $|b_{\sigma(i)}| \leq B_i$  for all  $i$ , we say that  $\sigma$  permutes  $r$ , or  $\sigma$  is a possible permutation for the  $a_i^{b_i}$ . Finally we say that a permutation  $\sigma \in S_n$  is permissible if

$$|\{r \in R : \sigma \text{ permutes } r\}| \gg A_1 \dots A_n B_1 \dots B_n.$$

The main result of this paper is the following

**Theorem.** *There exists a set  $E \subset \{(a_1, \dots, a_n, b_1, \dots, b_n), 1 \leq a_i \leq A_i, |b_i| \leq B_i\}$  satisfying*

$$|E| \sim 2^n A_1 A_2 \dots A_n B_1 B_2 \dots B_n,$$

as  $A_i, B_i \rightarrow \infty$ , such that any rational number  $r \in \{a_1^{b_1} a_2^{b_2} \dots a_n^{b_n} : (a_1, \dots, a_n, b_1, \dots, b_n) \in E\}$  has a unique representation in  $E$  up to permissible permutations.

From this result we can deduce that  $|R|$  is asymptotic to the cardinality of the set of  $2n$ -tuples  $\{(a_1, \dots, a_n, b_1, \dots, b_n), 1 \leq a_i \leq A_i, |b_i| \leq B_i\}$  modulo permissible permutations. In the case  $A_i = A, B_i = B$ , every permutation is permissible and we deduce the following Corollary

**Corollary 1.** *As  $A, B \rightarrow \infty$ , we have*

$$|\{r \in \mathbb{Q} : r = a_1^{b_1} a_2^{b_2} \dots a_n^{b_n}, 1 \leq a_i \leq A, |b_i| \leq B\}| = \frac{2^n A^n B^n}{n!} + o(A^n B^n).$$

Now suppose that  $A_i = o(A_{i+1})$  for all  $1 \leq i \leq n-1$ , or  $B_i = o(B_{i+1})$  for all  $1 \leq i \leq n-1$ . For a non-identity permutation  $\sigma \in S_n$ , there exists  $j$  for which  $\sigma(j) \neq j$ . Therefore if  $\sigma$  permutes  $r = a_1^{b_1} a_2^{b_2} \dots a_n^{b_n}$ , we must have  $1 \leq a_j, a_{\sigma(j)} \leq \min(A_j, A_{\sigma(j)})$  and  $-\min(B_j, B_{\sigma(j)}) \leq b_j, b_{\sigma(j)} \leq \min(B_j, B_{\sigma(j)})$ . And so we deduce that

$$\begin{aligned} & |\{r \in \mathbb{Q} : r = a_1^{b_1} a_2^{b_2} \dots a_n^{b_n}, 1 \leq a_i \leq A_i, |b_i| \leq B_i : \sigma \text{ permutes } r\}| \\ & \leq 2^n A_1 \dots A_n B_1 \dots B_n \left( \frac{\min(A_j, A_{\sigma(j)}) \min(B_j, B_{\sigma(j)})}{\max(A_j, A_{\sigma(j)}) \max(B_j, B_{\sigma(j)})} \right) = o(A_1 \dots A_n B_1 \dots B_n), \end{aligned}$$

by our assumption on the  $A_i$  and  $B_i$ . Thus in this case no permutation  $\sigma \neq 1$  is permissible. Therefore we have

**Corollary 2.** *If  $A_i = o(A_{i+1})$  for all  $1 \leq i \leq n-1$ , or  $B_i = o(B_{i+1})$  for all  $1 \leq i \leq n-1$ , then*

$$|\{r \in \mathbb{Q} : r = a_1^{b_1} a_2^{b_2} \dots a_n^{b_n}, 1 \leq a_i \leq A_i, |b_i| \leq B_i\}| \sim 2^n A_1 \dots A_n B_1 \dots B_n,$$

as  $A_i, B_i \rightarrow \infty$ .

We can observe that Corollaries 1 and 2 correspond to extreme cases: in Corollary 1 all permutations are permissible, while none is permissible in Corollary 2. Indeed we can prove

**Corollary 3.** *As  $A_i, B_i \rightarrow \infty$ , we have*

$$\frac{2^n}{n!} A_1 \dots A_n B_1 \dots B_n \lesssim |R| \lesssim 2^n A_1 \dots A_n B_1 \dots B_n.$$

*Moreover the two bounds are optimal.*

*Proof.* From the Theorem we have that

$$|R| \sim \sum_{\substack{1 \leq a_1 \leq A_1 \\ |b_1| \leq B_1}} \dots \sum_{\substack{1 \leq a_n \leq A_n \\ |b_n| \leq B_n}} \frac{1}{|\{\sigma \in S_n : \sigma \text{ is possible for the } a_i^{b_i}\}|}.$$

The result follows from the fact that  $1 \leq |\{\sigma \in S_n : \sigma \text{ is possible for the } a_i^{b_i}\}| \leq n!$ .

