

LORENTZIAN POLYNOMIALS

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ABSTRACT. We study the class of Lorentzian polynomials. The class contains homogeneous stable polynomials as well as volume polynomials of convex bodies and projective varieties. We prove that the Hessian of a nonzero Lorentzian polynomial has exactly one positive eigenvalue at any point on the positive orthant. This property can be seen as an analog of the Hodge-Riemann relations for Lorentzian polynomials.

Lorentzian polynomials are intimately connected to matroid theory and negative dependence properties. We show that matroids, and more generally M -convex sets, are characterized by the Lorentzian property, and develop a theory around Lorentzian polynomials. In particular, we provide a large class of linear operators that preserve the Lorentzian property and prove that Lorentzian measures enjoy several negative dependence properties. We also prove that the class of tropicalized Lorentzian polynomials coincides with the class of M -convex functions in the sense of discrete convex analysis. The tropical connection is used to produce Lorentzian polynomials from M -convex functions.

We give two applications of the general theory. First, we prove that the homogenized multivariate Tutte polynomial of a matroid is Lorentzian whenever the parameter q satisfies $0 < q \leq 1$. Consequences are proofs of the strongest Mason's conjecture from 1972 and negative dependence properties of the random cluster model in statistical physics. Second, we prove that the multivariate characteristic polynomial of an M -matrix is Lorentzian. This refines a result of Holtz who proved that the coefficients of the characteristic polynomial of an M -matrix form an ultra log-concave sequence.

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1. INTRODUCTION

Let H_n^d be the space of degree d homogeneous polynomials in n variables with real coefficients. Inspired by Hodge's index theorem for projective varieties, we introduce a class of polynomials with remarkable properties. Let $\mathring{L}_n^2 \subseteq H_n^2$ be the open subset of quadratic forms with positive coefficients that have the *Lorentz signature* $(+, -, \dots, -)$. For d larger than 2, we define $\mathring{L}_n^d \subseteq H_n^d$ by setting

$$\mathring{L}_n^d = \left\{ f \in H_n^d \mid \partial_i f \in \mathring{L}_n^{d-1} \text{ for all } i \right\},$$

where ∂_i is the partial derivative with respect to the i -th variable. Thus f belongs to \mathring{L}_n^d if and only if all polynomials of the form $\partial_{i_1} \partial_{i_2} \cdots \partial_{i_{d-2}} f$ belongs to \mathring{L}_n^2 . The polynomials in \mathring{L}_n^d are called *strictly Lorentzian*, and the limits of strictly Lorentzian polynomials are called *Lorentzian*. The class of Lorentzian polynomials contains the class of homogeneous stable polynomials (Section 2) as well as volume polynomials of convex bodies and projective varieties (Sections 9 and 10).

Lorentzian polynomials link discrete and continuous notions of convexity. Let $L_n^2 \subseteq H_n^2$ be the closed subset of quadratic forms with nonnegative coefficients that have at most one positive eigenvalue. We write $\text{supp}(f) \subseteq \mathbb{N}^n$ for the support of $f \in H_n^d$, and write e_i for the i -th standard unit vector in \mathbb{N}^n . For d larger than 2, we define $L_n^d \subseteq H_n^d$ by setting

$$L_n^d = \left\{ f \in M_n^d \mid \partial_i f \in L_n^{d-1} \text{ for all } i \right\},$$

where $M_n^d \subseteq H_n^d$ is the set of polynomials with nonnegative coefficients whose supports are *M-convex* in the sense of discrete convex analysis [Mur03]: For any index i and any $\alpha, \beta \in \text{supp}(f)$ whose i -th coordinates satisfy $\alpha_i > \beta_i$, there is an index j satisfying

$$\alpha_j < \beta_j \text{ and } \alpha - e_i + e_j \in \text{supp}(f) \text{ and } \beta - e_j + e_i \in \text{supp}(f).$$

Since $f \in M_n^d$ implies $\partial_i f \in M_n^{d-1}$, we have

$$L_n^d = \left\{ f \in M_n^d \mid \partial_{i_1} \partial_{i_2} \cdots \partial_{i_{d-2}} f \in L_n^2 \text{ for all } i_1, i_2, \dots, i_{d-2} \right\}.$$

In Section 5, we show that L_n^d is the closure of \mathring{L}_n^d in H_n^d . In other words, L_n^d is the set of Lorentzian polynomials in H_n^d . This generalizes a result of Choe *et al.* that the support of any homogenous multi-affine stable polynomial is the set of bases of a matroid [COSW04].

Lorentzian polynomials are intimately connected to matroid theory and discrete convex analysis. We show that matroids, and more generally M-convex sets, are characterized by the Lorentzian property. Let $\mathbb{P}H_n^d$ be the projectivization of the vector space H_n^d , and let L_J be the set of polynomials in L_n^d with support J . We denote the images of $\mathring{L}_n^d \setminus 0$, $L_n^d \setminus 0$, and $L_J \setminus 0$ in $\mathbb{P}H_n^d$ by $\mathbb{P}\mathring{L}_n^d$, $\mathbb{P}L_n^d$, and $\mathbb{P}L_J$ respectively, and write

$$\mathbb{P}L_n^d = \coprod_J \mathbb{P}L_J,$$

where the union is over all M-convex subsets of the d -th discrete simplex in \mathbb{N}^n . We prove that $\mathbb{P}L_n^d$ is a compact contractible subset of $\mathbb{P}H_n^d$ with contractible interior $\mathbb{P}\mathring{L}_n^d$ (Theorems 2.10 and 5.1). In addition, we show that $\mathbb{P}L_J$ is nonempty and contractible for every nonempty M-convex set J (Theorem 7.1 and Proposition 8.12). Similarly, writing \underline{H}_n^d for the space of multi-affine degree d homogeneous polynomials in n variables and \underline{L}_n^d for the corresponding set of multi-affine Lorentzian polynomials, we have

$$\mathbb{P}\underline{L}_n^d = \coprod_B \mathbb{P}\underline{L}_B,$$

where the union is over all rank d matroids on the n -element set $[n]$. The space $\mathbb{P}\underline{L}_n^d$ is compact and contractible (Theorem 2.10 and Corollary 6.5), and $\mathbb{P}\underline{L}_B$ is nonempty and contractible for every matroid B (Theorem 7.1 and Proposition 8.12).

In Section 4, we prove that Lorentzian polynomials satisfy a formal version of the Hodge-Riemann relations: The Hessian of any nonzero Lorentzian polynomial has exactly one positive eigenvalue at any point on the positive orthant. In Section 5, we use this result to show that the classes of strongly log-concave [Gur09], completely log-concave [AOV], and Lorentzian polynomials are identical for homogeneous polynomials (Theorem 5.3). This enables us to affirmatively answer two questions of Gurvits on strongly log-concave polynomials (Corollaries 5.4 and 5.5).

In Section 6, we describe a large class of linear operators preserving the class of Lorentzian polynomials, thus providing a toolbox for working with Lorentzian polynomials. We give a Lorentzian analog of a theorem of Borcea and Brändén for stable polynomials [BB09], who characterized linear operators preserving stable polynomials (Theorem 6.2). It follows from our result that any homogeneous linear operator that preserves stable polynomials and polynomials with nonnegative coefficients also preserves Lorentzian polynomials (Theorem 6.4).

In Section 8, we strengthen the connection between Lorentzian polynomials and discrete convex analysis. The effective domain of a function $\nu : \mathbb{N}^n \rightarrow \mathbb{R} \cup \{\infty\}$, denoted $\text{dom}(\nu)$, is the subset of \mathbb{N}^n where ν is finite. For a positive real parameter q , we consider the polynomial

$$f_q^\nu(w) = \sum_{\alpha \in \text{dom}(\nu)} \frac{q^{\nu(\alpha)}}{\alpha!} w^\alpha, \quad w = (w_1, \dots, w_n).$$

In Theorem 8.2, we show that f_q^ν is Lorentzian for all $0 < q \leq 1$ if and only if the function ν is M-convex in the sense of discrete convex analysis [Mur03]: For any index i and any $\alpha, \beta \in \text{dom}(\nu)$

whose i -th coordinates satisfy $\alpha_i > \beta_i$, there is an index j satisfying

$$\alpha_j < \beta_j \text{ and } \nu(\alpha) + \nu(\beta) \geq \nu(\alpha - e_i + e_j) + \nu(\beta - e_j + e_i).$$

In particular, $J \subseteq \mathbb{N}^n$ is M-convex if and only if its generating function $\sum_{\alpha \in J} \frac{1}{\alpha!} w^\alpha$ is a Lorentzian polynomial (Theorem 7.1). Another special case of Theorem 8.2 is the statement that a homogeneous polynomial with nonnegative coefficients is Lorentzian if the natural logarithms of its normalized coefficients form an M-concave function (Corollary 8.3).

Working over the field of formal Puiseux series \mathbb{K} , we show that tropicalizations of Lorentzian polynomials over \mathbb{K} are M-convex, and that *all* M-convex functions are limits of tropicalizations of Lorentzian polynomials over \mathbb{K} (Corollary 8.15). This generalizes a result of Brändén [Brä10], who showed that tropicalizations of homogeneous stable polynomials over \mathbb{K} are M-convex.¹ In particular, for any matroid M with the set of bases B , the *Dressian* of all valuated matroids on M can be identified with the tropicalization of the space of Lorentzian polynomials over \mathbb{K} with support B . This contrasts the case of stable polynomials. For example, when M is the Fano plane, there is no stable polynomial whose support is B [Brä07].

In Sections 9 and 10, we show that the volume polynomials of convex bodies and projective varieties are Lorentzian. It follows that, for any d -dimensional projective variety Y , the set of all $\alpha \in \mathbb{N}^n$ satisfying the conditions

$$\alpha_1 + \cdots + \alpha_n = d \text{ and } \underbrace{(H_1 \cdots H_1)}_{\alpha_1} \cdots \underbrace{(H_n \cdots H_n)}_{\alpha_n} \neq 0$$

is M-convex for any nef divisors H_1, \dots, H_n on Y (Corollary 10.2).

In Section 11, we show that the homogenized multivariate Tutte polynomial of any matroid is Lorentzian. A consequence is proof of a conjecture of Mason from 1972 on the enumeration of independent sets [Mas72]: For any matroid M on $[n]$ and any positive integer k ,

$$\frac{I_k(M)^2}{\binom{n}{k}^2} \geq \frac{I_{k+1}(M)}{\binom{n}{k+1}} \frac{I_{k-1}(M)}{\binom{n}{k-1}},$$

where $I_k(M)$ is the number of k -element independent sets of M . The Lorentzian property reveals several other inequalities satisfied by the coefficients of the classical *Tutte polynomial*

$$T_M(x, y) = \sum_{A \subseteq [n]} (x-1)^{\text{rk}_M([n]) - \text{rk}_M(A)} (y-1)^{|A| - \text{rk}_M(A)},$$

where $\text{rk}_M : \{0, 1\}^n \rightarrow \mathbb{N}$ is the rank function of M . For example, if we write

$$w^{\text{rk}_M([n])} T_M\left(1 + \frac{q}{w}, 1 + w\right) = \sum_{k=0}^n c_q^k(M) w^k,$$

then the sequence $c_q^k(M)$ is ultra log-concave for every $0 \leq q \leq 1$.

¹In [Brä10], the field of formal Puiseux series with real exponents was used. The tropicalization used in [Brä10] differs from ours by a sign.

An $n \times n$ matrix is an *M-matrix* if all the off-diagonal entries are nonpositive and all the principal minors are positive. The class of M-matrices shares many properties of hermitian positive definite matrices and appears naturally in mathematical economics and computational biology [BP94]. In Section 12, we show that the multivariate characteristic polynomial of any M-matrix is Lorentzian. This strengthens a theorem of Holtz [Hol05], who proved that the coefficients of the characteristic polynomial of any M-matrix form an ultra log-concave sequence.

In Section 13, we define a class of discrete probability measures called *Lorentzian measures*. This class properly contains the class of strongly Rayleigh measures studied in [BBL09]. We show that the class enjoys several negative dependence properties and prove that the class is closed under the symmetric exclusion process. As an example, we show that the uniform measure μ_M on $\{0, 1\}^n$ concentrated on the independent sets of a matroid M is Lorentzian (Proposition 13.6). A conjecture of Kahn [Kah00] and Grimmett-Winkler [GW04] states that, for any graphic matroid M and distinct elements i and j ,

$$\Pr(F \text{ contains } i \text{ and } j) \leq \Pr(F \text{ contains } i) \Pr(F \text{ contains } j),$$

where F is an independent set of M chosen uniformly at random. The Lorentzian property of the measure μ_M shows that, for any matroid M and distinct elements i and j ,

$$\Pr(F \text{ contains } i \text{ and } j) \leq 2\Pr(F \text{ contains } i) \Pr(F \text{ contains } j),$$

where F is an independent set of M chosen uniformly at random.²

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2. THE SPACE OF LORENTZIAN POLYNOMIALS

2.1. Let n and d be nonnegative integers, and set $[n] = \{1, \dots, n\}$. We write H_n^d for the set of degree d homogeneous polynomials in $\mathbb{R}[w_1, \dots, w_n]$, and write $P_n^d \subseteq H_n^d$ for the subset of polynomials all of whose coefficients are positive. The *Hessian* of $f \in \mathbb{R}[w_1, \dots, w_n]$ is the symmetric matrix

$$\mathcal{H}_f(w) = \left(\partial_i \partial_j f \right)_{i,j=1}^n,$$

where ∂_i stands for the partial derivative $\frac{\partial}{\partial w_i}$. For $\alpha \in \mathbb{N}^n$, we write

$$\alpha = \sum_{i=1}^n \alpha_i e_i \quad \text{and} \quad |\alpha|_1 = \sum_{i=1}^n \alpha_i,$$

where α_i is a nonnegative integer and e_i is the standard unit vector in \mathbb{N}^n . We set

$$w^\alpha = w_1^{\alpha_1} \cdots w_n^{\alpha_n} \quad \text{and} \quad \partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}.$$

²In their upcoming work, Nima Anari, Kuikui Liu, Shayan Oveis Gharan and Cynthia Vinzant have independently developed methods that partially overlap with our work.

We define the d -th discrete simplex $\Delta_n^d \subseteq \mathbb{N}^n$ by

$$\Delta_n^d = \left\{ \alpha \in \mathbb{N}^n \mid |\alpha|_1 = d \right\},$$

and define the boolean cube $\{0, 1\}^n \subseteq \mathbb{N}^n$ by

$$\{0, 1\}^n = \left\{ \sum_{i \in S} e_i \in \mathbb{N}^n \mid S \subseteq [n] \right\}.$$

The intersection of the d -th discrete simplex and the boolean cube will be denoted

$$\begin{bmatrix} n \\ d \end{bmatrix} = \{0, 1\}^n \cap \Delta_n^d.$$

The cardinality of $\begin{bmatrix} n \\ d \end{bmatrix}$ is the binomial coefficient $\binom{n}{d}$, and the sets $\begin{bmatrix} n \\ 1 \end{bmatrix}$ and $[n]$ can be naturally identified with each other. We often identify a subset S of $[n]$ with the zero-one vector $\sum_{i \in S} e_i$ in \mathbb{N}^n . For example, we write w^S for the square-free monomial $\prod_{i \in S} w_i$.

Definition 2.1 (Lorentzian polynomials). We set $\mathring{L}_n^0 = P_n^0$, $\mathring{L}_n^1 = P_n^1$, and

$$\mathring{L}_n^2 = \left\{ f \in P_n^2 \mid \mathcal{H}_f \text{ is nonsingular and has exactly one positive eigenvalue} \right\}.$$

For d larger than 2, we define \mathring{L}_n^d recursively by setting

$$\mathring{L}_n^d = \left\{ f \in P_n^d \mid \partial_i f \in \mathring{L}_n^{d-1} \text{ for all } i \in [n] \right\}.$$

The polynomials in \mathring{L}_n^d are called *strictly Lorentzian*, and the limits of strictly Lorentzian polynomials are called *Lorentzian*.

We define a topology on the space of homogeneous polynomials H_n^d using the Euclidean norm for the coefficients. Clearly, \mathring{L}_n^d is an open subset of H_n^d , and the space \mathring{L}_n^2 may be identified with the set of $n \times n$ symmetric matrices with positive entries that have the *Lorentz signature* $(+, -, \dots, -)$. Unwinding the definition, we have

$$\mathring{L}_n^d = \left\{ f \in P_n^d \mid \partial^\alpha f \in \mathring{L}_n^2 \text{ for every } \alpha \in \Delta_n^{d-2} \right\}.$$

Recall that a polynomial f in $\mathbb{R}[w_1, \dots, w_n]$ is *stable* if f is non-vanishing on \mathcal{H}^n or identically zero, where \mathcal{H} is the open upper half plane in \mathbb{C} . Let S_n^d be the set of degree d homogeneous stable polynomials in n variables with nonnegative coefficients. When f is homogeneous and has nonnegative coefficients, the stability of f is equivalent to any one of the following statements on univariate polynomials in the variable x [BBL09, Theorem 4.5]:

- For any $u \in \mathbb{R}_{>0}^n$, $f(xu - v)$ has only real zeros for all $v \in \mathbb{R}^n$.
- For some $u \in \mathbb{R}_{>0}^n$, $f(xu - v)$ has only real zeros for all $v \in \mathbb{R}^n$.
- For any $u \in \mathbb{R}_{\geq 0}^n$ with $f(u) > 0$, $f(xu - v)$ has only real zeros for all $v \in \mathbb{R}^n$.
- For some $u \in \mathbb{R}_{\geq 0}^n$ with $f(u) > 0$, $f(xu - v)$ has only real zeros for all $v \in \mathbb{R}^n$.

We refer to [Wag11] and [Pem12] for background on stable polynomials. We will use the fact that any polynomial $f \in S_n^d$ is the limit of polynomials in the interior of S_n^d , that is, of *strictly stable polynomials* [Nui68].

Proposition 2.2. Any polynomial in S_n^d is Lorentzian.

Proof. Since ∂_i is an open map sending S_n^d to S_n^{d-1} , homogeneous strictly stable polynomials are strictly Lorentzian, and homogeneous stable polynomials are Lorentzian. \square

All the nonzero coefficients of a homogeneous stable polynomial have the same sign [COSW04, Theorem 6.1]. Thus, any homogeneous stable polynomial is a constant multiple of a Lorentzian polynomial. For example, determinantal polynomials of the form

$$f(w_1, \dots, w_n) = \det(w_1 A_1 + \dots + w_n A_n),$$

where A_1, \dots, A_n are positive semidefinite matrices, are Lorentzian.

Example 2.3. Consider the homogeneous bivariate polynomial with positive coefficients

$$f = \sum_{k=0}^d a_k w_1^k w_2^{d-k}.$$

Computing partial derivatives of f reveals that f is strictly Lorentzian if and only if

$$\frac{a_k^2}{\binom{d}{k}^2} > \frac{a_{k-1}}{\binom{d}{k-1}} \frac{a_{k+1}}{\binom{d}{k+1}} \text{ for all } 0 < k < d.$$

On the other hand, f is stable if and only if the univariate polynomial $f|_{w_2=1}$ has only real zeros. Thus a Lorentzian polynomial need not be stable. For example, consider the cubic form

$$f = 2w_1^3 + 12w_1^2 w_2 + 18w_1 w_2^2 + \theta w_2^3,$$

where θ is a real parameter. A straightforward computation shows that

$$f \text{ is Lorentzian if and only if } 0 \leq \theta \leq 9, \text{ and } f \text{ is stable if and only if } 0 \leq \theta \leq 8.$$

Example 2.4. Clearly, if f is in the closure of \mathring{L}_n^d in H_n^d , then f has nonnegative coefficients and

$$\partial^\alpha f \text{ has at most one positive eigenvalue for every } \alpha \in \Delta_n^{d-2}.$$

The bivariate cubic $f = w_1^3 + w_2^3$ shows that the converse fails. In this case, $\partial_1 f$ and $\partial_2 f$ are Lorentzian, but f is not Lorentzian.

The following alternative characterization of \mathring{L}_n^2 is well-known. See, for example, [Gre81].

Lemma 2.5. The following conditions are equivalent for any $f \in P_n^2$.

