Subrank, Partition Rank and Slice Rank

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$T \in \mathbb{F}^{n \times n \times n}$

<u>Definition</u> Tensor rank $R(T)$

minimize $\rightarrow$

$$T = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i$$

equiv.:

$$T = U \otimes V \otimes W \cdot \sum_{i=1}^{r} e_i \otimes e_i \otimes e_i$$

<u>Applications</u>

- Matrix multiplication
- Arithmetic complexity [Raz]

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Definition Subrank $Q(T)$

$\leftarrow$ maximize

$$\sum_{i=1}^{s} e_i \otimes e_i \otimes e_i = U \otimes V \otimes W \cdot T$$

Applications

- Matrix multiplication
- Additive combinatorics

For <u>generic</u> $T \in \mathbb{F}^{n\times n\times n}$, $R(T) \approx n^2$ (maximal)

Easy: $Q(T) \leq n$

$$\boxed{\text{Recall: } \sum_{i=1}^{s} e_i \otimes e_i \otimes e_i = U \otimes V \otimes W \cdot T}$$

maximize

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① Generic subrank

$$\boxed{\begin{array}{l}\underline{\text{Theorem}}\ [\text{Kopparty, Z}] \\ \text{For generic } T \in \mathbb{F}^{n\times n\times n},\ Q(T) \leq 3n^{2/3}\end{array}}$$

<u>Note</u> Surprisingly small, in particular given that generic rank is maximal.

Definition Slice rank $SR(T)$ $u_i \otimes v_i \otimes w_i$

$$T = \sum_{i=1}^{a} \sum_{j} u_i \otimes v_{ij} \otimes w_{ij} + \sum_{i=1}^{b} \sum_{j} u'_{ij} \otimes v'_i \otimes w'_{ij}$$

$$+ \sum_{i=1}^{c} \sum_{j} u''_{ij} \otimes v''_{ij} \otimes w''_i$$

minimize $a+b+c$

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Remark

- $Q(T) \leq SR(T)$
- for generic T, $Q(T) \leq 3n^{2/3}$ while $SR(T) = n$
- Subrank and slice rank very different generically!

$\boxed{\text{Theorem. For generic } T \in \mathbb{F}^{n\times n\times n}: \quad Q(T) \le 3n^{2/3}}$

$S \subseteq [n] \times [n] \times [n]$

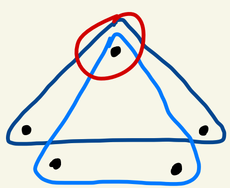
Maximal points: $\mathrm{Max}(S) = \{$ coordinate-wise maximal points in $S \}$

Example:

$S = \{(2,1,1), (1,2,1), (1,2,2)\}, \quad \mathrm{Max}(S) = \{(2,1,1), (1,2,2)\}$

Cover number: $\mathrm{cov}(S) =$ vertex cover number of S as 3-partite hypergraph.

Example: $S = \{(1,1,1), (1,2,2)\}$ $\qquad$ $S = \{(1,1,2), (2,2,1)\}$



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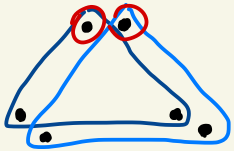

2

$T \in \mathbb{F}^{n \times n \times n}$

Support: $\mathrm{Supp}(T) \subseteq [n] \times [n] \times [n]$

Action: $g \in GL_n^{\times 3}$, $g \cdot T \in \mathbb{F}^{n \times n \times n}$

Cover number: $\mathrm{cov}(T) := \max_{g \in GL_n^{\times 3}} \mathrm{cov}(\mathrm{Max}(\mathrm{Supp}(g \cdot T)))$

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Example: $T = \sum_{i=1}^{s} e_i \otimes e_i \otimes e_i \in \mathbb{F}^{n \times n \times n} \rightsquigarrow g \cdot T = \sum_{i=1}^{s} e_i \otimes e_i \otimes e_{s-i}$

$$\mathrm{Supp}(g \cdot T) = \{(i, i, s-i) : i \in [s]\} \overset{!}{=} \mathrm{Max}(\mathrm{Supp}(T))$$

$$s \leq \mathrm{cov}(T)$$

$\underline{\text{Lemma 1}}$ $\quad Q(T) \leq \mathrm{cov}(T)$ $\qquad$ Follows from

$\boxed{\text{Lemma 2} \quad \mathrm{cov}(T) \leq 3n^{2/3} \text{ for generic } T \in \mathbb{F}^{n\times n\times n}.}$

Proof sketch

① There is a nonempty open $\mathcal{U} \subseteq \mathbb{F}^{n\times n\times n}$ such that

$$\forall T \in \mathcal{U} \;\; \forall g \in GL_n^{\times 3} \quad |\mathrm{Supp}(g \cdot T)| \geq n^3 - 3n^2 \qquad \text{[Bürgisser]}$$

Lemma 2 $\mathrm{cov}(T) \leq 3n^{2/3}$ for generic $T \in \mathbb{F}^{n\times n\times n}$.

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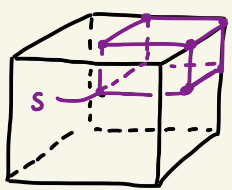
② For $S \subseteq [n]\times[n]\times[n]$, if $|S| \geq n^3 - 3n^2$, then

$$\mathrm{Max}(S) \subseteq \left\{ s \in [n]^3 : \prod_i (n - s_i + 1) \leq 3n^2 + 1 \right\}$$



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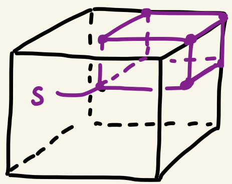

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③ $\mathrm{cov}(\quad) \leq 3n^{2/3}$ since $\forall s \ \exists i \ \ s_i \leq n^{2/3}$ $\approx \square$