For the simple case  $n = 2$ , there is only one non-trivial permutation  $\sigma = (12)$ . This permutation is possible only if  $1 \leq a_1, a_2 \leq \min(A_1, A_2)$  and  $|b_1|, |b_2| \leq \min(B_1, B_2)$ . Then by the Theorem, and after a simple calculation we deduce that

$$\begin{aligned} & |\{r \in \mathbb{Q} : r = a_1^{b_1} a_2^{b_2}, 1 \leq a_1 \leq A_1, 1 \leq a_2 \leq A_2, |b_1| \leq B_1, |b_2| \leq B_2\}| \\ & \sim 4A_1 A_2 B_1 B_2 - 2 \min(A_1, A_2)^2 \min(B_1, B_2)^2. \end{aligned}$$

In general the size of  $|R|$  is asymptotic to an homogeneous polynomial of degree  $2n$  in the variables  $A_1, \dots, A_n, B_1, \dots, B_n$ . Moreover it's also necessary to order the  $A_i$ 's and  $B_i$ 's, so without loss of generality we assume that  $A_1 \leq A_2 \leq \dots \leq A_n$  and  $B_{\pi(1)} \leq B_{\pi(2)} \leq \dots \leq B_{\pi(n)}$ , where  $\pi \in S_n$  is a permutation.

We prove the following

**Proposition.** *Suppose that  $A_1 \leq A_2 \leq \dots \leq A_n$  and  $B_{\pi(1)} \leq B_{\pi(2)} \leq \dots \leq B_{\pi(n)}$ , where  $\pi \in S_n$  is a permutation. Also let  $A_0 = B_{\pi(0)} = 1$ .*

*Then  $|R|$  is asymptotic to*

$$2^n \sum_{\substack{i_1=1 \\ 1 \leq j_1 \leq \pi^{-1}(1)}} \sum_{\substack{1 \leq i_2 \leq 2 \\ 1 \leq j_2 \leq \pi^{-1}(2)}} \dots \sum_{\substack{1 \leq i_n \leq n \\ 1 \leq j_n \leq \pi^{-1}(n)}} \frac{\prod_{k=1}^n (A_{i_k} - A_{i_{k-1}})(B_{\pi(j_k)} - B_{\pi(j_{k-1})})}{|\{\sigma \in S_n : i_{\sigma(l)} \leq l, j_{\sigma(l)} \leq \pi^{-1}(l), \forall 1 \leq l \leq n\}|},$$

*as  $A_i, B_i \rightarrow \infty$ .*

### Acknowledgments.

I sincerely thank my advisor, Professor Andrew Granville, for suggesting the problem, for many valuable discussions, and for his encouragement during the various stages of this work.

## 2. PRELIMINARY LEMMAS

Let  $C$  be a positive real number. We say that the  $n$ -tuple  $(a_1, \dots, a_n)$  satisfies condition  $(1_C)$ , if there exists a prime  $p$ , such that  $p^k | a_1 a_2 \dots a_n$  where  $k \geq 2$ , and  $p^k \geq C$ .

**Lemma 1.** *We have*

$$|\{(a_1, \dots, a_n), 1 \leq a_i \leq A_i : \text{which satisfy } (1_C)\}| \ll_n \frac{A_1 \dots A_n (\log C)^n}{\sqrt{C}}.$$

*Proof.* First we have

$$(1) \quad \begin{aligned} & |\{(a_1, \dots, a_n), 1 \leq a_i \leq A_i : \text{which satisfy } (1_C)\}| \\ & \leq \sum_p |\{(a_1, \dots, a_n), 1 \leq a_i \leq A_i : \exists k \geq 2, p^k \geq C, \text{ and } p^k | a_1 a_2 \dots a_n\}|. \end{aligned}$$

Case 1.  $p \leq \sqrt{C}$

In this case pick  $k$  to be the smallest integer such that  $p^k \geq C$ , ie  $k = \lceil \log C / \log p \rceil + 1$ . Then the number of  $(a_1, \dots, a_n)$  such that  $p^k | a_1 a_2 \dots a_n$  is equal to

$$\sum_{d_1 d_2 \dots d_n = p^k} \prod_{i=1}^n \sum_{\substack{1 \leq a_i \leq A_i \\ d_i | a_i}} 1 \leq d_n(p^k) \frac{A_1 \dots A_n}{p^k} \leq d_n(p^k) \frac{A_1 \dots A_n}{C}.$$