- (1) The Hessian of f has the Lorentzian signature $(+, -, \dots, -)$, that is, $f \in \mathring{L}_n^2$.
- (2) For any nonzero $u \in \mathbb{R}_{\geq 0}^n$, $(u^T \mathcal{H}_f v)^2 > (u^T \mathcal{H}_f u)(v^T \mathcal{H}_f v)$ for any $v \in \mathbb{R}^n$ not parallel to u .
- (3) For some $u \in \mathbb{R}_{\geq 0}^n$, $(u^T \mathcal{H}_f v)^2 > (u^T \mathcal{H}_f u)(v^T \mathcal{H}_f v)$ for any $v \in \mathbb{R}^n$ not parallel to u .

- (4) For any nonzero $u \in \mathbb{R}_{\geq 0}^n$, the univariate polynomial $f(xu - v)$ in x has two distinct real zeros for any $v \in \mathbb{R}^n$ not parallel to u .
- (5) For some $u \in \mathbb{R}_{\geq 0}^n$, the univariate polynomial $f(xu - v)$ in x has two distinct real zeros for any $v \in \mathbb{R}^n$ not parallel to u .

Therefore, a homogeneous quadratic polynomial with positive coefficients is strictly Lorentzian if and only if it is strictly stable. It follows that a homogeneous quadratic polynomial with non-negative coefficients is Lorentzian if and only if it is stable.

Proof. We prove (1) \Rightarrow (2). Since all the entries of \mathcal{H}_f are positive, $u^T \mathcal{H}_f u > 0$ for any nonzero $u \in \mathbb{R}_{\geq 0}^n$. By Cauchy's interlacing theorem, for any $v \in \mathbb{R}^n$ not parallel to u , the restriction of \mathcal{H}_f to the plane spanned by u, v has signature $(+, -)$. It follows that

$$\det \begin{pmatrix} u^T \mathcal{H}_f u & u^T \mathcal{H}_f v \\ u^T \mathcal{H}_f v & v^T \mathcal{H}_f v \end{pmatrix} = (u^T \mathcal{H}_f u)(v^T \mathcal{H}_f v) - (u^T \mathcal{H}_f v)^2 < 0.$$

We prove (3) \Rightarrow (1). Let u be the nonnegative vector in the statement (3). Then \mathcal{H}_f is negative definite on the hyperplane $\{v \in \mathbb{R}^n \mid u^T \mathcal{H}_f v = 0\}$. Since $u^T \mathcal{H}_f u$ is necessarily positive, \mathcal{H}_f has the Lorentz signature.

The remaining implications follows from the fact that the univariate polynomial $\frac{1}{2}f(xu - v)$ has the discriminant $(u^T \mathcal{H}_f v)^2 - (u^T \mathcal{H}_f u)(v^T \mathcal{H}_f v)$. \square

The same argument shows that a nonzero homogeneous quadratic polynomial f with non-negative coefficients is stable if and only if \mathcal{H}_f has exactly one positive eigenvalue.

2.2. Matroid theory captures various combinatorial notions of independence. A *matroid* M on $[n]$ is a nonempty family of subsets B of $[n]$, called the *set of bases* of M , that satisfies the *exchange property*:

For any $B_1, B_2 \in B$ and $i \in B_1 \setminus B_2$, there is $j \in B_2 \setminus B_1$ such that $(B_1 \setminus i) \cup j \in B$.

We refer to [Oxl11] for background on matroid theory. More generally, following [Mur03], we define a subset $J \subseteq \mathbb{N}^n$ to be *M-convex* if it satisfies any one of the following equivalent conditions³:

- For any $\alpha, \beta \in J$ and any index i satisfying $\alpha_i > \beta_i$, there is an index j satisfying

$$\alpha_j < \beta_j \text{ and } \alpha - e_i + e_j \in J.$$

- For any $\alpha, \beta \in J$ and any index i satisfying $\alpha_i > \beta_i$, there is an index j satisfying

$$\alpha_j < \beta_j \text{ and } \alpha - e_i + e_j \in J \text{ and } \beta - e_j + e_i \in J.$$

³The class of M-convex sets is essentially identical to the class of generalized polymatroids in the sense of [Fuj05]. We refer to [Mur03, Section 4.7] for more details.

The first condition is called the *exchange property* for M-convex sets, and the second condition is called the *symmetric exchange property* for M-convex sets. A proof of the equivalence can be found in [Mur03, Chapter 4]. Note that any M-convex subset of \mathbb{N}^n is necessarily contained in the discrete simplex Δ_n^d for some d . We refer to [Mur03] for a comprehensive treatment of M-convex sets.

Let f be a polynomial in $\mathbb{R}[w_1, \dots, w_n]$. We write f in the normalized form

$$f = \sum_{\alpha \in \mathbb{N}^n} \frac{c_\alpha}{\alpha!} w^\alpha, \text{ where } \alpha! = \prod_{i=1}^n \alpha_i!.$$

The *support* of the polynomial f is the subset of \mathbb{N}^n defined by

$$\text{supp}(f) = \left\{ \alpha \in \mathbb{N}^n \mid c_\alpha \neq 0 \right\}.$$

We write M_n^d for the set of all degree d homogeneous polynomials in $\mathbb{R}_{\geq 0}[w_1, \dots, w_n]$ whose supports are M-convex. Note that, in our convention, the empty subset of \mathbb{N}^n is an M-convex set. Thus the zero polynomial belongs to M_n^d , and $f \in M_n^d$ implies $\partial_i f \in M_n^{d-1}$.

Definition 2.6. We set $L_n^0 = S_n^0$, $L_n^1 = S_n^1$, and $L_n^2 = S_n^2$. For d larger than 2, we define

$$L_n^d = \left\{ f \in M_n^d \mid \partial_i f \in L_n^{d-1} \text{ for all } i \in [n] \right\} = \left\{ f \in M_n^d \mid \partial^\alpha f \in L_n^2 \text{ for every } \alpha \in \Delta_n^{d-2} \right\}.$$

Clearly, L_n^d contains \mathring{L}_n^d . In Theorem 5.1, we show that L_n^d is the closure of \mathring{L}_n^d in H_n^d . In other words, L_n^d is exactly the set of degree d Lorentzian polynomials in n variables. In this section, we show that \mathring{L}_n^d is contractible and its closure contains L_n^d . We fix i, j in $[n]$ and a degree d homogeneous polynomial f in $\mathbb{R}[w_1, \dots, w_n]$.

Proposition 2.7. If $f \in L_n^d$, then $(1 + \theta w_i \partial_j) f \in L_n^d$ for every nonnegative real number θ .

We prepare the proof of Proposition 2.7 with two lemmas.

Lemma 2.8. If $f \in M_n^d$, then $(1 + \theta w_i \partial_j) f \in M_n^d$ for every nonnegative real number θ .

Proof. We may suppose $\theta = 1$ and $j = n$. We use two combinatorial lemmas from [KMT07]. Introduce a new variable w_{n+1} , and set

$$g(w_1, \dots, w_n, w_{n+1}) = f(w_1, \dots, w_n + w_{n+1}) = \sum_{k=0}^d \frac{1}{k!} w_{n+1}^k \partial_n^k f(w_1, \dots, w_n).$$

By [KMT07, Lemma 6], the support of g is M-convex. In terms of [KMT07], the support of g is obtained from the support of f by an elementary splitting, and the operation of splitting preserves M-convexity. Therefore, g belongs to M_{n+1}^d . Since the intersection of an M-convex set with a cartesian product of intervals is M-convex, it follows that

$$(1 + w_{n+1} \partial_n) f \in M_{n+1}^d.$$

By [KMT07, Lemma 9], the above displayed inclusion implies

$$(1 + w_i \partial_n) f \in M_n^d.$$

In terms of [KMT07], the support of $(1 + w_i \partial_n) f$ is obtained from the support of $(1 + w_{n+1} \partial_n) f$ by an elementary aggregation, and the operation of aggregation preserves M-convexity. \square

For stable polynomials f and g in $\mathbb{R}[w_1, \dots, w_n]$, we define a relation $f < g$ by

$$f < g \iff g + w_{n+1} f \text{ is a stable polynomial in } \mathbb{R}[w_1, \dots, w_n, w_{n+1}].$$

If f and g are univariate polynomials with leading coefficients of the same sign, then $f < g$ if and only if the zeros of f interlace the zeros of g [BB10, Lemma 2.2]. In general, we have

$$f < g \iff f(xu - v) < g(xu - v) \text{ for all } u \in \mathbb{R}_{\geq 0}^n \text{ and } v \in \mathbb{R}^n.$$

For later use, we record here basic properties of stable polynomials and the relation $<$.

Lemma 2.9. Let f, g_1, g_2, h_1, h_2 be stable polynomials satisfying $h_1 < f < g_1, h_2 < f < g_2$.

- (1) The derivative $\partial_i f$ is stable and $\partial_i f < f$.
- (2) The diagonalization $f(w_1, w_1, w_3, \dots, w_n)$ is stable.
- (3) The dilation $f(a_1 w_1, \dots, a_n w_n)$ is stable for any $a \in \mathbb{R}_{\geq 0}^n$.
- (4) If f is not identically zero, then $f < \theta_1 g_1 + \theta_2 g_2$ for any $\theta_1, \theta_2 \geq 0$.
- (5) If f is not identically zero, then $\theta_1 h_1 + \theta_2 h_2 < f$ for any $\theta_1, \theta_2 \geq 0$.

The proof of Lemma 2.9 can be found in [Wag11, Section 2] and [BB10, Section 2].

Proof of Proposition 2.7. When $d \leq 2$, Lemma 2.9 imply Proposition 2.7. When $d \geq 3$, set

$$g = (1 + \theta w_i \partial_j) f.$$

By Lemma 2.8, the support of g is M-convex. Therefore, it is enough to prove that $\partial^\alpha g$ is stable for all $\alpha \in \Delta_n^{d-2}$. We give separate arguments when $\alpha_i = 0$ and $\alpha_i > 0$. If $\alpha_i = 0$, then

$$\partial^\alpha g = \partial^\alpha f + \theta w_i \partial_j \partial^\alpha f.$$

In this case, (1) and (2) of Lemma 2.9 for $\partial^\alpha f$ show that $\partial^\alpha g$ is stable. If $\alpha_i > 0$, then

$$\begin{aligned} \partial^\alpha g &= \partial^\alpha f + \theta \alpha_i \partial^{\alpha - e_i} \partial_j f + \theta w_i \partial^\alpha \partial_j f \\ &= \partial_i \left(\partial^{\alpha - e_i} f \right) + \theta \alpha_i \partial_j \left(\partial^{\alpha - e_i} f \right) + \theta w_i \partial_i \partial_j \left(\partial^{\alpha - e_i} f \right). \end{aligned}$$

If $\partial^\alpha g$ is not stable, then the cubic form $\partial^{\alpha - e_i} f$ contradicts Proposition 2.7. Thus the statement is reduced to the case $d = 3$ and $\alpha = e_i$. In this case, we have

$$\partial^\alpha g = \partial_i f + \theta \partial_j f + \theta w_i \partial_i \partial_j f.$$

The following special cases are easy to handle:

- (I) If $\partial_i \partial_j f$ is not identically zero, then $\partial^\alpha g$ is stable by (1), (2), and (4) of Lemma 2.9.
- (II) If $\partial_i \partial_j f$ and $(\partial_i f)(\partial_j f)$ are both identically zero, then $\partial^\alpha g$ is stable because $f \in L_n^3$.

We prove the statement for $d = 3$ in the remaining case: $\partial_i \partial_j f$ is identically zero and $(\partial_i f)(\partial_j f)$ is not identically zero. In this case, there are monomials $w_i w_{i'} w_{i''}$ and $w_j w_{j'} w_{j''}$ in the support of f . Let s be a nonnegative real parameter, and introduce the cubic form

$$h_s = f + s w_i \partial_{i'} f.$$

Since $\partial_i \partial_{i'} f$ is not identically zero, (I) shows that $h_s \in L_n^3$. We claim that $\partial_i \partial_j h_s$ is not identically zero when s is positive. For the claim, it is enough to show that $\partial_{i'} \partial_j f$ is not identically zero. We apply the symmetric exchange property for f , the monomials $w_i w_{i'} w_{i''}$, $w_j w_{j'} w_{j''}$, and the variable w_i : We see that the monomial $w_j w_{i'} w_{i''}$ must be in the support of f , since no monomial in the support of f is divisible by $w_i w_j$. By (I), we have

$$h_s + \theta w_i \partial_j h_s \in L_n^3 \text{ for every positive real number } s.$$

Since the support of g is M-convex and the stability is a closed condition, the above implies

$$g = \lim_{s \rightarrow 0} (h_s + \theta w_i \partial_j h_s) \in L_n^3.$$

This completes the proof of Proposition 2.7. \square

We use Proposition 2.7 to show that any nonnegative linear change of variables preserves L_n^d .

Theorem 2.10. If $f \in L_n^d$, then $f(Av) \in L_m^d$ for any $n \times m$ matrix A with nonnegative entries.

Proof. Fix $f = f(w_1, \dots, w_n) \in L_n^d$. Note that Theorem 2.10 follows from its two special cases:

(I) $f(w_1, \dots, w_{n-1}, w_n + w_{n+1})$ is in L_{n+1}^d , and

(II) $f(w_1, \dots, w_{n-1}, \theta w_n)$ is in L_n^d for any $\theta \geq 0$,

As observed in the proof of Lemma 2.8, we have

$$f(w_1, \dots, w_{n-1}, w_n + w_{n+1}) \in M_{n+1}^d.$$

Therefore,⁴ the first assertion follows from Proposition 2.7:

$$\lim_{k \rightarrow \infty} \left(1 + \frac{w_{n+1} \partial_n}{k} \right)^k f = f(w_1, \dots, w_{n-1}, w_n + w_{n+1}) \in L_{n+1}^d.$$

For the second assertion, note from the definition of M-convexity that

$$f(w_1, \dots, w_{n-1}, 0) \in M_n^d.$$

Thus the second assertion for $\theta = 0$ follows from the case $\theta > 0$, which is trivial to verify. \square

Theorem 2.10 can be used to show that taking directional derivatives in nonnegative directions takes polynomials in L_n^d to polynomials in L_n^{d-1} .

Corollary 2.11. If $f \in L_n^d$, then $\sum_{i=1}^n a_i \partial_i f \in L_n^{d-1}$ for any $a_1, \dots, a_n \geq 0$.

⁴It is necessary to check the inclusion in M_{n+1}^d in advance because we have not yet proved that L_{n+1}^d is closed.

Proof. We apply Theorem 2.10 to f and the matrix with column vectors e_1, \dots, e_n and $\sum_{i=1}^n a_i e_i$:

$$g := f(w_1 + a_1 w_{n+1}, \dots, w_n + a_n w_{n+1}) \in L_{n+1}^d, \text{ and hence } \partial_{n+1} g \in L_{n+1}^{d-1}.$$

Applying Theorem 2.10 to $\partial_{n+1} g$ and the matrix with column vectors e_1, \dots, e_n and 0, we get

$$\partial_{n+1} g|_{w_{n+1}=0} = \sum_{i=1}^n a_i \partial_i f \in L_n^{d-1}. \quad \square$$

Let θ be a nonnegative real parameter. We define a linear operator $T_n(\theta, -)$ by

$$T_n(\theta, f) = \prod_{i=1}^{n-1} (1 + \theta w_i \partial_n)^d f.$$

By Proposition 2.7, if $f \in L_n^d$, then $T_n(\theta, f) \in L_n^d$. In addition, if $f \in P_n^d$, then $T_n(\theta, f) \in P_n^d$.

Lemma 2.12. If $f \in L_n^d \cap P_n^d$, then $T_n(\theta, f) \in \mathring{L}_n^d$ for every positive real number θ .

Proof. Let e_i be the i -th standard unit vector in \mathbb{R}^n , and let v be any vector in \mathbb{R}^n not parallel to e_n . From here on, in this proof, all polynomials are restricted to the line $x e_n - v$ and considered as univariate polynomials in x .

Let α be an arbitrary element of Δ_n^{d-2} . By Lemma 2.5, it is enough to show that the quadratic polynomial $\partial^\alpha T_n(\theta, f)$ has two distinct real zeros. Using Proposition 2.7, we can deduce the preceding statement from the following claims:

- (I) If $\partial^\alpha f$ has two distinct real zeros, then $\partial^\alpha (1 + \theta w_i \partial_n) f$ has two distinct zeros.
- (II) If v_i is nonzero, then $\partial^\alpha (1 + \theta w_i \partial_n)^d f$ has two distinct real zeros.

We first prove (I). Suppose $\partial^\alpha f$ has two distinct real zeros, and set $g = (1 + \theta w_i \partial_n) f$. Note that

$$\partial^\alpha g = \partial^\alpha f + \theta \alpha_i \partial^{\alpha - e_i + e_n} f + \theta w_i \partial^{\alpha + e_n} f.$$

Let c be the unique zero of $\partial^{\alpha + e_n} f$. Since c strictly interlaces two distinct zeros of $\partial^\alpha f$, we have

$$\partial^\alpha f|_{x=c} < 0.$$

Similarly, since $\partial^{\alpha - e_i + e_n} f$ has only real zeros and $\partial^{\alpha + e_n} f < \partial^{\alpha - e_i + e_n} f$, we have

$$\partial^{\alpha - e_i + e_n} f|_{x=c} \leq 0.$$

Thus $\partial^\alpha g|_{x=c} < 0$, and hence $\partial^\alpha g$ has two distinct real zeros. This completes the proof of (I).

Before proving (II), we strengthen (I) as follows:

- (III) A multiple zero of $\partial^\alpha g$ is necessarily a multiple zero of $\partial^\alpha f$.

Suppose $\partial^\alpha g$ has a multiple zero. Using (I), we know that $\partial^\alpha f$ has a multiple zero, say c . Clearly, c must be also a zero of $\partial^{\alpha + e_n} f$. Since c interlaces the two (not necessarily distinct) zeros of $\partial^{\alpha - e_i + e_n} f$, we have

$$\partial^\alpha g|_{x=c} = \theta \alpha_i \partial^{\alpha - e_i + e_n} f|_{x=c} \leq 0.$$

Therefore, if c is not a zero of $\partial^\alpha g$, then $\partial^\alpha g$ has two distinct zeros, contradicting the hypothesis that $\partial^\alpha g$ has a multiple zero. This completes the proof of (III).

We prove (II). Suppose $\partial^\alpha (1 + \theta w_i \partial_n)^d f$ has a multiple zero, say c . Using (III), we know that the number c is a multiple zero of $\partial^\alpha (1 + \theta w_i \partial_n)^k f$ for all $0 \leq k \leq d$.

Expanding the k -th power and using the linearity of ∂^α , we deduce that

the number c is a zero of $\partial^\alpha w_i^k \partial_n^k f$ for all $0 \leq k \leq d$.

However, since f has positive coefficients, the value of $\partial^\alpha w_i^{\alpha_i+2} \partial_n^{\alpha_i+2} f$ at c is a positive multiple of v_i^2 , and hence v_i must be zero. This completes the proof of (II). \square

We use Lemma 2.12 to prove the main result of this section.

Theorem 2.13. The space \mathring{L}_n^d is contractible, and its closure contains L_n^d .