Now  $d_n(p^k) = \binom{n+k-1}{k}$ , and by Stirling's formula, for  $k$  large enough we have

$$\begin{aligned} \log d_n(p^k) &= (n+k-1 + \frac{1}{2}) \log(n+k-1) - (k + \frac{1}{2}) \log k - (n-1 + \frac{1}{2}) \log(n-1) + O(1) \\ &\leq (k + \frac{1}{2}) \log \left(1 + \frac{n-1}{k}\right) + (n-1) \log \left(\frac{n-1+k}{n-1}\right) \\ &\leq n \log k. \end{aligned}$$

Then summing over these primes gives

$$(2) \quad \sum_{p \leq \sqrt{C}} |\{(a_1, \dots, a_n), 1 \leq a_i \leq A_i : p^k | a_1 a_2 \dots a_n\}| = O_n \left( \frac{A_1 \dots A_n (\log C)^n}{\sqrt{C}} \right).$$

Case 2.  $p > \sqrt{C}$

In this case pick  $k = 2$ . Then the number of  $(a_1, \dots, a_n)$  such that  $p^2 | a_1 a_2 \dots a_n$  is  $O(A_1 \dots A_n / p^2)$ , where the constant involved in the  $O$  depends only on  $n$ . Therefore summing over these primes gives

$$(3) \quad \sum_{p > \sqrt{C}} |\{(a_1, \dots, a_n), 1 \leq a_i \leq A_i : p^2 | a_1 a_2 \dots a_n\}| = O_n \left( \frac{A_1 \dots A_n}{\sqrt{C}} \right).$$

Thus combining (1), (2) and (3) gives the result.

We say that  $(a_1, \dots, a_n)$  satisfies condition  $(2_C)$  if at least one of the  $a_i$  is  $C$ -smooth: that is has all its prime factors lying below  $C$ .

**Lemma 2.** *Write  $C^{u_i} = A_i$  for all  $1 \leq i \leq n$ . Then uniformly for  $\min_{1 \leq i \leq n} A_i \geq C \geq 2$ , we have*

$$|\{(a_1, \dots, a_n), 1 \leq a_i \leq A_i : \text{which satisfy } (2_C)\}| \ll_n A_1 A_2 \dots A_n \left( \sum_{i=1}^n e^{-u_i/2} \right).$$

*Proof.* We have that

$$|\{(a_1, \dots, a_n), 1 \leq a_i \leq A_i : \text{which satisfy } (2_C)\}| \ll_n A_1 A_2 \dots A_n \sum_{i=1}^n \frac{\Psi(A_i, C)}{A_i},$$

where  $\Psi(x, y)$  is the number of  $y$ -smooth positive integers below  $x$ . The result follows by the following Theorem of de Bruijn [3]

$$\Psi(A_i, C) \ll A_i e^{-u_i/2},$$

uniformly for  $A_i \geq C \geq 2$ .

We say that  $(b_1, b_2, \dots, b_n)$  satisfy condition  $(3_C)$ , if there exists an  $n$ -tuple of integers  $|c_i| \leq 2 \log C$  not all zero, such that  $c_1 b_1 + c_2 b_2 + \dots + c_n b_n = 0$ .

**Lemma 3.** *We have that*

$$|\{(b_1, \dots, b_n), |b_i| \leq B_i : \text{which satisfy condition } (3_C)\}| \leq B_1 B_2 \dots B_n \sum_{i=1}^n \left( \frac{(9 \log C)^n}{B_i} \right).$$

*Proof.* We note that

$$\begin{aligned} & |\{(b_1, \dots, b_n), |b_i| \leq B_i : \text{which satisfy condition } (3_C)\}| \\ & \leq \sum_{\substack{|c_i| \leq 2 \log C \\ (c_1, \dots, c_n) \neq (0, \dots, 0)}} |\{(b_1, \dots, b_n), |b_i| \leq B_i : c_1 b_1 + c_2 b_2 + \dots + c_n b_n = 0\}| \\ & \leq \sum_{\substack{|c_i| \leq 2 \log C \\ (c_1, \dots, c_n) \neq (0, \dots, 0)}} (2B_1 + 1) \dots (2B_n + 1) \sum_{i=1}^n \left( \frac{1}{2B_i + 1} \right) \leq B_1 B_2 \dots B_n \sum_{i=1}^n \left( \frac{(9 \log C)^n}{B_i} \right). \end{aligned}$$