Proof. Let f be a polynomial in L_n^d that is not identically zero, and let θ be a real parameter satisfying $0 \leq \theta \leq 1$. By Theorem 2.10, we have

$$S(\theta, f) := \frac{1}{|f|_1} f\left((1-\theta)w_1 + \theta(w_1 + \cdots + w_n), \dots, (1-\theta)w_n + \theta(w_1 + \cdots + w_n)\right) \in L_n^d,$$

where $|f|_1$ is the sum of all coefficients of f . Since $S(\theta, f)$ belongs to P_n^d when $0 < \theta \leq 1$, Lemma 2.12 shows that we have a homotopy

$$T_n(\theta, S(\theta, f)) \in \mathring{L}_n^d, \quad 0 < \theta \leq 1,$$

that deforms f to the polynomial $T_n(1, (w_1 + \cdots + w_n)^d)$. It follows that \mathring{L}_n^d is contractible and the closure of \mathring{L}_n^d in the space of homogeneous polynomials H_n^d contains L_n^d . \square

Remark 2.14. Let $\mathbb{P}H_n^d$ be the projectivization of the vector space H_n^d , and let L_J be the set of polynomials in L_n^d with support J . Writing $\mathbb{P}\mathring{L}_n^d$, $\mathbb{P}L_n^d$, and $\mathbb{P}L_J$ for the images of $\mathring{L}_n^d \setminus \{0\}$, $L_n^d \setminus \{0\}$, and L_J in $\mathbb{P}H_n^d$ respectively, we have

$$\mathbb{P}L_n^d = \coprod_J \mathbb{P}L_J,$$

where the union is over all nonempty M-convex subsets of Δ_n^d . By Theorems 2.10 and 5.1, $\mathbb{P}L_n^d$ is a compact contractible subset of $\mathbb{P}H_n^d$ with contractible interior $\mathbb{P}\mathring{L}_n^d$. By Theorem 7.1, $\mathbb{P}L_J$ is nonempty for every nonempty M-convex subset J of Δ_n^d . In addition, by Proposition 8.12, $\mathbb{P}L_J$ is contractible for every nonempty M-convex subset J of Δ_n^d .

3. INDEPENDENCE AND NEGATIVE DEPENDENCE

Let c be a fixed positive real number, and let f be a polynomial in $\mathbb{R}[w_1, \dots, w_n]$. In this section, the polynomial f is not necessarily homogeneous. As before, we write e_i for the i -th standard unit vector in \mathbb{R}^n .

Definition 3.1. We say that f is c -Rayleigh if f has nonnegative coefficients and

$$\partial^\alpha f(w) \partial^{\alpha+e_i+e_j} f(w) \leq c \partial^{\alpha+e_i} f(w) \partial^{\alpha+e_j} f(w) \text{ for all } i, j \in [n], \alpha \in \mathbb{N}^n, w \in \mathbb{R}_{\geq 0}^n.$$

When f is the partition function of a discrete probability measure μ , the c -Rayleigh condition captures a negative dependence property of μ . More precisely, when f is *multi-affine*, that is, when f has degree at most one in each variable, the c -Rayleigh condition for f is equivalent to

$$f(w) \partial_i \partial_j f(w) \leq c \partial_i f(w) \partial_j f(w) \text{ for all distinct } i, j \in [n], \text{ and } w \in \mathbb{R}_{\geq 0}^n.$$

Thus the 1-Rayleigh property of multi-affine polynomials is equivalent to the Rayleigh property for discrete probability measures studied in [Wag08] and [BBL09]. In Corollary 4.5, we prove that any Lorentzian polynomial is 2-Rayleigh.

In Theorem 3.5, we show that the support of any homogeneous c -Rayleigh polynomial is M -convex. The notion of M^\natural -convexity will be useful for the proof: A subset $J^\natural \subseteq \mathbb{N}^n$ is said to be M^\natural -convex if there is an M -convex set J in \mathbb{N}^{n+1} such that

$$J^\natural = \{(\alpha_1, \dots, \alpha_n) \mid (\alpha_1, \dots, \alpha_n, \alpha_{n+1}) \in J\}.$$

The projection from J to J^\natural should be bijective for any such J , as the M -convexity of J implies that J is in Δ_n^d for some d . We refer to [Mur03, Section 4.7] for more on M^\natural -convex sets.

We prepare the proof of Theorem 3.5 with three lemmas. Verification of the first lemma is routine and will be omitted.

Lemma 3.2. The following polynomials are c -Rayleigh whenever f is c -Rayleigh:

- (1) The *contraction* $\partial_i f$ of f .
- (2) The *deletion* $f \setminus i$ of f , the polynomial obtained from f by evaluating $w_i = 0$.
- (3) The *diagonalization* $f(w_1, w_1, w_3, \dots, w_n)$.
- (4) The *dilation* $f(a_1 w_1, \dots, a_n w_n)$, for $(a_1, \dots, a_n) \in \mathbb{R}_{\geq 0}^n$.
- (5) The *translation* $f(a_1 + w_1, \dots, a_n + w_n)$, for $(a_1, \dots, a_n) \in \mathbb{R}_{\geq 0}^n$.

We introduce a partial order \leq on \mathbb{N}^n by setting

$$\alpha \leq \beta \iff \alpha_i \leq \beta_i \text{ for all } i \in [n].$$

We say that a subset J^\natural of \mathbb{N}^n is *interval convex* if the following implication holds:

$$(\alpha \in J^\natural, \beta \in J^\natural, \alpha \leq \gamma \leq \beta) \implies \gamma \in J^\natural.$$

The *augmentation property* for $J^\natural \subseteq \mathbb{N}^n$ is the implication

$$(\alpha \in J^\natural, \beta \in J^\natural, |\alpha|_1 < |\beta|_1) \implies (\alpha_j < \beta_j \text{ and } \alpha + e_j \in J^\natural \text{ for some } j \in [n]).$$

Lemma 3.3. Let J^\natural be an interval convex subset of \mathbb{N}^n containing 0. Then J^\natural is M^\natural -convex if and only if J^\natural satisfies the augmentation property.

Therefore, a nonempty interval convex subset of $\{0, 1\}^n$ containing 0 is M^\natural -convex if and only if it is the collection of independent sets of a matroid on $[n]$.

Proof. Let d be any sufficiently large positive integer, and set

$$J = \left\{ (\alpha_1, \dots, \alpha_n, d - \alpha_1 - \dots - \alpha_n) \in \mathbb{N}^{n+1} \mid (\alpha_1, \dots, \alpha_n) \in J^\natural \right\}.$$

The “only if” direction is straightforward: If J^\natural is M^\natural -convex, then J is M -convex, and the augmentation property for J^\natural is a special case of the exchange property for J .

We prove the “if” direction by checking the exchange property for J . Let α and β be elements of J , and let i be an index satisfying $\alpha_i > \beta_i$. We claim that there is an index j satisfying

$$\alpha_j < \beta_j \text{ and } \alpha - e_i + e_j \in J.$$

By the augmentation property for J^\natural , it is enough to justify the claim when $i \neq n+1$. When $\alpha_{n+1} < \beta_{n+1}$, then we may take $j = n+1$, again by the augmentation property for M^\natural .

Suppose $\alpha_{n+1} \geq \beta_{n+1}$. In this case, we consider the element $\gamma = \alpha - e_i + e_{n+1}$. The element γ belongs to J , because J^\natural is an interval convex set containing 0. We have $\gamma_{n+1} > \beta_{n+1}$, and hence the augmentation property for J^\natural gives an index j satisfying

$$\gamma_j < \beta_j \text{ and } \alpha - e_i + e_j = \gamma - e_{n+1} + e_j \in J.$$

This index j is necessarily different from i because $\alpha_i > \beta_i$. It follows that $\alpha_j = \gamma_j < \beta_j$, and the M -convexity of J is proved. \square

Lemma 3.4. Let f be a c -Rayleigh polynomial in $\mathbb{R}[w_1, \dots, w_n]$.

- (1) The support of f is interval convex.
- (2) If $f(0)$ is nonzero, then $\text{supp}(f)$ is M^\natural -convex.

Proof. Suppose that $\alpha \leq \gamma \leq \beta$ is a counterexample to (1) with minimal $|\beta|_1$. We have

$$\alpha = 0,$$

since otherwise the contraction $\partial_i f$ for any i satisfying $\alpha_i \neq 0$ is a smaller counterexample to (1). In addition, γ is necessarily a unit vector, say

$$\gamma = e_i,$$

since otherwise the contraction $\partial_i f$ for any i satisfying $\gamma_i \neq 0$ is a smaller counterexample to (1). Suppose e_j is in the support of f for some j . In this case, we should have

$$\beta = e_i + e_j,$$

since otherwise $\partial_j f$ is a smaller counterexample. However, the above implies

$$\partial_i f(0) = 0 \text{ and } f(0) \partial_i \partial_j f(0) > 0,$$

contradicting the c -Rayleigh property of f . Therefore, no e_j is in the support of f . By (3) of Lemma 3.2, the following univariate polynomial is c -Rayleigh:

$$g(w_1) = f(w_1, w_1, \dots, w_1) = a_1 + a_2 w_1^k + a_3 w_1^{k+1} + \dots, \quad k \geq 2.$$

The preceding analysis shows that $k \geq 2$ and $a_1, a_2 > 0$ in the above expression. However,

$$(\partial_1 g)^2 = a_2^2 k^2 w_1^{2k-2} + \text{higher order terms}, \quad (\partial_1^2 f)g = a_1 a_2 k(k-1) w_1^{k-2} + \text{higher order terms},$$

contradicting the c -Rayleigh property of g for sufficiently small positive w_1 . This proves (1).

Suppose f is a counterexample to (2) with minimal number of variables n . We may suppose in addition that f has minimal degree d among all such examples. By Lemma 3.3 and (1) of the current lemma, we know that the support of f fails to have the augmentation property. In other words, there are $\alpha, \beta \in \text{supp}(f)$ such that $|\alpha|_1 < |\beta|_1$ and

$$\alpha_i < \beta_i \implies \alpha + e_i \notin \text{supp}(f).$$

For any γ , write $S(\gamma)$ for the set of indices i such that $\gamma_i > 0$. If i is in the intersection of $S(\alpha)$ and $S(\beta)$, then $\partial_i f$ is a counterexample to (2) that has degree less than d , and hence

$$S(\alpha) \cap S(\beta) = \emptyset.$$

Similarly, if i is not in the union of $S(\alpha)$ and $S(\beta)$, then $f \setminus i$ is a counterexample to (2) that involves less than n variables, and hence

$$S(\alpha) \cup S(\beta) = [n].$$

In addition, we should have $|S(\beta)| = 1$, since otherwise we get a counterexample to (2) that involves less than n variables by identifying all the variables in $S(\beta)$. Therefore, after replacing β with its multiple if necessary, we may suppose that

$$\beta = d e_n.$$

Let T be the set of all γ in the support of f such that

$$|\gamma|_1 < d \quad \text{and} \quad \gamma + e_n \notin \text{supp}(f).$$

This set T is nonempty because it contains α . Let U be the set of elements in T with largest possible n -th coordinate, and take an element γ in U with smallest possible $|\gamma|_1$. From (1), we know that γ is not a multiple of e_n . Therefore, there is an index $j < n$ such that

$$\gamma - e_j \in \text{supp}(f).$$

Since $|\gamma - e_j|_1 < |\gamma|_1$, the element $\gamma - e_j$ cannot be in T , and hence

$$\gamma - e_j + e_n \in \text{supp}(f).$$

Since γ is an element of U , the element $\gamma - e_j + e_n$ cannot be in T , and hence

$$\gamma - e_j + 2e_n \in \text{supp}(f).$$

Let g be the bivariate c -Rayleigh polynomial obtained from $\partial^{\gamma-e_j} f$ by setting $w_i = 0$ for all i other than j and n . By construction, we have

$$0, e_j, e_n, 2e_n \in \text{supp}(g) \text{ and } e_j + e_n \notin \text{supp}(g).$$

Since the support of g is interval convex by (1), we may write

$$g(w_j, w_n) = h(w_j) + r(w_n),$$

where h and g are univariate polynomials satisfying $\deg h \geq 1$ and $\deg r \geq 2$. We have

$$(\partial_n g)^2 = \left(\frac{dr}{dw_n} \right)^2 \text{ and } (\partial_n^2 g)g = \frac{d^2 r}{dw_n^2} (h(w_j) + r(w_n)),$$

which contradicts the c -Rayleigh property of g for fixed w_n and large w_j . This proves (2). \square

Theorem 3.5. If f is homogeneous and c -Rayleigh, then the support of f is M-convex.

Proof. By (5) of Lemma 3.2 and (2) of Lemma 3.4, the support of the translation

$$g(w_1, \dots, w_n) = f(w_1 + 1, \dots, w_n + 1)$$

is M^\natural -convex. In other words, the support J of the homogenization of g is M-convex. Since the intersection of an M-convex set with a coordinate hyperplane is M-convex, this implies the M-convexity of the support of f . \square

A multi-affine polynomial f is said to be *strongly Rayleigh* if

$$f(w) \partial_i \partial_j f(w) \leq \partial_i f(w) \partial_j f(w) \text{ for all distinct } i, j \in [n], \text{ and } w \in \mathbb{R}^n.$$

Clearly, any strongly Rayleigh multi-affine polynomial is 1-Rayleigh. Since a multi-affine polynomial is stable if and only if it is strongly Rayleigh [Brä07, Theorem 5.6], Theorem 3.5 extends the following theorem of Choe *et al.* [COSW04, Theorem 7.1]: If f is a nonzero homogeneous stable multi-affine polynomial, then the support of f is the set of bases of a matroid.

4. HODGE-RIEMANN RELATIONS FOR LORENTZIAN POLYNOMIALS

Let f be a nonzero degree $d \geq 2$ homogeneous polynomial in $\mathbb{R}_{\geq 0}[w_1, \dots, w_n]$. The following proposition may be seen as an analog of the Hodge-Riemann relations for homogeneous stable polynomials.⁵

Proposition 4.1. If f is in $S_n^d \setminus 0$, then $\mathcal{H}_f(w)$ has exactly one positive eigenvalue for all $w \in \mathbb{R}_{>0}^n$. Moreover, if f is in the interior of S_n^d , then $\mathcal{H}_f(w)$ is nonsingular for all $w \in \mathbb{R}_{>0}^n$.

In Theorem 4.3, we extend the above result to Lorentzian polynomials.

⁵We refer to [Huh18] for a survey of the Hodge-Riemann relations in combinatorial contexts.

Proof. Fix a vector $w \in \mathbb{R}_{>0}^n$. By Lemma 2.5, the Hessian of f has exactly one positive eigenvalue at w if and only if the following quadratic polynomial in z is stable:

$$z^T \mathcal{H}_f(w) z = \sum_{1 \leq i, j \leq n} z_i z_j \partial_i \partial_j f(w).$$

The above is the quadratic part of the stable polynomial $f(z+w)$, and hence is stable by [BBL09, Lemma 4.16].

Moreover, if f is strictly stable, then $f_\epsilon = f \pm \epsilon(w_1^d + \dots + w_n^d)$ is stable for all sufficiently small positive ϵ . Therefore, by the result obtained in the previous paragraph, the matrix

$$\mathcal{H}_{f_\epsilon}(w) = \mathcal{H}_f(w) \pm d(d-1)\epsilon \operatorname{diag}(w_1^{d-2}, \dots, w_n^{d-2})$$

has exactly one positive eigenvalue for all sufficiently small positive ϵ , and hence $\mathcal{H}_f(w)$ is nonsingular. \square

Lemma 4.2. If $\mathcal{H}_{\partial_i f}(w)$ has exactly one positive eigenvalue for every $i \in [n]$ and $w \in \mathbb{R}_{>0}^n$, then

$$\ker \mathcal{H}_f(w) = \bigcap_{i=1}^n \ker \mathcal{H}_{\partial_i f}(w).$$

Proof. We may suppose $d \geq 3$. Fix $w \in \mathbb{R}_{>0}^n$, and write \mathcal{H}_f for $\mathcal{H}_f(w)$. By Euler's formula for homogeneous functions,

$$(d-2)\mathcal{H}_f = \sum_{i=1}^n w_i \mathcal{H}_{\partial_i f}.$$

It follows that the kernel of \mathcal{H}_f contains the intersection of the kernels of $\mathcal{H}_{\partial_i f}$.

For the other inclusion, let z be a vector in the kernel of \mathcal{H}_f . By Euler's formula again,

$$(d-2)e_i^T \mathcal{H}_f = w^T \mathcal{H}_{\partial_i f},$$

and hence $w^T \mathcal{H}_{\partial_i f} z = 0$. We have $w^T \mathcal{H}_{\partial_i f} w > 0$ because $\partial_i f$ is nonzero and has nonnegative coefficients. It follows that $\mathcal{H}_{\partial_i f}$ is negative semidefinite on the kernel of $w^T \mathcal{H}_{\partial_i f}$. In particular,

$$z^T \mathcal{H}_{\partial_i f} z \leq 0, \text{ with equality if and only if } \mathcal{H}_{\partial_i f} z = 0.$$

To conclude, we write zero as the positive linear combination

$$0 = (d-2) \left(z^T \mathcal{H}_f z \right) = \sum_{i=1}^n w_i \left(z^T \mathcal{H}_{\partial_i f} z \right).$$

Since every summand in the right-hand side is non-positive by the previous analysis, we must have $z^T \mathcal{H}_{\partial_i f} z = 0$ for every i , and hence $\mathcal{H}_{\partial_i f} z = 0$ for every i . \square

We are now ready to prove an analog of the Hodge-Riemann relation for Lorentzian polynomials.

Theorem 4.3. Let f be a nonzero homogeneous polynomial in $\mathbb{R}[w_1, \dots, w_n]$ of degree $d \geq 2$.

(1) If f is in \mathring{L}_n^d , then $\mathcal{H}_f(w)$ is nonsingular for all $w \in \mathbb{R}_{>0}^n$.

(2) If f is in L_n^d , then $\mathcal{H}_f(w)$ has exactly one positive eigenvalue for all $w \in \mathbb{R}_{>0}^n$.

Proof. By Theorem 2.13, L_n^d is in the closure of \mathring{L}_n^d . Therefore, we may suppose $f \in \mathring{L}_n^d$ in (2). We prove (1) and (2) simultaneously by induction on d under this assumption. The base case $d = 2$ is trivial. We suppose that $d \geq 3$ and that the theorem holds for \mathring{L}_n^{d-1} .

That (1) holds for \mathring{L}_n^d follows from induction and Lemma 4.2. Using Proposition 4.1, we see that (2) holds for stable polynomials in \mathring{L}_n^d . Since \mathring{L}_n^d is connected by Theorem 2.13, the continuity of eigenvalues and the validity of (1) together implies (2). \square

We use Theorem 4.3 to show that all polynomials in L_n^d are $2\left(1 - \frac{1}{d}\right)$ -Rayleigh.

Lemma 4.4. If $\mathcal{H}_f(w)$ has exactly one positive eigenvalue for all $w \in \mathbb{R}_{>0}^n$, then

$$f(w) \partial_i \partial_j f(w) \leq 2\left(1 - \frac{1}{d}\right) \partial_i f(w) \partial_j f(w) \quad \text{for all } w \in \mathbb{R}_{\geq 0}^n \text{ and } i, j \in [n].$$

Proof. Fix $w \in \mathbb{R}_{>0}^n$, and write \mathcal{H} for $\mathcal{H}_f(w)$. By Euler's formula for homogeneous functions,

$$w^T \mathcal{H} w = d(d-1)f(w) \quad \text{and} \quad w^T \mathcal{H} e_i = (d-1)\partial_i f(w),$$

where e_i is the i -th standard unit vector in \mathbb{R}^n . Let t be a real parameter, and consider the restriction of \mathcal{H} to the plane spanned by w and $v_t = e_i + te_j$. By Cauchy's interlacing theorem, the restriction of \mathcal{H} has exactly one positive eigenvalue. In particular, the determinant of the restriction must be nonpositive:

$$(w^T \mathcal{H} v_t)^2 - (w^T \mathcal{H} w) \cdot (v_t^T \mathcal{H} v_t) \geq 0 \quad \text{for all } t \in \mathbb{R}.$$

In other words, for all $t \in \mathbb{R}$, we have

$$(d-1)^2(\partial_i f + t\partial_j f)^2 - d(d-1)f(\partial_i^2 f + 2t\partial_i \partial_j f + t^2\partial_j^2 f) \geq 0.$$

It follows that, for all $t \in \mathbb{R}$, we have

$$(d-1)^2(\partial_i f + t\partial_j f)^2 - 2td(d-1)f\partial_i \partial_j f \geq 0.$$

Therefore, the discriminant of the above quadratic polynomial in t should be nonpositive:

$$f\partial_i \partial_j f - 2\left(1 - \frac{1}{d}\right)\partial_i f \partial_j f \leq 0.$$

This completes the proof of Lemma 4.4. \square

Corollary 4.5. Any polynomial f in L_n^d is $2\left(1 - \frac{1}{d}\right)$ -Rayleigh.