### 3. PROOF OF THE RESULTS

*Proof of the Theorem.* We begin by choosing  $C := \min(B_1, \dots, B_n, \log A_1, \dots, \log A_n)$ . We consider the following set

$$\begin{aligned} E := & \{(a_1, \dots, a_n, b_1, \dots, b_n), 1 \leq a_i \leq A_i, |b_i| \leq B_i : \\ & (a_i) \text{ don't satisfy any of } (1_C), (2_C), (b_i) \text{ don't satisfy } (3_C)\}. \end{aligned}$$

Then by our choice of  $C$ , if we combine Lemmas 1, 2 and 3, we observe that

$$|E| = 2^n A_1 \dots A_n B_1 \dots B_n (1 + o(1)).$$

Therefore it remains to prove that any representation of a rational number  $r$  as  $a_1^{b_1} a_2^{b_2} \dots a_n^{b_n}$  where  $(a_1, \dots, a_n, b_1, \dots, b_n)$  belongs to  $E$ , is unique up to possible permutations of the  $a_i^{b_i}$ , and finally we can consider only permissible permutations (since the number of  $r \in R$  which can be permuted by a non-permissible permutation is negligible).

We begin by considering the following equation

$$(4) \quad a_1^{b_1} a_2^{b_2} \dots a_n^{b_n} = e_1^{f_1} e_2^{f_2} \dots e_n^{f_n},$$

where  $(a_1, \dots, a_n, b_1, \dots, b_n)$  and  $(e_1, \dots, e_n, f_1, \dots, f_n)$  are in  $E$ . If for some  $i$ ,  $a_i$  contains a prime factor  $p$  such that  $p^2 \nmid a_1 a_2 \dots a_n$  and  $p^2 \nmid e_1 e_2 \dots e_n$ , then  $b_i \in \{f_1, f_2, \dots, f_n\}$ . Now suppose that there exists  $1 \leq j \leq n$  such that  $b_j \notin \{f_1, f_2, \dots, f_n\}$ , then for all the primes  $p$  that divides  $a_j$ , there exists  $k \geq 2$  for which  $p^k \mid a_1 a_2 \dots a_n$  or  $p^k \mid e_1 e_2 \dots e_n$ , but the  $(a_i)$  and the  $(e_i)$  don't satisfy condition  $(1_C)$  and so we must have  $p^k \leq C$ , which implies that  $a_j$  is  $C$ -smooth; however this contradicts the fact that the  $(a_i)$  do not satisfy condition  $(2_C)$ . Therefore we deduce that

$$\{b_1, b_2, \dots, b_n\} = \{f_1, f_2, \dots, f_n\}.$$

Then up to permutations, we have that  $b_i = f_i$ , and so equation (4) become

$$(5) \quad a_1^{b_1} a_2^{b_2} \dots a_n^{b_n} = e_1^{b_1} e_2^{b_2} \dots e_n^{b_n}.$$

Let  $p$  be any prime dividing  $a_1 a_2 \dots a_n$ , and let  $\alpha_i \geq 0$  and  $\beta_i \geq 0$  be the corresponding powers of  $p$  in  $a_i$  and  $e_i$  respectively, and let  $c_i = \alpha_i - \beta_i$ . Then equation (5) implies that

$$c_1 b_1 + c_2 b_2 + \dots + c_n b_n = 0.$$

Now the  $(a_i)$  and the  $(e_i)$  do not satisfy condition  $(1_C)$ , and so  $0 \leq \alpha_i, \beta_i \leq \log C / \log 2 \leq 2 \log C$ , which implies that  $|c_i| \leq 2 \log C$ . And since the  $(b_i)$  do not satisfy condition  $(3_C)$ , we deduce that  $c_i = 0$ , and then  $\alpha_i = \beta_i$  for all  $1 \leq i \leq n$ . Since this is true for every prime factor of  $a_1 a_2 \dots a_n$ , we must have  $a_i = e_i$  for all  $1 \leq i \leq n$ , and our Theorem is proved.

*Proof of the Proposition.* We want to count the number of elements  $r = (r_1, \dots, r_n)$ , where  $r_i = (a_i, b_i) \in [1, A_i] \times [-B_i, B_i] \cap \mathbb{Z} \times \mathbb{Z}$ , modulo possible permutations of the  $r_i$ 's.

Since the number of  $r$  for which some  $b_i$  is 0, is  $o(A_1 \dots A_n B_1 \dots B_n)$ , we can suppose that all the  $b_i$ 's are positive by symmetry.