Proof. Theorem 4.3 and Lemma 4.4 show that, for any $\alpha \in \mathbb{N}^n$, any $w \in \mathbb{R}_{\geq 0}^n$, and any $i, j \in [n]$,

$$\partial^\alpha f(w) \partial^{\alpha+e_i+e_j} f(w) \leq 2\left(1 - \frac{1}{d-|\alpha|_1}\right) \partial^{\alpha+e_i} f(w) \partial^{\alpha+e_j} f(w).$$

Since $2\left(1 - \frac{1}{d}\right)$ is an increasing function of d , the conclusion follows. \square

5. CHARACTERIZATIONS OF LORENTZIAN POLYNOMIALS

We may now give a complete and useful description of the space of Lorentzian polynomials. As before, we write H_n^d for the space of degree d homogeneous polynomials in n variables.

Theorem 5.1. The closure of \mathring{L}_n^d in H_n^d is L_n^d . In particular, L_n^d is a closed subspace of H_n^d .

Proof. By Theorem 2.13, the closure of \mathring{L}_n^d contains L_n^d . The other inclusion follows from Theorem 3.5 and Corollary 4.5. \square

Therefore, a degree d homogeneous polynomial f with nonnegative coefficients is Lorentzian if and only if the support of f is M-convex and $\partial^\alpha f$ has at most one positive eigenvalue for every $\alpha \in \Delta_n^{d-2}$. In other words, Definitions 2.1 and 2.6 define the same class of polynomials.

Example 5.2. A sequence of nonnegative numbers a_0, a_1, \dots, a_d is said to be *ultra log-concave* if

$$\frac{a_k^2}{\binom{d}{k}^2} \geq \frac{a_{k-1}}{\binom{d}{k-1}} \frac{a_{k+1}}{\binom{d}{k+1}} \quad \text{for all } 0 < k < d.$$

The sequence is said to have *no internal zeros* if

$$a_{k_1} a_{k_3} > 0 \implies a_{k_2} > 0 \quad \text{for all } 0 \leq k_1 < k_2 < k_3 \leq d.$$

A bivariate homogeneous polynomial $\sum_{k=0}^d a_k w_1^k w_2^{d-k}$ with nonnegative coefficients is Lorentzian if and only if the sequence a_k is ultra log-concave and has no internal zeros.

Let f be a polynomial in n variables with nonnegative coefficients. In [Gur09], Gurvits defines f to be *strongly log-concave* if, for all $\alpha \in \mathbb{N}^n$,

$$\partial^\alpha f \text{ is identically zero or } \log(\partial^\alpha f) \text{ is concave on } \mathbb{R}_{>0}^n.$$

In [AOV], Anari *et al.* define f to be *completely log-concave* if, for all $m \in \mathbb{N}$ and any $m \times n$ matrix (a_{ij}) with nonnegative entries,

$$\left(\prod_{i=1}^m D_i \right) f \text{ is identically zero or } \log \left(\left(\prod_{i=1}^m D_i \right) f \right) \text{ is concave on } \mathbb{R}_{>0}^n,$$

where D_i is the differential operator $\sum_{j=1}^n a_{ij} \partial_j$. We show that the two notions agree with each other and with the Lorentzian property for homogeneous polynomials.

Theorem 5.3. The following conditions are equivalent for any homogeneous polynomial f .

- (1) f is completely log-concave.
- (2) f is strongly log-concave.
- (3) f is Lorentzian.

Recall that the support of any Lorentzian polynomial is M-convex. Theorem 5.3 shows that the same holds for any strongly log-concave homogeneous polynomial. This answers a question of Gurvits [Gur09, Section 4.5 (iii)].

Corollary 5.4. The support of any strongly log-concave homogeneous polynomial is M-convex.

Similarly, we can use Theorem 5.3 to show that the class of strongly log-concave homogeneous polynomials is closed under multiplication. This answers another question of Gurvits [Gur09, Section 4.5 (iv)] for homogeneous polynomials.

Corollary 5.5. The product of strongly log-concave homogeneous polynomials is strongly log-concave.

Proof. Let $f(w)$ be an element of L_n^d , and let $g(w)$ be an element of L_n^e . It is straightforward to check that $f(w)g(u)$ is an element of L_{n+n}^{d+e} , where u is a set of variables disjoint from w . It follows that $f(w)g(w)$ is an element of L_n^{d+e} , since setting $u = w$ preserves the Lorentzian property by Theorem 2.10. \square

Corollary 5.5 extends the following theorem of Liggett [Lig97, Theorem 2]: The convolution product of two ultra log-concave sequences with no internal zeros is an ultra log-concave with no internal zeros.

To prove Theorem 5.3, we use the following proposition. Let f be a homogeneous polynomial in n variables of degree $d \geq 2$.

Proposition 5.6. The following are equivalent for any $w \in \mathbb{R}^n$ satisfying $f(w) > 0$.

- (1) The Hessian of $f^{1/d}$ is negative semidefinite at w .
- (2) The Hessian of $\log f$ is negative semidefinite at w .
- (3) The Hessian of f has exactly one positive eigenvalue at w .

The equivalence of (2) and (3) appears in [AOV].

Proof. We fix w throughout the proof. For $n \times n$ symmetric matrices A and B , we write $A < B$ to mean the following interlacing relationship between the eigenvalues of A and B :

$$\lambda_1(A) \leq \lambda_1(B) \leq \lambda_2(A) \leq \lambda_2(B) \leq \dots \leq \lambda_n(A) \leq \lambda_n(B).$$

Let $\mathcal{H}_1, \mathcal{H}_2$, and \mathcal{H}_3 for the Hessians of $f^{1/d}$, $\log f$, and f , respectively. We have

$$df^{-1/d} \mathcal{H}_1 = \mathcal{H}_2 + \frac{1}{d} f^{-2} (\text{grad} f)(\text{grad} f)^T \quad \text{and} \quad \mathcal{H}_2 = f^{-1} \mathcal{H}_3 - f^{-2} (\text{grad} f)(\text{grad} f)^T.$$

Since $(\text{grad} f)(\text{grad} f)^T$ is positive semidefinite of rank one, Cauchy's interlacing theorem shows

$$\mathcal{H}_2 < \mathcal{H}_1 \quad \text{and} \quad \mathcal{H}_2 < \mathcal{H}_3 \quad \text{and} \quad \mathcal{H}_1 < \mathcal{H}_3.$$

Since $w^T \mathcal{H}_3 w = d(d-1)f$, \mathcal{H}_3 has at least one positive eigenvalue, and hence (1) \Rightarrow (2) \Rightarrow (3).

For (3) \Rightarrow (1), suppose that \mathcal{H}_3 has exactly one positive eigenvalue. We introduce a positive real parameter ϵ and consider the polynomial

$$f_\epsilon = f - \epsilon(w_1^d + \dots + w_n^d).$$

We write $\mathcal{H}_{3,\epsilon}$ for the Hessian of f_ϵ , and write $\mathcal{H}_{1,\epsilon}$ for the Hessian of $f_\epsilon^{1/d}$.

Note that $\mathcal{H}_{3,\epsilon}$ is nonsingular and has exactly one positive eigenvalue for all sufficiently small positive ϵ . In addition, we have $\mathcal{H}_{1,\epsilon} < \mathcal{H}_{3,\epsilon}$, and hence $\mathcal{H}_{1,\epsilon}$ has at most one nonnegative eigenvalue for all sufficiently small positive ϵ . However, by Euler's formula for homogeneous functions, we have

$$w^T \mathcal{H}_{1,\epsilon} w = 0,$$

so that 0 is the only nonnegative eigenvalue of $\mathcal{H}_{1,\epsilon}$ for any such ϵ . The implication (3) \Rightarrow (1) now follows by limiting ϵ to 0. \square

It follows that, for any nonzero degree $d \geq 2$ homogeneous polynomial f with nonnegative coefficients, the following conditions are equivalent:

- The function $f^{1/d}$ is concave on $\mathbb{R}_{>0}^n$.
- The function $\log f$ is concave on $\mathbb{R}_{>0}^n$.
- The Hessian of f has exactly one positive eigenvalue on $\mathbb{R}_{>0}^n$.

Proof of Theorem 5.3. We may suppose that f has degree $d \geq 2$. Clearly, completely log-concave polynomials are strongly log-concave.

Suppose f is a strongly log-concave homogeneous polynomial of degree d . By Proposition 5.6, either $\partial^\alpha f$ is identically zero or the Hessian of $\partial^\alpha f$ has exactly one positive eigenvalue on $\mathbb{R}_{>0}^n$ for all $\alpha \in \mathbb{N}^n$. By Lemma 4.4, f is $2(1 - \frac{1}{d})$ -Rayleigh, and hence, by Theorem 3.5, the support of f is M-convex. Therefore, by Theorem 5.1, f is Lorentzian.

Suppose f is a nonzero Lorentzian polynomial. Theorem 4.3 and Proposition 5.6 together show that $\log f$ is concave on $\mathbb{R}_{>0}^n$. Therefore, it is enough to prove that $(\sum_{i=1}^n a_i \partial_i) f$ is Lorentzian for any nonnegative numbers a_1, \dots, a_n . This is a direct consequence of Theorem 5.1 and Corollary 2.11. \square

Since L_n^d is the set of degree d Lorentzian polynomials in n variables, Corollary 4.5 shows that any degree d Lorentzian polynomial is $2(1 - \frac{1}{d})$ -Rayleigh. We show that the bound in Corollary 4.5 is optimal.

Proposition 5.7. When $n \leq 2$, all polynomials in L_n^d are 1-Rayleigh. When $n \geq 3$, we have

$$\left(\text{all polynomials in } L_n^d \text{ are } c\text{-Rayleigh} \right) \implies c \geq 2\left(1 - \frac{1}{d}\right).$$

In other words, for any $n \geq 3$ and any $c < 2(1 - \frac{1}{d})$, there is $f \in L_n^d$ that is not c -Rayleigh.

Proof. We first show by induction that, for any homogeneous bivariate polynomial $f = f(w_1, w_2)$ with nonnegative coefficients, we have

$$f(w) \left(\partial_1 \partial_2 f(w) \right) \leq \left(\partial_1 f(w) \right) \left(\partial_2 f(w) \right) \text{ for any } w \in \mathbb{R}_{\geq 0}^2.$$

We use the obvious fact that, for any homogeneous polynomial with nonnegative coefficients h ,

$$(\deg(h) + 1)h \geq (1 + w_i \partial_i)h \text{ for any } w \in \mathbb{R}_{\geq 0}^2.$$

Since f is bivariate, we may write $f = c_1 w_1^d + c_2 w_2^d + w_1 w_2 g$. We have

$$\begin{aligned} \partial_1 f \partial_2 f - f \partial_1 \partial_2 f &= d^2 c_1 c_2 w_1^{d-1} w_2^{d-1} \\ &\quad + d c_1 w_1^d (1 + w_2 \partial_2)g - c_1 w_1^d (1 + w_1 \partial_1)(1 + w_2 \partial_2)g \\ &\quad + d c_2 w_2^d (1 + w_1 \partial_1)g - c_2 w_2^d (1 + w_1 \partial_1)(1 + w_2 \partial_2)g \\ &\quad + w_1 w_2 (g + w_1 \partial_1 g)(g + w_2 \partial_2 g) - w_1 w_2 g (1 + w_1 \partial_1)(1 + w_2 \partial_2)g. \end{aligned}$$

The summand in the second line is nonnegative on $\mathbb{R}_{\geq 0}^2$ by the mentioned fact for $(1 + w_2 \partial_2)g$. The summand in the third line is nonnegative on $\mathbb{R}_{\geq 0}^2$ by the mentioned fact for $(1 + w_1 \partial_1)g$. The summand in the fourth line is nonnegative on $\mathbb{R}_{\geq 0}^2$ by the induction hypothesis applied to g .

We next show that, for any bivariate Lorentzian polynomial $f = f(w_1, w_2)$, we have

$$f(w) \left(\partial_1 \partial_1 f(w) \right) \leq \left(\partial_1 f(w) \right) \left(\partial_1 f(w) \right) \text{ for any } w \in \mathbb{R}_{\geq 0}^2.$$

Since f is homogeneous, it is enough to prove the inequality when $w_2 = 1$. In this case, the inequality follows from the concavity of the function $\log f$ restricted to the line $w_2 = 1$. This completes the proof that any bivariate Lorentzian polynomial is 1-Rayleigh.

To see the second statement, consider the polynomial

$$f = 2 \left(1 - \frac{1}{d} \right) w_1^d + w_1^{d-1} w_2 + w_1^{d-1} w_3 + w_1^{d-2} w_2 w_3.$$

It is straightforward to check that f is in L_n^d . If f is c -Rayleigh, then, for any $w \in \mathbb{R}_{\geq 0}^n$,

$$w_1^{2d-4} \left(2 \left(1 - \frac{1}{d} \right) w_1^2 + w_1 w_2 + w_1 w_3 + w_2 w_3 \right) \leq c w_1^{2d-4} (w_1 + w_2) (w_1 + w_3).$$

The lower bound of c is obtained by setting $w_1 = 1, w_2 = 0, w_3 = 0$. □

6. LINEAR OPERATORS PRESERVING LORENTZIAN POLYNOMIALS

We describe a large class of linear operators that preserve the Lorentzian property. An analog was achieved for the class of stable polynomials in [BB09, Theorem 2.2], where the linear operators preserving stability were characterized. For an element κ of \mathbb{N}^n , we set

$$\begin{aligned} \mathbb{R}_\kappa[w_i] &= \left\{ \text{polynomials in } \mathbb{R}[w_i]_{1 \leq i \leq n} \text{ of degree at most } \kappa_i \text{ in } w_i \text{ for every } i \right\}, \\ \mathbb{R}_\kappa^a[w_{ij}] &= \left\{ \text{multi-affine polynomials in } \mathbb{R}[w_{ij}]_{1 \leq i \leq n, 1 \leq j \leq \kappa_i} \right\}. \end{aligned}$$

The *projection operator* $\Pi_\kappa^\downarrow : \mathbb{R}_\kappa^a[w_{ij}] \rightarrow \mathbb{R}_\kappa[w_i]$ is the linear map that substitutes each w_{ij} by w_i :

$$\Pi_\kappa^\downarrow(g) = g|_{w_{ij}=w_i}.$$

The *polarization operator* $\Pi_\kappa^\uparrow : \mathbb{R}_\kappa[w_i] \rightarrow \mathbb{R}_\kappa^a[w_{ij}]$ is the linear map that sends w^α to the product

$$\frac{1}{\binom{\kappa}{\alpha}} \prod_{i=1}^n (\text{elementary symmetric polynomial of degree } \alpha_i \text{ in the variables } \{w_{ij}\}_{1 \leq j \leq \kappa_i}),$$

where $\binom{\kappa}{\alpha}$ stands for the product of binomial coefficients $\prod_{i=1}^n \binom{\kappa_i}{\alpha_i}$. Note that

- for every f , we have $\Pi_\kappa^\downarrow \circ \Pi_\kappa^\uparrow(f) = f$, and
- for every f and every i , the polynomial $\Pi_\kappa^\uparrow(f)$ is symmetric in the variables $\{w_{ij}\}_{1 \leq j \leq \kappa_i}$.

The above properties characterize Π_κ^\uparrow among the linear operators from $\mathbb{R}_\kappa[w_i]$ to $\mathbb{R}_\kappa^a[w_{ij}]$.

Proposition 6.1. The operators Π_κ^\downarrow and Π_κ^\uparrow preserve the Lorentzian property.

In other words, $\Pi_\kappa^\uparrow(f)$ is a Lorentzian polynomial for any Lorentzian polynomial $f \in \mathbb{R}_\kappa[w_i]$, and $\Pi_\kappa^\downarrow(g)$ is a Lorentzian polynomial for any Lorentzian polynomial $g \in \mathbb{R}_\kappa^a[w_{ij}]$.

Proof. The statement for Π_κ^\downarrow follows from Theorem 2.10. We prove the statement for Π_κ^\uparrow . It is enough to prove that $\Pi_\kappa^\uparrow(f)$ is Lorentzian when $f \in \mathring{L}_n^d \cap \mathbb{R}_\kappa[w_i]$ for $d \geq 2$.

Set $k = |\kappa|_1$, and identify \mathbb{N}^k with the set of all monomials in w_{ij} . Since $f \in \mathring{L}_n^d$, we have

$$\text{supp}(\Pi_\kappa^\uparrow(f)) = \begin{bmatrix} k \\ d \end{bmatrix},$$

which is clearly M-convex. Therefore, by Theorem 5.1, it remains to show that the quadratic form $\partial^\beta \Pi_\kappa^\uparrow(f)$ is stable for any $\beta \in \begin{bmatrix} k \\ d-2 \end{bmatrix}$.

Define α by the equality $\Pi_\kappa^\downarrow(w^\beta) = w^\alpha$. Note that, after renaming the variables if necessary, the β -th partial derivative of $\Pi_\kappa^\uparrow(f)$ is a positive multiple of a polarization of the α -th partial derivative of f :

$$\partial^\beta \Pi_\kappa^\uparrow(f) = \frac{(\kappa - \alpha)!}{\kappa!} \Pi_{\kappa - \alpha}^\uparrow(\partial^\alpha f).$$

Since the operator $\Pi_{\kappa - \alpha}^\uparrow$ preserves stability [BB09, Proposition 3.4], the conclusion follows from the stability of the quadratic form $\partial^\alpha f$. \square

Let κ be an element of \mathbb{N}^n , let γ be an element of \mathbb{N}^m , and set $k = |\kappa|_1$. In the remainder of this section, we fix a linear operator

$$T : \mathbb{R}_\kappa[w_i] \rightarrow \mathbb{R}_\gamma[w_i],$$

and suppose that the linear operator T is *homogeneous of degree ℓ* for some $\ell \in \mathbb{Z}$:

$$(0 \leq \alpha \leq \kappa \text{ and } T(w^\alpha) \neq 0) \implies \deg T(w^\alpha) = \deg w^\alpha + \ell.$$

The *symbol* of T is a homogeneous polynomial of degree $k + \ell$ in $m + n$ variables defined by

$$g_T(w, u) = \sum_{0 \leq \alpha \leq \kappa} \binom{\kappa}{\alpha} T(w^\alpha) u^{\kappa - \alpha}.$$

We show that T preserves the Lorentzian property if its symbol g_T is Lorentzian.

Theorem 6.2. If $g_T \in L_{m+n}^{k+\ell}$ and $f \in L_n^d \cap \mathbb{R}_\kappa[w_i]$, then $T(f) \in L_m^{d+\ell}$.

When $n = 2$, Theorem 6.2 provides a large class of linear operators that preserve the ultra log-concavity of sequences of nonnegative numbers with no internal zeros. We prepare the proof of Theorem 6.2 with a special case.

Lemma 6.3. Let $T = T_{w_1, w_2} : \mathbb{R}_{(1, \dots, 1)}[w_i] \rightarrow \mathbb{R}_{(1, \dots, 1)}[w_i]$ be the linear operator defined by

$$T(w^S) = \begin{cases} w^{S \setminus 1} & \text{if } 1 \in S \text{ and } 2 \in S, \\ w^{S \setminus 1} & \text{if } 1 \in S \text{ and } 2 \notin S, \\ w^{S \setminus 2} & \text{if } 1 \notin S \text{ and } 2 \in S, \\ 0 & \text{if } 1 \notin S \text{ and } 2 \notin S, \end{cases} \quad \text{for all } S \subseteq [n].$$

Then T preserves the Lorentzian property.

Proof. It is enough to prove that $T(f) \in L_n^d$ when $f \in \mathring{L}_n^{d+1} \cap \mathbb{R}_{(1, \dots, 1)}[w_i]$ for $d \geq 2$. In this case,

$$\text{supp}(T(f)) = \{d\text{-element subsets of } [n] \text{ not containing } 1\},$$

which is clearly M-convex. Therefore, by Theorem 5.1, it suffices to show that the quadratic form $\partial^S T(f)$ is stable for any $S \in \binom{[n]}{d-2}$ not containing 1. We write h for the Lorentzian polynomial $f|_{w_1=0}$. Since f is multi-affine, we have

$$f = h + w_1 \partial_1 f \quad \text{and} \quad T(f) = \partial_2 h + \partial_1 f.$$

We give separate arguments when $2 \in S$ and $2 \notin S$. If S contains 2, then

$$\partial^S T(f) = \partial^{S \cup 1} f,$$

and hence $\partial^S T(f)$ is stable. If S does not contain 2, then

- the linear form $\partial^S \partial_1 \partial_2 f = \partial^{S \cup 1 \cup 2} f$ is not identically zero, because $f \in \mathring{L}_n^{d+1}$,
- we have $\partial^{S \cup 1 \cup 2} f < \partial^{S \cup 2} h$, because $\partial^{S \cup 2} f$ is stable, and
- we have $\partial^{S \cup 1 \cup 2} f < \partial^{S \cup 1} f$, by Lemma 2.9 (1).

Therefore, by Lemma 2.9 (4), the quadratic form $\partial^S T(f) = \partial^{S \cup 2} h + \partial^{S \cup 1} f$ is stable. \square

Proof of Theorem 6.2. The polarization of $T : \mathbb{R}_\kappa[w_i] \rightarrow \mathbb{R}_\gamma[w_i]$ is the operator $\Pi^\uparrow(T)$ defined by

$$\Pi^\uparrow(T) = \Pi_\gamma^\uparrow \circ T \circ \Pi_\kappa^\downarrow.$$

We write $\gamma \oplus \kappa$ for the concatenation of γ and κ in \mathbb{N}^{m+n} . By [BB09, Lemma 3.5], the symbol of the polarization is the polarization of the symbol⁶:

$$g_{\Pi^\uparrow(T)} = \Pi_{\gamma \oplus \kappa}^\uparrow(g_T).$$

Therefore, by Proposition 6.1, the proof reduces to the case $\kappa = (1, \dots, 1)$ and $\gamma = (1, \dots, 1)$.

⁶The statement was proved in [BB09, Lemma 3.5] when $m = n$. Clearly, this special case implies the general case.

Suppose $f(v)$ is a multi-affine polynomial in L_n^d and $g_T(w, u)$ is a multi-affine polynomial in $L_{m+n}^{\ell+n}$. Since the product of Lorentzian polynomials is Lorentzian by Corollary 5.5, we have

$$g_T(w, u)f(v) = \sum_{S \subseteq [n]} T(w^S)u^{[n] \setminus S} f(v) \in L_{m+n+n}^{d+\ell+n}.$$

Applying the operator in Lemma 6.3 for the pair of variables (u_i, v_i) for $i = 1, \dots, n$, we have

$$\prod_{i=1}^n T_{u_i, v_i}(g_T(w, u)f(v)) = \sum_{S \subseteq [n]} T(w^S)(\partial^S f)(v) \in L_{m+n}^{d+\ell}.$$

We substitute every v_i by zero in the displayed equation to get

$$\left[\sum_{S \subseteq [n]} T(w^S)(\partial^S f)(v) \right]_{v=0} = T(f(w)).$$

Theorem 2.10 shows that the right-hand side belongs to $L_m^{d+\ell}$, completing the proof. \square

We remark that there are homogeneous linear operators T preserving the Lorentzian property whose symbols are not Lorentzian. This contrasts the analog of Theorem 6.2 for stable polynomials [BB09, Theorem 2.2]. As an example, consider the linear operator $T : \mathbb{R}_{(1,1)}[w_1, w_2] \rightarrow \mathbb{R}_{(1,1)}[w_1, w_2]$ defined by

$$T(1) = 0, \quad T(w_1) = w_1, \quad T(w_2) = w_2, \quad T(w_1 w_2) = w_1 w_2.$$

The symbol of T is not Lorentzian because its support is not M-convex. The operator T preserves Lorentzian polynomials but does not preserve (non-homogeneous) stable polynomials.

Theorem 6.4. If T is a homogeneous linear operator that preserves stable polynomials and polynomials with nonnegative coefficients, then T preserves Lorentzian polynomials.

Proof. According to [BB09, Theorem 2.2], T preserves stable polynomials if and only if either

- (I) the rank of T is not greater than two and T is of the form

$$T(f) = \alpha(f)P + \beta(f)Q,$$

where α, β are linear functionals and P, Q are stable polynomials satisfying $P < Q$,

- (II) the polynomial $g_T(w, u)$ is stable, or

- (III) the polynomial $g_T(w, -u)$ is stable.

Suppose one of the three conditions, and suppose in addition that T preserves polynomials with nonnegative coefficients.

Suppose (I) holds. In this case, the image of T is contained in the set of stable polynomials [BB10, Theorem 1.6]. By Proposition 2.2, homogeneous stable polynomials with nonnegative coefficients are Lorentzian. Since T preserves polynomials with nonnegative coefficients, $T(f)$ is Lorentzian whenever f is a homogeneous polynomial with nonnegative coefficients.

Suppose (II) holds. Since T preserves polynomials with nonnegative coefficients, $g_T(w, u)$ is Lorentzian by Proposition 2.2. Therefore, by Theorem 6.2, $T(f)$ is Lorentzian whenever f is Lorentzian.

Suppose (III) holds. Since all the nonzero coefficients of a homogeneous stable polynomial have the same sign [COSW04, Theorem 6.1], we have

$$g_T(w, -v) = g_T(w, v) \text{ or } g_T(w, -v) = -g_T(w, v).$$

In both cases, $g_T(w, v)$ is stable and has nonnegative coefficients. Thus $g_T(w, v)$ is Lorentzian, and the conclusion follows from Theorem 6.2. \square

The *multi-affine part* of a polynomial $\sum_{\alpha \in \mathbb{N}^n} c_\alpha w^\alpha$ is the polynomial $\sum_{\alpha \in \{0,1\}^n} c_\alpha w^\alpha$.

Corollary 6.5. The multi-affine part of any Lorentzian polynomial is a Lorentzian polynomial.

Proof. Clearly, taking the multi-affine part is a homogeneous linear operator that preserves polynomials with nonnegative coefficients. Since this operator also preserves stable polynomials [COSW04, Proposition 4.17], the proof follows from Theorem 6.4. \square

Remark 6.6. Corollary 6.5 can be used to obtain a multi-affine analog of Remark 2.14. Write \underline{H}_n^d for the space of multi-affine degree d homogeneous polynomials in n variables, and write \underline{L}_n^d for the corresponding set of multi-affine Lorentzian polynomials. Let $\mathbb{P}\underline{H}_n^d$ be the projectivization of the vector space \underline{H}_n^d , and let \underline{L}_B be the set of polynomials in \underline{L}_n^d with support B . We identify a rank d matroid M on $[n]$ with its set of bases $B \subseteq \binom{[n]}{d}$. Writing $\mathbb{P}\underline{L}_n^d$ and $\mathbb{P}\underline{L}_B$ for the images of $\underline{L}_n^d \setminus \{0\}$ and \underline{L}_B in $\mathbb{P}\underline{H}_n^d$ respectively, we have

$$\mathbb{P}\underline{L}_n^d = \bigsqcup_B \mathbb{P}\underline{L}_B,$$

where the union is over all rank d matroids on $[n]$. By Theorem 2.10 and Corollary 6.5, $\mathbb{P}\underline{L}_n^d$ is a compact contractible subset of $\mathbb{P}\underline{H}_n^d$. By Theorem 7.1, $\mathbb{P}\underline{L}_B$ is nonempty for every matroid $B \subseteq \binom{[n]}{d}$. In addition, by Proposition 8.12, $\mathbb{P}\underline{L}_B$ is contractible for every matroid $B \subseteq \binom{[n]}{d}$.

The *symmetric exclusion process* is one of the main models considered in interacting particle systems. It is a continuous time Markov chain which models particles that jump symmetrically between sites, where each site may be occupied by at most one particle [Lig10]. A problem that has attracted much attention is to find negative dependence properties that are preserved under the symmetric exclusion process. In [BBL09, Theorem 4.20], it was proved that strongly Rayleigh measures are preserved under the symmetric exclusion process. In other words, if $f = f(w_1, w_2, \dots, w_n)$ is a stable multi-affine polynomial with nonnegative coefficients, then the multi-affine polynomial $\Phi_\theta^{1,2}(f)$ defined by

$$\Phi_\theta^{1,2}(f) = (1 - \theta)f(w_1, w_2, w_3, \dots, w_n) + \theta f(w_2, w_1, w_3, \dots, w_n)$$

is stable for all $0 \leq \theta \leq 1$. We prove an analog for Lorentzian polynomials.

Theorem 6.7. Let $f = f(w_1, w_2, \dots, w_n)$ be a multi-affine polynomial with nonnegative coefficients. If the homogenization of f is a Lorentzian polynomial, then the homogenization of $\Phi_\theta^{1,2}(f)$ is a Lorentzian polynomial for all $0 \leq \theta \leq 1$.

Proof. Recall that a polynomial with nonnegative coefficients is stable if and only if its homogenization is stable [BBL09, Theorem 4.5]. Clearly, $\Phi_\theta^{1,2}$ is homogeneous and preserves polynomials with nonnegative coefficients. Since $\Phi_\theta^{1,2}$ preserves stability of multi-affine polynomials by [BBL09, Theorem 4.20], the statement follows from Theorem 6.4. \square

7. MATROIDS, M-CONVEX SETS, AND LORENTZIAN POLYNOMIALS

The *generating function* of a subset $J \subseteq \mathbb{N}^n$ is, by definition,

$$f_J = \sum_{\alpha \in J} \frac{w^\alpha}{\alpha!}, \text{ where } \alpha! = \prod_{i=1}^n \alpha_i!.$$

We characterize matroids and M-convex sets in terms of their generating functions.

Theorem 7.1. The following are equivalent for any nonempty $J \subseteq \mathbb{N}^n$.

- (1) There is a Lorentzian polynomial whose support is J .
- (2) There is a homogeneous 2-Rayleigh polynomial whose support is J .
- (3) There is a homogeneous c -Rayleigh polynomial whose support is J for some $c > 0$.
- (4) The generating function f_J is a Lorentzian polynomial.
- (5) The generating function f_J is a homogeneous 2-Rayleigh polynomial.
- (6) The generating function f_J is a homogeneous c -Rayleigh polynomial for some $c > 0$.
- (7) J is M-convex.

When $J \subseteq \{0, 1\}^n$, any one of the above conditions is equivalent to

- (8) J is the set of bases of a matroid on $[n]$.

The implication (8) \Rightarrow (4) goes back to [HW17, Remark 15]. See also [AOV, Theorem 4.2] and [HSW18, Section 3]. The equivalence (7) \Leftrightarrow (4) will be generalized to M-convex functions in Theorem 8.2.

We prepare the proof of Theorem 7.1 with an analysis of the quadratic case.

Lemma 7.2. The following conditions are equivalent for any $n \times n$ symmetric matrix A with entries in $\{0, 1\}$.

- (1) The quadratic polynomial $w^T A w$ is Lorentzian.
- (2) The support of the quadratic polynomial $w^T A w$ is M-convex.

Proof. Theorem 5.1 implies (1) \Rightarrow (2). We prove (2) \Rightarrow (1). We may and will suppose that no column of A is zero. Let J be the M-convex support of $w^T A w$, and set

$$S = \left\{ i \in [n] \mid 2e_i \in J \right\}.$$

The exchange property for J shows that $e_i + e_j \in J$ for every $i \in S$ and $j \in [n]$. In addition, again by the exchange property for J ,

$$B := \left\{ e_i + e_j \in J \mid i \notin S \text{ and } j \notin S \right\}$$

is the set of bases of a rank 2 matroid on $[n] \setminus S$ without loops. Writing $S_1 \cup \dots \cup S_k$ for the decomposition of $[n] \setminus S$ into parallel classes in the matroid, we have

$$w^T A w = \left(\sum_{j \in [n]} w_j \right)^2 - \left(\sum_{j \in S_1} w_j \right)^2 - \dots - \left(\sum_{j \in S_k} w_j \right)^2,$$

and hence $w^T A w$ is a Lorentzian polynomial. \square

Proof of Theorem 7.1. Theorem 3.5, Theorem 5.1, and Corollary 4.5 show that

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (7) \text{ and } (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (7).$$

Since (4) \Rightarrow (1), we only need to prove (7) \Rightarrow (4).

If J is an M-convex subset of \mathbb{N}^n , then f_J is a homogeneous polynomial of some degree d . Suppose $d \geq 2$, and let α be an element of Δ_n^{d-2} . Note that, in general, the support of $\partial^\alpha f_J$ is M-convex whenever the support of f_J is M-convex. Therefore, $\partial^\alpha f_J$ is Lorentzian by Lemma 7.2, and hence f_J is Lorentzian by Theorem 5.1. \square

Let J be the set of bases of a matroid M on $[n]$. If M is regular [FM92], if M is representable over the finite fields \mathbb{F}_3 and \mathbb{F}_4 [COSW04], if the rank of M is at most 3 [Wag05], or if the number of elements n is at most 7 [Wag05], then f_J is 1-Rayleigh. Seymour and Welsh found the first example of a matroid whose basis generating function is not 1-Rayleigh [SW75]. We propose the following improvement of Theorem 7.1.

Conjecture 7.3. The following conditions are equivalent for any nonempty $J \subseteq \{0, 1\}^n$.

- (1) J is the set of bases of a matroid on $[n]$.
- (2) The generating function f_J is a homogeneous $\frac{8}{7}$ -Rayleigh polynomial.

The constant $\frac{8}{7}$ is best possible: For any positive real number $c < \frac{8}{7}$, there is a matroid whose basis generating function is not c -Rayleigh [HSW18, Theorem 7].

8. VALUATED MATROIDS, M-CONVEX FUNCTIONS, AND LORENTZIAN POLYNOMIALS

8.1. Let ν be a function from \mathbb{N}^n to $\mathbb{R} \cup \{\infty\}$. The *effective domain* of ν is, by definition,

$$\text{dom}(\nu) = \left\{ \alpha \in \mathbb{N}^n \mid \nu(\alpha) < \infty \right\}.$$

The function ν is said to be *M-convex* if satisfies the *symmetric exchange property*:

(1) For any $\alpha, \beta \in \text{dom}(\nu)$ and any i satisfying $\alpha_i > \beta_i$, there is j satisfying

$$\alpha_j < \beta_j \text{ and } \nu(\alpha) + \nu(\beta) \geq \nu(\alpha - e_i + e_j) + \nu(\beta - e_j + e_i).$$

Note that the effective domain of an M-convex function on \mathbb{N}^n is an M-convex subset of \mathbb{N}^n . In particular, the effective domain of an M-convex function on \mathbb{N}^n is contained in Δ_n^d for some d . In this case, we identify ν with its restriction to Δ_n^d . When the effective domain of ν is M-convex, the symmetric exchange property for ν is equivalent to the following *local exchange property*:

(2) For any $\alpha, \beta \in \text{dom}(\nu)$ with $|\alpha - \beta|_1 = 4$, there are i and j satisfying

$$\alpha_i > \beta_i, \alpha_j < \beta_j \text{ and } \nu(\alpha) + \nu(\beta) \geq \nu(\alpha - e_i + e_j) + \nu(\beta - e_j + e_i).$$

A proof of the equivalence of the two exchange properties can be found in [Mur03, Section 6.2].

Example 8.1. The indicator function of $J \subseteq \mathbb{N}^n$ is the function $\nu_J : \mathbb{N}^n \rightarrow \mathbb{R} \cup \{\infty\}$ defined by

$$\nu_J(\alpha) = \begin{cases} 0 & \text{if } \alpha \in J, \\ \infty & \text{if } \alpha \notin J. \end{cases}$$

Clearly, $J \subseteq \mathbb{N}^n$ is M-convex if and only if the indicator function ν_J is M-convex.

A function $\nu : \mathbb{N}^n \rightarrow \mathbb{R} \cup \{-\infty\}$ is said to be *M-concave* if $-\nu$ is M-convex. The *effective domain* of an M-concave function is

$$\text{dom}(\nu) = \left\{ \alpha \in \mathbb{N}^n \mid \nu(\alpha) > -\infty \right\}.$$

A *valuated matroid* on $[n]$ is an M-concave function on \mathbb{N}^n whose effective domain is a nonempty subset of $\{0, 1\}^n$. The effective domain of a valuated matroid ν on $[n]$ is the set of bases of a matroid on $[n]$, the *underlying matroid* of ν .

In this section, we prove that the class of tropicalized Lorentzian polynomials coincides with the class of M-convex functions. The tropical connection is used to produce Lorentzian polynomials from M-convex functions. First, we state a classical version of the result. For any function $\nu : \Delta_n^d \rightarrow \mathbb{R} \cup \{\infty\}$ and a positive real number q , we define

$$f_q^\nu(w) = \sum_{\alpha \in \text{dom}(\nu)} \frac{q^{\nu(\alpha)}}{\alpha!} w^\alpha \text{ and } g_q^\nu(w) = \sum_{\alpha \in \text{dom}(\nu)} \binom{\delta}{\alpha} q^{\nu(\alpha)} w^\alpha,$$

where $\delta = (d, \dots, d)$ and $\binom{\delta}{\alpha}$ is the product of binomial coefficients $\prod_{i=1}^n \binom{d}{\alpha_i}$. When ν is the indicator function of $J \subseteq \mathbb{N}^n$, the polynomial f_q^ν is independent of q and equal to the generating function f_J considered in Section 7.

Theorem 8.2. The following conditions are equivalent for $\nu : \Delta_n^d \rightarrow \mathbb{R} \cup \{\infty\}$.

- (1) The function ν is M-convex.
- (2) The polynomial $f_q^\nu(w)$ is Lorentzian for all $0 < q \leq 1$.
- (3) The polynomial $g_q^\nu(w)$ is Lorentzian for all $0 < q \leq 1$.

Theorem 8.2 provides a useful sufficient condition for a homogeneous polynomial to be Lorentzian. Let f be an arbitrary homogeneous polynomial with nonnegative real coefficients written in the normalized form

$$f = \sum_{\alpha \in \Delta_n^d} \frac{c_\alpha}{\alpha!} w^\alpha.$$

We define a discrete function ν_f using natural logarithms of the normalized coefficients:

$$\nu_f : \Delta_n^d \longrightarrow \mathbb{R} \cup \{-\infty\}, \quad \alpha \longmapsto \log(c_\alpha).$$

Corollary 8.3. If ν_f is an M-concave function, then f is a Lorentzian polynomial.

Proof. By Theorem 8.2, the polynomial $\sum_{\alpha \in \text{dom}(\nu_f)} \frac{q^{-\nu_f(\alpha)}}{\alpha!} w^\alpha$ is Lorentzian when $q = e^{-1}$. \square

We note that the converse of Corollary 8.3 does not hold. For example, the polynomial

$$f = \prod_{i=1}^{n-1} (w_i + w_n)$$

is Lorentzian, being a product of Lorentzian polynomials. However, ν_f fails to be M-concave when $n \geq 2$.

8.2. We formulate a tropical counterpart of Theorem 8.2. Let $\mathbb{C}((t))_{\text{conv}}$ be the field of Laurent series with complex coefficients that have a positive radius of convergence around 0. By definition, any nonzero element of $\mathbb{C}((t))_{\text{conv}}$ is a series of the form

$$s(t) = c_1 t^{a_1} + c_2 t^{a_2} + c_3 t^{a_3} + \dots,$$

where c_1, c_2, \dots are nonzero complex numbers and $a_1 < a_2 < \dots$ are integers, that converges on a punctured open disk centered at 0. Let $\mathbb{R}((t))_{\text{conv}}$ be the subfield of elements that have real coefficients. We define the fields of real and complex Puiseux series by

$$\mathbb{K} = \bigcup_{k \geq 1} \mathbb{R}((t^{1/k}))_{\text{conv}} \quad \text{and} \quad \overline{\mathbb{K}} = \bigcup_{k \geq 1} \mathbb{C}((t^{1/k}))_{\text{conv}}.$$

Any nonzero element of $\overline{\mathbb{K}}$ is a series of the form

$$s(t) = c_1 t^{a_1} + c_2 t^{a_2} + c_3 t^{a_3} + \dots,$$

where c_1, c_2, \dots are nonzero complex numbers and $a_1 < a_2 < \dots$ are rational numbers that have a common denominator. The *leading coefficient* of $s(t)$ is c_1 , and the *leading exponent* of $s(t)$

is a_1 . A nonzero element of \mathbb{K} is *positive* if its leading coefficient is positive. The *valuation map* is the function

$$\text{val} : \overline{\mathbb{K}} \longrightarrow \mathbb{R} \cup \{\infty\},$$

that takes the zero element to ∞ and a nonzero element to its leading exponent. For a nonzero element $s(t) \in \mathbb{K}$, we have

$$\text{val}(s(t)) = \lim_{t \rightarrow 0^+} \log_t(s(t)).$$

Since the theory of real closed fields has quantifier elimination [Mar02, Section 3.3], for any first-order formula $\varphi(x_1, \dots, x_m)$ in the language of ordered fields and any $s_1(t), \dots, s_m(t) \in \mathbb{K}$, we have

$$\begin{aligned} \left(\varphi(s_1(t), \dots, s_m(t)) \text{ holds in } \mathbb{K} \right) &\iff \\ \left(\varphi(s_1(q), \dots, s_m(q)) \text{ holds in } \mathbb{R} \text{ for all sufficiently small positive real numbers } q \right). \end{aligned}$$

In particular, Tarski's principle holds for \mathbb{K} : A first-order sentence in the language of ordered fields holds in \mathbb{K} if and only if it holds in \mathbb{R} .