Moreover let  $R_i := [1, A_i] \times [1, B_i] \cap \mathbb{Z} \times \mathbb{Z}$ , and define the following distinct discrete sets  $R_{ij} := [A_{i-1}, A_i] \times [B_{\pi(j-1)}, B_{\pi(j)}] \cap \mathbb{Z} \times \mathbb{Z}$ , for  $1 \leq i, j \leq n$ .

For every  $1 \leq k \leq n$ , we have

$$(6) \quad R_k = \bigsqcup_{\substack{1 \leq i_k \leq k \\ 1 \leq j_k \leq \pi^{-1}(k)}} R_{i_k j_k}.$$

This implies

$$R_1 \times R_2 \times \dots \times R_n = \bigsqcup_{\substack{i_1=1 \\ 1 \leq j_1 \leq \pi^{-1}(1)}} \bigsqcup_{\substack{1 \leq i_2 \leq 2 \\ 1 \leq j_2 \leq \pi^{-1}(2)}} \dots \bigsqcup_{\substack{1 \leq i_n \leq n \\ 1 \leq j_n \leq \pi^{-1}(n)}} R_{i_1 j_1} \times R_{i_2 j_2} \times \dots \times R_{i_n j_n}.$$

Now consider the elements  $r \in R_{i_1 j_1} \times R_{i_2 j_2} \dots \times R_{i_n j_n}$ , with  $1 \leq i_k \leq k$  and  $1 \leq j_k \leq \pi^{-1}(k)$  being fixed. If  $\sigma \in S_n$  permutes  $r$ , then  $r_{\sigma(k)} \in R_k$  for all  $1 \leq k \leq n$ , but  $r_{\sigma(k)} \in R_{i_{\sigma(k)} j_{\sigma(k)}}$  also, which implies that  $R_{i_{\sigma(k)} j_{\sigma(k)}} \cap R_k \neq \emptyset$ . From (6) this is equivalent to  $R_{i_{\sigma(k)} j_{\sigma(k)}} \subseteq R_k$ , and thus to the fact that  $i_{\sigma(k)} \leq k$  and  $j_{\sigma(k)} \leq \pi^{-1}(k)$  for all  $1 \leq k \leq n$ . Therefore for any  $r \in R_{i_1 j_1} \times R_{i_2 j_2} \dots \times R_{i_n j_n}$ , the number of  $\sigma \in S_n$  which permutes  $r$  is constant and equal to

$$|\{\sigma \in S_n : i_{\sigma(l)} \leq l, j_{\sigma(l)} \leq \pi^{-1}(l), \forall 1 \leq l \leq n\}|.$$

Thus the number of elements in  $R_1 \times R_2 \dots \times R_n$ , modulo possible permutations is

$$\sum_{\substack{i_1=1 \\ 1 \leq j_1 \leq \pi^{-1}(1)}} \sum_{\substack{1 \leq i_2 \leq 2 \\ 1 \leq j_2 \leq \pi^{-1}(2)}} \dots \sum_{\substack{1 \leq i_n \leq n \\ 1 \leq j_n \leq \pi^{-1}(n)}} \frac{\prod_{k=1}^n (A_{i_k} - A_{i_k-1})(B_{\pi(j_k)} - B_{\pi(j_k-1)})}{|\{\sigma \in S_n : i_{\sigma(l)} \leq l, j_{\sigma(l)} \leq \pi^{-1}(l), \forall 1 \leq l \leq n\}|},$$

which implies the result.

## REFERENCES

- [1] A. Baker, *Linear forms in the logarithms of algebraic numbers I*, *Mathematika* **13**, 1966, 204-216.
- [2] ———, *Linear forms in the logarithms of algebraic numbers II, III*, *Mathematika* **14**, 1967, 102-107, 220-228.
- [3] N.G. de Bruijn, *On the number of positive integers  $\leq x$  and free of prime factors  $> y$ . II*, *Nederl. Akad. Wetensch. Proc. Ser. A* **69**, 239-247= *Indag. Math* **28**, 1966, 239-247.
- [4] S. Lang, *Elliptic curves: Diophantine analysis*, *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, **231**. Springer-Verlag, Berlin-New York, 1978, xi+261 pp.
- [5] G. Tenenbaum, *Introduction to analytic and probabilistic number theory*, *Cambridge Studies in Advanced Mathematics*, **46**. Cambridge University Press, Cambridge, 1995, xvi+448 pp.

DÉPARTEMENT DE MATHÉMATIQUES ET STATISTIQUE, UNIVERSITÉ DE MONTRÉAL, CP 6128 SUCC  
CENTRE-VILLE, MONTRÉAL, QC H3C 3J7, CANADA

*E-mail address:* Lamzouri@dms.umontreal.ca