Definition 8.4. Let $f_t = \sum_{\alpha \in \Delta_n^d} s_\alpha(t) w^\alpha$ be a nonzero homogeneous polynomial with coefficients in $\mathbb{K}_{\geq 0}$. The *tropicalization* of f_t is the discrete function defined by

$$\text{trop}(f_t) : \Delta_n^d \longrightarrow \mathbb{R} \cup \{\infty\}, \quad \alpha \longmapsto \text{val}(s_\alpha(t)).$$

We say that f_t is *log-concave on $\mathbb{K}_{>0}^n$* if the function $\log(f_q)$ is concave on $\mathbb{R}_{>0}^n$ for all sufficiently small positive real numbers q .

Note that the support of f_t is the effective domain of the tropicalization of f_t . We write $M_n^d(\mathbb{K})$ for the set of all degree d homogeneous polynomials in $\mathbb{K}_{\geq 0}[w_1, \dots, w_n]$ whose support is M-convex.

Definition 8.5 (Lorentzian polynomials over \mathbb{K}). We set $L_n^0(\mathbb{K}) = M_n^0(\mathbb{K})$, $L_n^1(\mathbb{K}) = M_n^1(\mathbb{K})$, and

$$L_n^2(\mathbb{K}) = \left\{ f_t \in M_n^2(\mathbb{K}) \mid \text{The Hessian of } f_t \text{ has at most one eigenvalue in } \mathbb{K}_{>0} \right\}.$$

For $d \geq 3$, we define $L_n^d(\mathbb{K})$ by setting

$$L_n^d(\mathbb{K}) = \left\{ f_t \in M_n^d(\mathbb{K}) \mid \partial^\alpha f_t \in L_n^2(\mathbb{K}) \text{ for all } \alpha \in \Delta_n^{d-2} \right\}.$$

The polynomials in $L_n^d(\mathbb{K})$ will be called *Lorentzian*.

By Proposition 5.6, the log-concavity of homogeneous polynomials can be expressed in the first-order language of ordered fields. It follows that the analog of Theorem 5.3 holds for any homogeneous polynomial f_t with coefficients in $\mathbb{K}_{\geq 0}$.

Theorem 8.6. The following conditions are equivalent for f_t .

(1) For any $m \in \mathbb{N}$ and any $m \times n$ matrix (a_{ij}) with entries in $\mathbb{K}_{\geq 0}$,

$$\left(\prod_{i=1}^m D_i\right) f_t \text{ is identically zero or } \left(\prod_{i=1}^m D_i\right) f_t \text{ is log-concave on } \mathbb{K}_{>0}^n,$$

where D_i is the differential operator $\sum_{j=1}^n a_{ij} \partial_j$.

(2) For any $\alpha \in \mathbb{N}^n$, the polynomial $\partial^\alpha f_t$ is identically zero or log-concave on $\mathbb{K}_{>0}^n$.

(3) The polynomial f_t is Lorentzian.

The field \mathbb{K} is real closed, and the field $\overline{\mathbb{K}}$ is algebraically closed [Spe05, Section 1.5]. Any element $s(t)$ of $\overline{\mathbb{K}}$ can be written as a sum

$$s(t) = p(t) + i q(t),$$

where $p(t) \in \mathbb{K}$ is the *real part* of $s(t)$ and $q(t) \in \mathbb{K}$ is the *imaginary part* of $s(t)$. The *open upper half plane* in $\overline{\mathbb{K}}$ is the set of elements in $\overline{\mathbb{K}}$ with positive imaginary parts. A polynomial f_t in $\mathbb{K}[w_1, \dots, w_n]$ is *stable* if f_t is non-vanishing on $\mathcal{H}_{\overline{\mathbb{K}}}^n$ or identically zero, where $\mathcal{H}_{\overline{\mathbb{K}}}$ is the open upper half plane in $\overline{\mathbb{K}}$. According to [Brä10, Theorem 4], tropicalizations of homogeneous stable polynomials over \mathbb{K} are M-convex functions.⁷ Here we prove that tropicalizations of Lorentzian polynomials over \mathbb{K} are M-convex, and that *all* M-convex functions are limits of tropicalizations of Lorentzian polynomials over \mathbb{K} .⁸

Theorem 8.7. The following conditions are equivalent for any function $\nu : \Delta_n^d \rightarrow \mathbb{Q} \cup \{\infty\}$.

- (i) The function ν is M-convex.
- (ii) There is a Lorentzian polynomial in $\mathbb{K}[w_1, \dots, w_n]$ whose tropicalization is ν .

Let M be a matroid with the set of bases B . The *Dressian* of M , denoted $\text{Dr}(M)$, is the tropical variety in \mathbb{R}^B obtained by intersecting the tropical hypersurfaces of the Plücker relations in \mathbb{R}^B [MS15, Section 4.4]. Since $\text{Dr}(M)$ is a rational polyhedral fan whose points bijectively correspond to the valuated matroids with underlying matroid M , Theorem 8.7 shows that

$$\text{Dr}(M) = \text{closure} \left\{ -\pi \text{ trop}(f_t) \mid f_t \text{ is a Lorentzian polynomial with } \text{supp}(f_t) = B \right\},$$

where π is the projection onto \mathbb{R}^B . We note that the corresponding statement for stable polynomials fails to hold. For example, when M is the Fano plane, there is no stable polynomial whose support is B [Brä07, Section 6].

⁷In [Brä10], the field of formal Puiseux series with real exponents $\mathbb{R}\{t\}$ containing \mathbb{K} was used. The tropicalization used in [Brä10] differs from ours by a sign.

⁸If $\mathbb{R}\{t\}$ is used instead of \mathbb{K} , then all M-convex functions are tropicalizations of Lorentzian polynomials. More precisely, a discrete function ν with values in $\mathbb{R} \cup \{\infty\}$ is M-convex if and only if there is a Lorentzian polynomial over $\mathbb{R}\{t\}$ whose tropicalization is ν . In this setting, the Dressian of a matroid M can be identified with the set of tropicalized Lorentzian polynomials f_t with $\text{supp}(f_t) = B$, where B is the set of bases of M .

8.3. The proofs of Theorems 8.2 and 8.7 take a path through the theory of phylogenetic trees and the problem of isometric embeddings of metric spaces in Hilbert spaces. A *phylogenetic tree* with n leaves is a tree with n labelled leaves and no vertices of degree 2. A function $d : \begin{bmatrix} n \\ 2 \end{bmatrix} \rightarrow \mathbb{R}$ is a *tree distance* if there is a phylogenetic tree τ with n leaves and edge weights $\ell_e \in \mathbb{R}$ such that

$$d(i, j) = \left(\text{the sum of all } \ell_e \text{ along the unique path in } \tau \text{ joining the leaves } i \text{ and } j \right).$$

The *space of phylogenetic trees* \mathcal{T}_n is the set of all tree distances in $\mathbb{R}^{\binom{n}{2}}$. The Fundamental Theorem of Phylogenetics shows that

$$\mathcal{T}_n = \text{Dr}(2, n),$$

where $\text{Dr}(2, n)$ is the Dressian of the rank 2 uniform matroid on $[n]$ [MS15, Section 4.3].

We give a spectral characterization of tree distances. For any function $d : \begin{bmatrix} n \\ 2 \end{bmatrix} \rightarrow \mathbb{R}$ and any positive real number q , we define an $n \times n$ symmetric matrix $H_q(d)$ by

$$H_q(d)_{ij} = \begin{cases} 0 & \text{if } i = j, \\ q^{d(i,j)} & \text{if } i \neq j. \end{cases}$$

We say that an $n \times n$ symmetric matrix H is *conditionally negative definite* if

$$(1, \dots, 1)w = 0 \implies w^T H w \leq 0.$$

Basic properties of conditionally negative definite matrices are collected in [BR97, Chapter 4].

Lemma 8.8. The following conditions are equivalent for any function $d : \begin{bmatrix} n \\ 2 \end{bmatrix} \rightarrow \mathbb{R}$.

- (1) The matrix $H_q(d)$ is conditionally negative semidefinite for all $q \geq 1$.
- (2) The matrix $H_q(d)$ has exactly one positive eigenvalue for all $q \geq 1$.
- (3) The function d is a tree distance.

Lemma 8.8 is closely linked to the problem of isometric embeddings of ultrametric spaces in Hilbert spaces. Let d be a metric on $[n]$. Since $d(i, i) = 0$ and $d(i, j) = d(j, i)$ for all i , we may identify d with a function $\begin{bmatrix} n \\ 2 \end{bmatrix} \rightarrow \mathbb{R}$. We define an $n \times n$ symmetric matrix $E(d)$ by

$$E(d)_{ij} = d(i, j)^2.$$

We say that d admits an *isometric embedding* into \mathbb{R}^m if there is $\phi : [n] \rightarrow \mathbb{R}^m$ such that

$$d(i, j) = \|\phi(i) - \phi(j)\|_2 \text{ for all } i, j \in [n],$$

where $\|\cdot\|_2$ is the standard Euclidean norm on \mathbb{R}^m . The following theorem of Schoenberg [Sch38] characterizes metrics on $[n]$ that admit an isometric embedding into some \mathbb{R}^m .

Theorem 8.9. A metric d on $[n]$ admits an isometric embedding into some \mathbb{R}^m if and only if the matrix $E(d)$ is conditionally negative semidefinite.

Recall that an *ultrametric* on $[n]$ is a metric d on $[n]$ such that

$$d(i, j) \leq \max \{d(i, k), d(j, k)\} \text{ for any } i, j, k \in [n].$$

Equivalently, d is an ultrametric if the maximum of $d(i, j), d(i, k), d(j, k)$ is attained at least twice for any $i, j, k \in [n]$. Any ultrametric is a tree distance given by a phylogenetic tree [MS15, Section 4.3]. In [TV83], Timan and Vestfrid proved that any separable ultrametric space is isometric to a subspace of ℓ_2 . We use the following special case.

Theorem 8.10. Any ultrametric on $[n]$ admits an isometric embedding into \mathbb{R}^{n-1} .

Proof of Lemma 8.8. Cauchy's interlacing theorem shows (1) \Rightarrow (2). We prove (2) \Rightarrow (3). We may suppose that d takes rational values. If (2) holds, then the quadratic polynomial $w^T H_q(d) w$ is stable for all $q \geq 1$. Therefore, by the quantifier elimination for the theory of real closed fields, the quadratic form

$$\sum_{i < j} t^{-d(i, j)} w_i w_j \in \mathbb{K}[w_1, \dots, w_n]$$

is stable. By [Brä10, Theorem 4], tropicalizations of stable polynomials are M-convex⁹, and hence the function $-d$ is M-convex. In other words, we have

$$d \in \text{Dr}(2, n) = \mathcal{T}_n.$$

For (3) \Rightarrow (1), we first consider the special case when d is an ultrametric on $[n]$. In this case, q^d is also an ultrametric on $[n]$ for all $q \geq 1$. It follows from Theorems 8.9 and 8.10 that $H_q(d)$ is conditionally negative semidefinite for all $q \geq 1$. In the general case, we use that \mathcal{T}_n is the sum of its linearity space with the space of ultrametries on $[n]$ [MS15, Lemma 4.3.9]. Thus, for any tree distance d on $[n]$, there is an ultrametric \underline{d} on $[n]$ and real numbers c_1, \dots, c_n such that

$$d = \underline{d} + \sum_{i=1}^n c_i \left(\sum_{i \neq j} e_{ij} \right) \in \mathbb{R}^{\binom{n}{2}}.$$

Therefore, the symmetric matrix $H_q(d)$ is congruent to $H_q(\underline{d})$, and the conclusion follows from the case of ultrametries. \square

8.4. We start the proof of Theorems 8.2 and 8.7 with a linear algebraic lemma. Let (a_{ij}) be an $n \times n$ symmetric matrix with entries in $\mathbb{R}_{>0}$.

Lemma 8.11. If (a_{ij}) has exactly one positive eigenvalue, then (a_{ij}^p) has exactly one positive eigenvalue for $0 \leq p \leq 1$.

Proof. If (v_i) is the Perron eigenvector of (a_{ij}) , then $(\frac{a_{ij}}{v_i v_j})$ is conditionally negative definite [BR97, Lemma 4.4.1]. Therefore, $(\frac{a_{ij}^p}{v_i^p v_j^p})$ is conditionally negative definite [BCR84, Corollary 2.10], and the conclusion follows. \square

⁹The tropicalization used in [Brä10] differs from ours by a sign.

Let f be a degree d homogeneous polynomial written in the normalized form

$$f = \sum_{\alpha \in \text{supp}(f)} \frac{c_\alpha}{\alpha!} w^\alpha.$$

For any nonnegative real number p , we define

$$R_p(f) = \sum_{\alpha \in \text{supp}(f)} \frac{c_\alpha^p}{\alpha!} w^\alpha.$$

We use Lemma 8.11 to construct a homotopy from any Lorentzian polynomial to the generating function of its support. The following proposition was proved in [ALOVII] for strongly log-concave multi-affine polynomials.

Proposition 8.12. If f is Lorentzian, then $R_p(f)$ is Lorentzian for all $0 \leq p \leq 1$.

Proof. Using the characterization of Lorentzian polynomials in Theorem 5.1, the proof reduces to the case of quadratic polynomials. Using Theorem 2.10, the proof further reduces to the case $f \in P_n^2$. In this case, the assertion is Lemma 8.11. \square

Set $m = nd$, and let $\nu : \Delta_n^d \rightarrow \mathbb{R} \cup \{\infty\}$ and $\mu : \Delta_m^d \rightarrow \mathbb{R} \cup \{\infty\}$ be arbitrary functions. Write e_{ij} for the standard unit vectors in \mathbb{R}^m with $1 \leq i \leq n$ and $1 \leq j \leq d$, and let ϕ be the linear map

$$\phi : \mathbb{R}^m \longrightarrow \mathbb{R}^n, \quad e_{ij} \longmapsto e_i.$$

We define the *polarization* of ν to be the function $\Pi^\uparrow \nu : \Delta_m^d \rightarrow \mathbb{R} \cup \{\infty\}$ satisfying

$$\text{dom}(\Pi^\uparrow \nu) \subseteq \begin{bmatrix} m \\ d \end{bmatrix} \text{ and } \Pi^\uparrow \nu = \nu \circ \phi \text{ on } \begin{bmatrix} m \\ d \end{bmatrix}.$$

We define the *projection* of μ to be the function $\Pi^\downarrow \mu : \Delta_n^d \rightarrow \mathbb{R} \cup \{\infty\}$ satisfying

$$\Pi^\downarrow \mu(\alpha) = \min \left\{ \mu(\beta) \mid \phi(\beta) = \alpha \right\}.$$

It is straightforward to check the symmetric exchange properties of $\Pi^\uparrow \nu$ and $\Pi^\downarrow \mu$ from the symmetric exchange properties of ν and μ .¹⁰

Lemma 8.13. Let $\nu : \Delta_n^d \rightarrow \mathbb{R} \cup \{\infty\}$ and $\mu : \Delta_m^d \rightarrow \mathbb{R} \cup \{\infty\}$ be arbitrary functions.

- (1) If ν is an M-convex function, then $\Pi^\uparrow \nu$ is an M-convex function.
- (2) If μ is an M-convex function, then $\Pi^\downarrow \mu$ is an M-convex function.

As a final preparation for the proof of Theorems 8.2 and 8.7, we show that any M-convex function on Δ_n^d can be approximated by M-convex functions whose effective domain is Δ_n^d .

¹⁰In the language of [KMT07], the polarization of ν is obtained from ν by splitting of variables and restricting to $\begin{bmatrix} m \\ d \end{bmatrix}$, and the projection of μ is obtained from μ by aggregation of variables.

Lemma 8.14. For any M-convex function $\nu : \Delta_n^d \rightarrow \mathbb{R} \cup \{\infty\}$, there is a sequence of M-convex functions $\nu_k : \Delta_n^d \rightarrow \mathbb{R}$ such that

$$\lim_{k \rightarrow \infty} \nu_k(\alpha) = \nu(\alpha) \text{ for all } \alpha \in \Delta_n^d.$$

The sequence ν_k can be chosen so that $\nu_k = \nu$ in $\text{dom}(\nu)$ and $\nu_k < \nu_{k+1}$ outside $\text{dom}(\nu)$.

Proof. Write e_{ij} for the standard unit vectors in \mathbb{R}^{n^2} . Let φ and ψ be the linear maps from \mathbb{R}^{n^2} to \mathbb{R}^n given by

$$\varphi(e_{ij}) = e_i \text{ and } \psi(e_{ij}) = e_j.$$

For any function $\mu : \Delta_n^d \rightarrow \mathbb{R} \cup \{\infty\}$, we define the function $\varphi^* \mu : \Delta_n^d \rightarrow \mathbb{R} \cup \{\infty\}$ by

$$\varphi^* \mu(\beta) = \mu(\varphi(\beta)).$$

For any function $\mu : \Delta_{n^2}^d \rightarrow \mathbb{R} \cup \{\infty\}$, we define the function $\psi_* \mu : \Delta_n^d \rightarrow \mathbb{R} \cup \{\infty\}$ by

$$\psi_* \mu(\alpha) = \min \left\{ \mu(\beta) \mid \psi(\beta) = \alpha \right\}.$$

Recall that the operations of splitting [KMT07, Section 4] and aggregation [KMT07, Section 5] preserve M-convexity of discrete functions. Therefore, φ^* and ψ_* preserve M-convexity. Now, given ν , set

$$\nu_k = \psi_*(\ell_k + \varphi^* \nu),$$

where ℓ_k is the restriction of the linear function on \mathbb{R}^{n^2} defined by

$$\ell_k(e_{ij}) = \begin{cases} 0 & \text{if } i = j, \\ k & \text{if } i \neq j. \end{cases}$$

It is straightforward to check that the sequence ν_k has the required properties. \square

Proof of Theorem 8.7, (ii) \Rightarrow (i). Let f_t be a polynomial in $L_n^d(\mathbb{K})$ whose tropicalization is ν . We show the M-convexity of ν by checking the local exchange property: For any $\alpha, \beta \in \text{dom}(\nu)$ with $|\alpha - \beta|_1 = 4$, there are i and j satisfying

$$\alpha_i > \beta_i, \quad \alpha_j < \beta_j \text{ and } \nu(\alpha) + \nu(\beta) \geq \nu(\alpha - e_i + e_j) + \nu(\beta - e_j + e_i).$$

Since $|\alpha - \beta|_1 = 4$, we can find γ in Δ_n^{d-2} and indices p, q, r, s in $[n]$ such that

$$\alpha = \gamma + e_p + e_q \text{ and } \beta = \gamma + e_r + e_s \text{ and } \{p, q\} \cap \{r, s\} = \emptyset.$$

Since $\partial^\gamma f_t$ is stable, the tropicalization of $\partial^\gamma f_t$ is M-convex by [Brä10, Theorem 4]. The conclusion follows from the local exchange property for the tropicalization of $\partial^\gamma f_t$. \square

Proof of Theorem 8.2. We prove (1) \Rightarrow (3). We first show the implication in the special case

$$\text{dom}(\nu) = \begin{bmatrix} n \\ d \end{bmatrix}.$$

Since $\text{dom}(\nu)$ is M-convex, it is enough to prove that $\partial^\alpha g_q^\nu$ has exactly one positive eigenvalue for all $\alpha \in \left[\begin{smallmatrix} n \\ d-2 \end{smallmatrix} \right]$, which is the content of Lemma 8.8. This proves the first special case. Now consider the second special case

$$\text{dom}(\nu) = \Delta_n^d.$$

By Lemma 8.13, the polarization $\Pi^\dagger \nu$ is an M-convex function with effective domain $\left[\begin{smallmatrix} nd \\ d \end{smallmatrix} \right]$, and hence we may apply the known implication (1) \Rightarrow (3) for $\Pi^\dagger \nu$. Therefore,

$$\Pi_\delta^\dagger(g_q^\nu) = \frac{1}{d^d} g_q^{\Pi^\dagger \nu} \text{ is a Lorentzian polynomial for } 0 < q \leq 1,$$

where $\delta = (d, \dots, d)$. Thus, by Proposition 6.1, the polynomial g_q^ν is Lorentzian for all $0 < q \leq 1$, and the second special case is proved. Next consider the third special case

$$\text{dom}(\nu) \text{ is an arbitrary M-convex set and } q = 1.$$

By Lemma 8.13, the effective domain of $\Pi^\dagger \nu$ is an M-convex set. Therefore, by Theorem 7.1,

$$\Pi_\delta^\dagger(g_1^\nu) = \frac{1}{d^d} g_1^{\Pi^\dagger \nu} \text{ is a Lorentzian polynomial.}$$

Thus, by Proposition 6.1, the polynomial g_1^ν is Lorentzian, and the third special case is proved. In the remaining case when $q < 1$ and the effective domain of ν is arbitrary, we express ν as the limit of M-convex functions ν_k with effective domain Δ_n^d using Lemma 8.14. Since $q < 1$, we have

$$g_q^\nu = \lim_{k \rightarrow \infty} g_q^{\nu_k}.$$

Thus the conclusion follows from the second special case applied to each $g_q^{\nu_k}$.

We prove (1) \Rightarrow (2). Introduce a positive real number p , and consider the M-convex function $\frac{\nu}{p}$. Applying the known implication (1) \Rightarrow (3), we see that the polynomial $g_q^{\nu/p}$ is Lorentzian for all $0 < q \leq 1$. Therefore, by Proposition 8.12,

$$R_p(g_q^{\nu/p}) = \sum_{\alpha \in \text{dom}(\nu)} (\alpha!)^p \binom{\delta}{\alpha}^p \frac{q^{\nu(\alpha)}}{\alpha!} w^\alpha$$

is Lorentzian for all $0 < p \leq 1$. Taking the limit p to zero, we have (2).

We prove (2) \Rightarrow (1) and (3) \Rightarrow (1). By the quantifier elimination for the theory of real closed fields, the polynomial f_t^ν with coefficients in \mathbb{K} is Lorentzian if (2) holds. Similarly, the polynomial g_t^ν is Lorentzian if (3) holds. Since

$$\nu = \text{trop}(f_t^\nu) = \text{trop}(g_t^\nu),$$

the conclusion follows from (ii) \Rightarrow (i) of Theorem 8.7. \square

Proof of Theorem 8.7, (i) \Rightarrow (ii). By Theorem 8.2, f_q^ν is Lorentzian for all sufficiently small positive real numbers q . Therefore, by the quantifier elimination for the theory of real closed fields, the polynomial f_t^ν is Lorentzian over \mathbb{K} . Clearly, the tropicalization of f_t^ν is ν . \square

Corollary 8.15. Tropicalizations of Lorentzian polynomials over \mathbb{K} are M-convex, and all M-convex functions are limits of tropicalizations of Lorentzian polynomials over \mathbb{K} .

Proof. By Theorem 8.7, it is enough to show that any M-convex function $\nu : \Delta_n^d \rightarrow \mathbb{R} \cup \{\infty\}$ is a limit of M-convex functions $\nu_k : \Delta_n^d \rightarrow \mathbb{Q} \cup \{\infty\}$. By Lemma 8.14, we may suppose that

$$\text{dom}(\nu) = \Delta_n^d.$$

In this case, by Lemma 8.13, the polarization $\Pi^\uparrow \nu$ is M-convex function satisfying

$$\text{dom}(\Pi^\uparrow \nu) = \left[\begin{smallmatrix} nd \\ d \end{smallmatrix} \right].$$

In other words, $-\Pi^\uparrow \nu$ is a valuated matroid whose underlying matroid is uniform of rank d on nd elements. Since the Dressian of the matroid is a rational polyhedral fan [MS15, Section 4.4], there are M-convex functions $\mu_k : \Delta_{nd}^d \rightarrow \mathbb{Q} \cup \{\infty\}$ satisfying

$$\Pi^\uparrow \nu = \lim_{k \rightarrow \infty} \mu_k.$$

By Lemma 8.13, $\nu = \Pi^\downarrow \Pi^\uparrow \nu$ is the limit of M-convex functions $\Pi^\downarrow \mu_k : \Delta_n^d \rightarrow \mathbb{Q} \cup \{\infty\}$. \square

9. CONVEX BODIES AND LORENTZIAN POLYNOMIALS

For any collection of convex bodies $K = (K_1, \dots, K_n)$ in \mathbb{R}^d , consider the function

$$\text{vol}_K : \mathbb{R}_{\geq 0}^n \longrightarrow \mathbb{R}, \quad w \longmapsto \text{vol}(w_1 K_1 + \dots + w_n K_n),$$

where $w_1 K_1 + \dots + w_n K_n$ is the *Minkowski sum* and vol is the Euclidean volume. Minkowski noticed that the function vol_K is a degree d homogeneous polynomial in $w = (w_1, \dots, w_n)$ with nonnegative coefficients. We may write

$$\text{vol}_K(w) = \sum_{1 \leq i_1, \dots, i_d \leq n} V(K_{i_1}, \dots, K_{i_d}) w_{i_1} \dots w_{i_d} = \sum_{\alpha \in \Delta_n^d} \frac{d!}{\alpha!} V_\alpha(K) w^\alpha,$$

where $V_\alpha(K)$ is, by definition, the *mixed volume*

$$V_\alpha(K) = V(\underbrace{K_1, \dots, K_1}_{\alpha_1}, \dots, \underbrace{K_n, \dots, K_n}_{\alpha_n}) := \frac{1}{d!} \partial^\alpha \text{vol}_K.$$

For any convex bodies C_0, C_1, \dots, C_d in \mathbb{R}^d , the mixed volume $V(C_1, C_2, \dots, C_d)$ is symmetric in its arguments and satisfies the relation

$$V(C_0 + C_1, C_2, \dots, C_d) = V(C_0, C_2, \dots, C_d) + V(C_1, C_2, \dots, C_d).$$

We refer to [Sch14] for background on mixed volumes.

Theorem 9.1. The *volume polynomial* vol_K is a Lorentzian polynomial for any $K = (K_1, \dots, K_n)$.

When combined with Theorem 5.1, Theorem 9.1 implies the following statement.

Corollary 9.2. The support of vol_K is an M-convex for any $K = (K_1, \dots, K_n)$.

In other words, the set of all $\alpha \in \Delta_n^d$ satisfying the non-vanishing condition

$$V(\underbrace{K_1, \dots, K_1}_{\alpha_1}, \dots, \underbrace{K_n, \dots, K_n}_{\alpha_n}) \neq 0$$

is M-convex for any convex bodies K_1, \dots, K_n in \mathbb{R}^d .

Example 9.3. The mixed volume $V(C_1, \dots, C_d)$ is positive precisely when there are line segments $\ell_i \subseteq C_i$ with linearly independent directions [Sch14, Theorem 5.1.8]. Thus, when K consists of n line segments in \mathbb{R}^d , Corollary 9.2 states the familiar fact that, for any configuration of n vectors $\mathcal{A} \subseteq \mathbb{R}^d$, the collection of linearly independent d -subsets of \mathcal{A} is the set of bases of a matroid.

Proof of Theorem 9.1. By continuity of the volume functional [Sch14, Theorem 1.8.20], we may suppose that every convex body in K is d -dimensional. In this case, every coefficient of vol_K is positive. Thus, by Theorem 5.1, it is enough to show that $\partial^\alpha \text{vol}_K$ is Lorentzian for every $\alpha \in \Delta_n^{d-2}$. For this we use a special case of the Brunn-Minkowski theorem [Sch14, Theorem 7.4.5]: For any convex bodies C_3, \dots, C_d in \mathbb{R}^d , the function

$$w \mapsto V\left(\sum_{i=1}^n w_i K_i, \sum_{i=1}^n w_i K_i, C_3, \dots, C_d\right)^{1/2}$$

is concave on $\mathbb{R}_{\geq 0}^n$. In particular, the function

$$\left(\frac{2!}{d!} \partial^\alpha \text{vol}_K(w)\right)^{1/2} = V\left(\sum_{i=1}^n w_i K_i, \sum_{i=1}^n w_i K_i, \underbrace{K_1, \dots, K_1}_{\alpha_1}, \dots, \underbrace{K_n, \dots, K_n}_{\alpha_n}\right)^{1/2}$$

is concave on $\mathbb{R}_{\geq 0}^n$ for every $\alpha \in \Delta_n^{d-2}$. The conclusion follows from Proposition 5.6. \square

The *Alexandrov–Fenchel inequality* [Sch14, Section 7.3] states that

$$V(C_1, C_2, C_3, \dots, C_d)^2 \geq V(C_1, C_1, C_3, \dots, C_d)V(C_2, C_2, C_3, \dots, C_d).$$

We show that an analog holds for any Lorentzian polynomial.

Proposition 9.4. If $f = \sum_{\alpha \in \Delta_n^d} \frac{c_\alpha}{\alpha!} w^\alpha$ is a Lorentzian polynomial, then

$$c_\alpha^2 \geq c_{\alpha+e_i-e_j} c_{\alpha-e_i+e_j} \quad \text{for any } i, j \in [n] \text{ and any } \alpha \in \Delta_n^d.$$

Proof. Consider the Lorentzian polynomial $\partial^{\alpha-e_i-e_j} f$. Substituting w_k by zero for all k other than i and j , we get the bivariate quadratic polynomial

$$\frac{1}{2} c_{\alpha+e_i-e_j} w_i^2 + c_\alpha w_i w_j + \frac{1}{2} c_{\alpha-e_i+e_j} w_j^2.$$

The displayed polynomial is Lorentzian by Theorem 2.10, and hence $c_\alpha^2 \geq c_{\alpha+e_i-e_j} c_{\alpha-e_i+e_j}$. \square

We may reformulate Proposition 9.4 as follows. Let f be a homogeneous polynomial of degree d in n variables. The *complete homogeneous form* of f is the multi-linear function $F_f : (\mathbb{R}^n)^d \rightarrow \mathbb{R}$ defined by

$$F_f(v_1, \dots, v_d) = \frac{1}{d!} \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_d} f(x_1 v_1 + \cdots + x_d v_d).$$

Note that the complete homogeneous form of f is symmetric in its arguments. By Euler's formula for homogeneous functions, we have

$$F_f(w, w, \dots, w) = f(w).$$

Proposition 9.5. If f is Lorentzian, then, for any $v_1 \in \mathbb{R}^n$ and $v_2, \dots, v_d \in \mathbb{R}_{\geq 0}^n$,

$$F_f(v_1, v_2, v_3, \dots, v_d)^2 \geq F_f(v_1, v_1, v_3, \dots, v_d) F_f(v_2, v_2, v_3, \dots, v_d).$$

Proof. For every $k = 1, \dots, d$, we write $v_k = (v_{k1}, v_{k2}, \dots, v_{kn})$, and set

$$D_k = v_{k1} \frac{\partial}{\partial w_1} + v_{k2} \frac{\partial}{\partial w_2} + \cdots + v_{kn} \frac{\partial}{\partial w_n}.$$

By Corollary 2.11, the quadratic polynomial $D_3 \cdots D_d f$ is Lorentzian. We may suppose that the Hessian \mathcal{H} of the quadratic polynomial is not identically zero and $v_2^T \mathcal{H} v_2 > 0$. Note that

$$v_i^T \mathcal{H} v_j = D_i D_j D_3 \cdots D_d f = d! F_f(v_i, v_j, v_3, \dots, v_d) \text{ for any } i \text{ and } j.$$

Since \mathcal{H} has exactly one positive eigenvalue, the conclusion follows from Cauchy's interlacing theorem. \square

10. PROJECTIVE VARIETIES AND LORENTZIAN POLYNOMIALS

Let Y be a d -dimensional irreducible projective variety over an algebraically closed field \mathbb{F} . If D_1, \dots, D_d are Cartier divisors on Y , the *intersection product* $(D_1 \cdots D_d)$ is an integer defined by the following properties:

- the product $(D_1 \cdots D_d)$ is symmetric and multilinear as a function of its arguments,
- the product $(D_1 \cdots D_d)$ depends only on the linear equivalence classes of the D_i , and
- if D_1, \dots, D_n are effective divisors meeting transversely at smooth points of Y , then

$$(D_1 \cdots D_d) = \#D_1 \cap \cdots \cap D_d.$$

Given an irreducible subvariety $X \subseteq Y$ of dimension k , the intersection product

$$(D_1 \cdots D_k \cdot X)$$

is then defined by replacing each divisor D_i with a linearly equivalent Cartier divisor whose support does not contain X and intersecting the restrictions of D_i in X . The definition of the intersection product linearly extends to \mathbb{R} -linear combination of Cartier divisors. If D is a Cartier divisor on Y , we write $(D)^d$ for the self-intersection $(D \cdots D)$. We refer to [Ful98] for background on intersection theory.

Let $H = (H_1, \dots, H_n)$ be a collection of Cartier divisors on Y . We define the *volume polynomial* of H by

$$\text{vol}_H(w) = (w_1 H_1 + \dots + w_n H_n)^d = \sum_{\alpha \in \Delta_n^d} \frac{d!}{\alpha!} V_\alpha(H) w^\alpha,$$

where $V_\alpha(H)$ is the intersection product

$$V_\alpha(H) = (\underbrace{H_1 \cdots H_1}_{\alpha_1} \cdots \underbrace{H_n \cdots H_n}_{\alpha_n}) = \frac{1}{d!} \partial^\alpha \text{vol}_H.$$

A Cartier divisor D on Y is said to be *nef* if $(D \cdot C) \geq 0$ for every irreducible curve C in Y .

Theorem 10.1. If H_1, \dots, H_n are nef divisors on Y , then $\text{vol}_H(w)$ is a Lorentzian polynomial.

When combined with Theorem 5.1, Theorem 10.1 implies the following statement.

Corollary 10.2. If H_1, \dots, H_n are nef divisors on Y , then the support of $\text{vol}_H(w)$ is M-convex.

In other words, the set of all $\alpha \in \Delta_n^d$ satisfying the non-vanishing condition

$$(\underbrace{H_1 \cdots H_1}_{\alpha_1} \cdots \underbrace{H_n \cdots H_n}_{\alpha_n}) \neq 0$$

is M-convex for any d -dimensional projective variety Y and any nef divisors H_1, \dots, H_n on Y .

Example 10.3. Let $\mathcal{A} = \{v_1, \dots, v_n\}$ be a collection of n vectors in \mathbb{F}^d . In [HW17, Section 4], one can find a d -dimensional projective variety $Y_{\mathcal{A}}$ and nef divisors H_1, \dots, H_n on $Y_{\mathcal{A}}$ such that

$$\text{vol}_H(w) = \sum_{\alpha \in \binom{[n]}{d}} c_\alpha w^\alpha,$$

where $c_\alpha = 1$ if α corresponds to a linearly independent subset of \mathcal{A} and $c_\alpha = 0$ if otherwise. Thus, in this case, Corollary 10.2 states the familiar fact that the collection of linearly independent d -subsets of $\mathcal{A} \subseteq \mathbb{F}^d$ is the set of bases of a matroid.

Proof of Theorem 10.1. By Kleiman's theorem [Laz04, Section 1.4], we may suppose that every divisor in H is very ample. In this case, every coefficient of vol_H is positive. Thus, by Theorem 5.1, it is enough to show that $\partial^\alpha \text{vol}_H$ is Lorentzian for every $\alpha \in \Delta_n^{d-2}$. Note that

$$\frac{2!}{d!} \partial^\alpha \text{vol}_H(w) = \left(\sum_{i=1}^n w_i H_i \cdot \sum_{i=1}^n w_i H_i \cdot \underbrace{H_1 \cdots H_1}_{\alpha_1} \cdots \underbrace{H_n \cdots H_n}_{\alpha_n} \right).$$

By Bertini's theorem [Laz04, Section 3.3], there is an irreducible surface $S \subseteq Y$ such that

$$\frac{2!}{d!} \partial^\alpha \text{vol}_H(w) = \left(\sum_{i=1}^n w_i H_i \cdot \sum_{i=1}^n w_i H_i \cdot S \right).$$

Now the Hodge index theorem [Ful98, Example 15.2.4] for any resolution of singularities of S implies that the displayed quadratic form has exactly one positive eigenvalue. \square

11. POTTS MODEL PARTITION FUNCTIONS AND LORENTZIAN POLYNOMIALS

The q -state Potts model, or the random-cluster model, of a graph is a much studied class of measures introduced by Fortuin and Kasteleyn [FK72]. We refer to [Gri06] for a comprehensive introduction to random-cluster models.

Let M be a matroid on $[n]$, and let rk_M be the rank function of M . For a nonnegative integer k and a positive real parameter q , consider the degree k homogeneous polynomial in n variables

$$Z_{q,M}^k(w) = \sum_{A \in \binom{[n]}{k}} q^{-\text{rk}_M(A)} w^A, \quad w = (w_1, \dots, w_n).$$

We define the *homogeneous multivariate Tutte polynomial* of M by

$$Z_{q,M}(w_0, w_1, \dots, w_n) = \sum_{k=0}^n Z_{q,M}^k(w) w_0^{n-k},$$

which is a homogeneous polynomial of degree n in $n+1$ variables. When M is the cycle matroid of a graph G , the polynomial obtained from $Z_{q,M}$ by setting $w_0 = 1$ is the partition function of the q -state Potts model associated to G [Sok05].

Theorem 11.1. For any matroid M and $0 < q \leq 1$, the polynomial $Z_{q,M}$ is Lorentzian.

We prepare the proof with two simple lemmas.

Lemma 11.2. The support of $Z_{q,M}$ is M -convex for all $0 < q \leq 1$.

Proof. Writing $Z_{q,M}^\natural$ for the polynomial obtained from $Z_{q,M}$ by setting $w_0 = 1$, we have

$$\text{supp}(Z_{q,M}^\natural) = \{0, 1\}^n.$$

It is straightforward to verify the augmentation property in Lemma 3.3 for $\{0, 1\}^n$. \square

For a nonnegative integer k and a subset $S \subseteq [n]$, we define a degree k homogeneous polynomial $e_S^k(w)$ by the equation

$$\sum_{k=0}^n e_S^k(w) = \sum_{A \subseteq S} w^A.$$

In other words, $e_S^k(w)$ is the k -th elementary symmetric polynomial in the variables $\{w_i\}_{i \in S}$.

Lemma 11.3. If $S_1 \sqcup \dots \sqcup S_m$ is a partition of $[n]$ into m nonempty parts, then

$$\frac{1}{n} e_{[n]}^1(w)^2 \leq e_{S_1}^1(w)^2 + \dots + e_{S_m}^1(w)^2 \quad \text{for all } w \in \mathbb{R}^n.$$

Proof. Since $m \leq n$, it is enough to prove the statement when $m = n$. In this case, we have

$$(w_1 + \dots + w_n)^2 \leq n(w_1^2 + \dots + w_n^2),$$

by the Cauchy-Schwarz inequality for the vectors $(1, \dots, 1)$ and (w_1, \dots, w_n) in \mathbb{R}^n . \square

The proof of Theorem 11.1 parallels that of Theorem 12.1.

Proof of Theorem 11.1. Let α be an element of Δ_{n+1}^{n-2} . By Theorem 5.1 and Lemma 11.2, the proof reduces to the statement that the quadratic form $\partial^\alpha Z_{q,M}$ is stable. We prove the statement by induction on n . The assertion is clear when $n = 1$, so suppose $n \geq 2$. When $i \neq 0$, we have

$$\partial_i Z_{q,M} = q^{-\text{rk}_M(i)} Z_{q,M/i},$$

where M/i is the contraction of M by i [Oxl11, Chapter 3]. Thus, it is enough to prove that the following quadratic form is stable:

$$\frac{n!}{2} w_0^2 + (n-1)! Z_{q,M}^1(w) w_0 + (n-2)! Z_{q,M}^2(w).$$

As in the proof of Theorem 12.1, it suffices to show that the discriminant of the displayed quadratic form with respect to w_0 is nonnegative:

$$Z_{q,M}^1(w)^2 \geq 2 \frac{n}{n-1} Z_{q,M}^2(w) \text{ for all } w \in \mathbb{R}^n.$$

We prove the inequality after making the change of variables

$$w_i \mapsto \begin{cases} w_i & \text{if } i \text{ is a loop in } M, \\ qw_i & \text{if } i \text{ is not a loop in } M. \end{cases}$$

Write $L \subseteq [n]$ for the set of loops and $P_1, \dots, P_\ell \subseteq [n] \setminus L$ for the parallel classes in M [Oxl11, Section 1.1]. The above change of variables gives

$$Z_{q,M}^1(w) = e_{[n]}^1(w) \text{ and } Z_{q,M}^2(w) = e_{[n]}^2(w) - (1-q)(e_{P_1}^2(w) + \dots + e_{P_\ell}^2(w)).$$

When $q = 1$, the desired inequality directly follows from the case $m = n$ of Lemma 11.3. Therefore, when proving the desired inequality for an arbitrary $0 < q \leq 1$, we may assume that

$$e_{P_1}^2(w) + \dots + e_{P_\ell}^2(w) < 0.$$

Therefore, exploiting the monotonicity of $Z_{q,M}^2$ in q , the desired inequality reduces to

$$(n-1)e_{[n]}^1(w)^2 - 2n(e_{[n]}^2(w) - e_{P_1}^2(w) - \dots - e_{P_\ell}^2(w)) \geq 0.$$

Note that the left-hand side of the above inequality simplifies to

$$n(e_{P_1}^1(w)^2 + \dots + e_{P_\ell}^1(w)^2 + \sum_{i \in L} w_i^2) - e_{[n]}^1(w)^2.$$

The conclusion now follows from Lemma 11.3. \square

Mason [Mas72] offered the following three conjectures of increasing strength. Several authors studied correlations in matroid theory partly in pursuit of these conjectures [SW75, Wag08, BBL09, KN10, KN11].

Conjecture 11.4. For any matroid M on $[n]$ and any positive integer k ,

- (1) $I_k(M)^2 \geq I_{k-1}(M)I_{k+1}(M)$,
- (2) $I_k(M)^2 \geq \frac{k+1}{k} I_{k-1}(M)I_{k+1}(M)$,
- (3) $I_k(M)^2 \geq \frac{k+1}{k} \frac{n-k+1}{n-k} I_{k-1}(M)I_{k+1}(M)$,

where $I_k(M)$ is the number of k -element independent sets of M .

Conjecture 11.4 (1) was proved in [AHK18], and Conjecture 11.4 (2) was proved in [HSW18]. Note that Conjecture 11.4 (3) may be written

$$\frac{I_k(M)^2}{\binom{n}{k}^2} \geq \frac{I_{k+1}(M)}{\binom{n}{k+1}} \frac{I_{k-1}(M)}{\binom{n}{k-1}},$$

and the equality holds when all $(k+1)$ -subsets of $[n]$ are independent in M . Conjecture 11.4 (3) is known to hold when n is at most 11 or k is at most 5 [KN11]. See [Sey75, Dow80, Mah85, Zha85, HK12, HS89, Len13] for other partial results.

Theorem 11.5. For any matroid M on $[n]$ and any positive integer k ,

$$\frac{I_k(M)^2}{\binom{n}{k}^2} \geq \frac{I_{k+1}(M)}{\binom{n}{k+1}} \frac{I_{k-1}(M)}{\binom{n}{k-1}},$$

where $I_k(M)$ is the number of k -element independent sets of M .

In [BH], direct proofs of Theorems 11.1 and 11.5 were given.¹¹ Here we deduce Theorem 11.5 from the Lorentzian property of

$$f_M(w_0, w_1, \dots, w_n) = \sum_{A \in \mathcal{J}(M)} w^A w_0^{n-|A|}, \quad w = (w_1, \dots, w_n),$$

where $\mathcal{J}(M)$ is the collection of independent sets of M .

Proof of Theorem 11.5. The polynomial f_M is Lorentzian by Theorem 11.1 and the identity

$$f_M(w_0, w_1, \dots, w_n) = \lim_{q \rightarrow 0} Z_{q,M}(w_0, qw_1, \dots, qw_n).$$

Therefore, by Theorem 2.10, the bivariate polynomial obtained from f_M by setting $w_1 = \dots = w_n$ is Lorentzian. The conclusion follows from the fact that a bivariate homogeneous polynomial with nonnegative coefficients is Lorentzian if and only if the sequence of coefficients form an ultra log-concave sequence with no internal zeros. \square

The *Tutte polynomial* of a matroid M on $[n]$ is the bivariate polynomial

$$T_M(x, y) = \sum_{A \subseteq [n]} (x-1)^{\text{rk}_M([n]) - \text{rk}_M(A)} (y-1)^{|A| - \text{rk}_M(A)}.$$

Theorem 11.1 reveals several nontrivial inequalities satisfied by the coefficients of the Tutte polynomial. For example, if we write

$$w^{\text{rk}_M([n])} T_M\left(1 + \frac{q}{w}, 1 + w\right) = \sum_{k=0}^n \left(\sum_{A \in \binom{[n]}{k}} q^{\text{rk}_M([n]) - \text{rk}_M(A)} \right) w^k = \sum_{k=0}^n c_q^k(M) w^k,$$

then the sequence $c_q^k(M)$ is ultra log-concave whenever $0 \leq q \leq 1$.

¹¹Independent proofs of Theorems 11.1 and 11.5 were given by Anari *et al.* in [ALOVII] and [ALOVIII].

12. M-MATRICES AND LORENTZIAN POLYNOMIALS

We write I_n for the $n \times n$ identity matrix, J_n for the $n \times n$ matrix all of whose entries are 1, and 1_n for the $n \times 1$ matrix all of whose entries are 1. Let $A = (a_{ij})$ be an $n \times n$ matrix with real entries. The following conditions are equivalent if $a_{ij} \leq 0$ for all $i \neq j$ [BP94, Chapter 6]:

- The real part of each nonzero eigenvalue of A is positive.
- The real part of each eigenvalue of A is nonnegative.
- All the principal minors of A are nonnegative.
- Every real eigenvalue of A is nonnegative.
- The matrix $A + \epsilon I_n$ is nonsingular for every $\epsilon > 0$.
- The univariate polynomial $\det(xI_n + A)$ has nonnegative coefficients.

The matrix A is an *M-matrix* if $a_{ij} \leq 0$ for all $i \neq j$ and if it satisfies any one of the above conditions. One can find 50 different characterizations of nonsingular M-matrices in [BP94, Chapter 6].

We define the *multivariate characteristic polynomial* of A by the equation

$$p_A(w_0, w_1, \dots, w_n) = \det(w_0 I_n + \text{diag}(w_1, \dots, w_n)A).$$

In [Hol05, Theorem 4], Holtz proved that the coefficients of the characteristic polynomial of an M-matrix form an ultra log-concave sequence. We will strengthen this result and prove that the multivariate characteristic polynomial of an M-matrix is Lorentzian.

Theorem 12.1. If A is an M-matrix, then p_A is a Lorentzian polynomial.

Using Example 5.2, we may recover the theorem of Holtz by setting $w_1 = \dots = w_n$.

Corollary 12.2. If A is an M-matrix, then the support of p_A is M-convex.

Since every M-matrix is a limit of nonsingular M-matrices, it is enough to prove Theorem 12.1 for nonsingular M-matrices.

Lemma 12.3. If A is a nonsingular M-matrix, the support of p_A is M-convex.

Proof. It is enough to prove that the support of p_A^\natural is M^\natural -convex, where

$$p_A^\natural(w_1, \dots, w_n) = p_A(1, w_1, \dots, w_n).$$

If A is a nonsingular M-matrix, then all the principal minors of A are positive, and hence

$$\text{supp}(p_A^\natural) = \{0, 1\}^n.$$

It is straightforward to verify the augmentation property in Lemma 3.3 for $\{0, 1\}^n$. □

We prepare the proof of Theorem 12.1 with a proposition on doubly sub-stochastic matrices. Recall that an $n \times n$ matrix $B = (b_{ij})$ with nonnegative entries is said to be *doubly sub-stochastic* if

$$\sum_{j=1}^n b_{ij} \leq 1 \text{ for every } i \quad \text{and} \quad \sum_{i=1}^n b_{ij} \leq 1 \text{ for every } j.$$

A *partial permutation matrix* is a zero-one matrix with at most one nonzero entry in each row and column. We use Mirsky's analog of the Birkhoff-von Neumann theorem for doubly sub-stochastic matrices [HJ94, Theorem 3.2.6]: The set of $n \times n$ doubly sub-stochastic matrix is equal to the convex hull of the $n \times n$ partial permutation matrices.

Lemma 12.4. For $n \geq 2$, define $n \times n$ matrices M_n and N_n by

$$M_n = \begin{pmatrix} 2 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 2 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & 1 \\ 1 & 0 & 0 & \cdots & 1 & 2 \end{pmatrix}, \quad N_n = \begin{pmatrix} 2 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 2 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 2 \end{pmatrix}.$$

Then the matrices $M_n - \frac{2}{n}J_n$ and $N_n - \frac{2}{n}J_n$ are positive semidefinite. Equivalently,

$$\underline{M}_{n+1} := \begin{pmatrix} M_n & 1_n \\ 1_n^T & \frac{n}{2} \end{pmatrix}, \quad \underline{N}_{n+1} := \begin{pmatrix} N_n & 1_n \\ 1_n^T & \frac{n}{2} \end{pmatrix}$$

are positive semidefinite.

Proof. We define symmetric matrices L_{n+1} and K_{n+1} by

$$L_{n+1} = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & \frac{1}{2} \\ 1 & 2 & 1 & \cdots & 0 & 0 & 1 \\ 0 & 1 & 2 & \cdots & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 1 & \frac{1}{2} \\ \frac{1}{2} & 1 & 1 & \cdots & 1 & \frac{1}{2} & \frac{n}{2} \end{pmatrix}, \quad K_{n+1} = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & \frac{1}{2} \\ 1 & 2 & 1 & \cdots & 0 & 0 & 1 \\ 0 & 1 & 2 & \cdots & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 2 & 1 \\ \frac{1}{2} & 1 & 1 & \cdots & 1 & 1 & \frac{n}{2} \end{pmatrix}.$$

As before, the subscript indicates the size of the matrix. We show, by induction on n , that the matrices L_{n+1} and K_{n+1} are positive semidefinite. It is straightforward to check that L_3 and K_3 are positive semidefinite. Perform the symmetric row and column elimination of L_{n+1} and

K_{n+1} based on their 1×1 entries, and notice that

$$L_{n+1} \simeq \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 2 & \cdots & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & 1 & \cdots & 1 & \frac{1}{2} & \frac{n}{2} - \frac{1}{4} \end{pmatrix}, \quad K_{n+1} \simeq \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 2 & \cdots & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 1 & 1 \\ 0 & \frac{1}{2} & 1 & \cdots & 1 & 1 & \frac{n}{2} - \frac{1}{4} \end{pmatrix},$$

where the symbol \simeq stands for the congruence relation for symmetric matrices. Since L_n is positive semidefinite, L_{n+1} is congruent to the sum of positive semidefinite matrices, and hence L_{n+1} is positive semidefinite. Similarly, since K_n is positive semidefinite, K_{n+1} is congruent to the sum of positive semidefinite matrices, and hence K_{n+1} is positive semidefinite.

We now prove that the symmetric matrices \underline{M}_{n+1} and \underline{N}_{n+1} are positive semidefinite. Perform the symmetric row and column elimination of \underline{M}_{n+1} and \underline{N}_{n+1} based on their 1×1 entries, and notice that

$$\underline{M}_{n+1} \simeq \begin{pmatrix} 2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & \cdots & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 2 & \cdots & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & 1 & 1 \\ 0 & -\frac{1}{2} & 0 & \cdots & 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 1 & \cdots & 1 & \frac{1}{2} & \frac{n-1}{2} \end{pmatrix}, \quad \underline{N}_{n+1} \simeq \begin{pmatrix} 2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & \cdots & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 2 & \cdots & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 2 & 1 \\ 0 & \frac{1}{2} & 1 & \cdots & 1 & 1 & \frac{n-1}{2} \end{pmatrix}.$$

Since L_n is positive semidefinite, \underline{M}_{n+1} is congruent to the sum of two positive semidefinite matrices, and hence \underline{M}_{n+1} is positive semidefinite. Similarly, since K_n is positive semidefinite, \underline{N}_{n+1} is congruent to the sum of two positive semidefinite matrices, and hence \underline{N}_{n+1} is positive semidefinite. \square

Proposition 12.5. If B is an $n \times n$ doubly sub-stochastic matrix, then $2I_n + B + B^T - \frac{2}{n}J_n$ is positive semidefinite.

Proof. Let C_n be the symmetric matrix $2I_n + B + B^T$, and let \underline{C}_{n+1} be the symmetric matrix

$$\underline{C}_{n+1} := \begin{pmatrix} C_n & 1_n \\ 1_n^T & \frac{n}{2} \end{pmatrix}.$$

It is enough to prove that \underline{C}_{n+1} is positive semidefinite. Since the convex hull of the partial permutation matrices is the set of doubly sub-stochastic matrix, the proof reduces to the case when B is a partial permutation matrix. We use the following extension of the cycle decomposition for partial permutations: For any partial permutation matrix B , there is a permutation matrix P

such that PBP^T is a block diagonal matrix, where each block diagonal is either zero, identity,

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Using the cyclic decomposition for B , we can express the matrix \underline{C}_{n+1} as a sum, where each summand is positive semidefinite by Lemma 12.4. \square

Proof of Theorem 12.1. Since every M-matrix is a limit of nonsingular M-matrices, we may suppose that A is a nonsingular M-matrix. For $k = 0, 1, \dots, n$, we set

$$p_A^k(w) = \sum_{\alpha \in \binom{[n]}{k}} A_\alpha w^\alpha, \quad w = (w_1, \dots, w_n),$$

where A_α is the principal minor of A corresponding to α , so that

$$p_A(w_0, w_1, \dots, w_n) = \sum_{k=0}^n p_A^k(w) w_0^{n-k}.$$

Lemma 12.3 shows that the support of p_A is M-convex. Therefore, by Theorem 5.1, it is enough to prove that $\partial_i(p_A)$ is Lorentzian for $i = 0, 1, \dots, n$. We prove this statement by induction on n . The assertion is clear when $n = 1$, so suppose $n \geq 2$. When $i \neq 0$, we have

$$\partial_i(p_A) = p_{A \setminus i},$$

where $A \setminus i$ is the M-matrix obtained from A by deleting the i -th row and column. Thus, it is enough to prove that the following quadratic form is stable:

$$\frac{n!}{2} w_0^2 + (n-1)! p_A^1(w) w_0 + (n-2)! p_A^2(w).$$

Recall that a homogeneous polynomial f with nonnegative coefficients in $n+1$ variables is stable if and only if the univariate polynomial $f(xu - v)$ has only real zeros for all $v \in \mathbb{R}^{n+1}$ for some $u \in \mathbb{R}_{\geq 0}^{n+1}$ satisfying $f(u) > 0$. Therefore, it suffices to show that the discriminant of the displayed quadratic form with respect to w_0 is nonnegative:

$$p_A^1(w)^2 \geq \frac{2n}{n-1} p_A^2(w) \quad \text{for all } w \in \mathbb{R}^n.$$

In terms of the entries of A , the displayed inequality is equivalent to the statement that the matrix $\left(a_{ij}a_{ji} - \frac{1}{n}a_{ii}a_{jj}\right)$ is positive semidefinite. According to the 29-th characterization of nonsingular M-matrices in [BP94, Chapter 6], there are positive diagonal matrices D and D' such that DAD' has all diagonal entries 1 and all row sums positive. Therefore, we may suppose that A has all diagonal entries 1 and all the row sums of A are positive. Under this assumption,

$$\left(a_{ij}a_{ji} - \frac{1}{n}a_{ii}a_{jj}\right) = I_n - B - \frac{1}{n}J_n,$$

where $-B$ is a symmetric doubly sub-stochastic matrix all of whose diagonal entries are zero. The conclusion follows from Proposition 12.5. \square

13. LORENTZIAN PROBABILITY MEASURES

There are numerous important examples of negatively dependent “repelling” random variables in probability theory, combinatorics, stochastic processes, and statistical mechanics. See, for example, [Pem00]. A theory of negative dependence for strongly Rayleigh measures was developed in [BBL09], but the theory is too restrictive for several applications. Here we introduce a broader class of discrete probability measures using the Lorentzian property.

A *discrete probability measure* μ on $\{0, 1\}^n$ is a probability measure on $\{0, 1\}^n$ such that all subsets of $\{0, 1\}^n$ are measurable. The *partition function* of μ is the polynomial

$$Z_\mu(w) = \sum_{S \subseteq [n]} \mu(\{S\}) \prod_{i \in S} w_i.$$

The following notions capture various aspects of negative dependence:

- The measure μ is *pairwise negatively correlated* (PNC) if for all distinct i and j in $[n]$,

$$\mu(\mathcal{E}_i \cap \mathcal{E}_j) \leq \mu(\mathcal{E}_i)\mu(\mathcal{E}_j),$$

where \mathcal{E}_i is the collection of all subsets of $[n]$ containing i .

- The measure μ is *ultra log-concave* (ULC) if for every positive integer $k < n$,

$$\frac{\mu\left(\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]\right)^2}{\binom{n}{k}^2} \geq \frac{\mu\left(\left[\begin{smallmatrix} n \\ k-1 \end{smallmatrix}\right]\right)}{\binom{n}{k-1}} \frac{\mu\left(\left[\begin{smallmatrix} n \\ k+1 \end{smallmatrix}\right]\right)}{\binom{n}{k+1}}.$$

- The measure μ is *strongly Rayleigh* if for all distinct i and j in $[n]$,

$$Z_\mu(w) \partial_i \partial_j Z_\mu(w) \leq \partial_i Z_\mu(w) \partial_j Z_\mu(w) \text{ for all } w \in \mathbb{R}^n.$$

Let P be a property of discrete probability measures. We say that μ has property \underline{P} if, for every $x \in \mathbb{R}_{>0}^n$, the discrete probability measure on $\{0, 1\}^n$ with the partition function

$$Z_\mu(x_1 w_1, \dots, x_n w_n) / Z_\mu(x_1, \dots, x_n)$$

has property P . The new discrete probability measure is said to be obtained from μ by applying the *external field* $x \in \mathbb{R}_{>0}^n$. For example, the property $\underline{\text{PNC}}$ for μ is equivalent to the 1-Rayleigh property

$$Z_\mu(w) \partial_i \partial_j Z_\mu(w) \leq \partial_i Z_\mu(w) \partial_j Z_\mu(w) \text{ for all distinct } i, j \text{ in } [n] \text{ and all } w \in \mathbb{R}_{>0}^n.$$

More generally, for a positive real number c , we say that μ is c -Rayleigh if

$$Z_\mu(w) \partial_i \partial_j Z_\mu(w) \leq c \partial_i Z_\mu(w) \partial_j Z_\mu(w) \text{ for all distinct } i, j \text{ in } [n] \text{ and all } w \in \mathbb{R}_{>0}^n.$$

Definition 13.1. A discrete probability measure μ on $\{0, 1\}^n$ is *Lorentzian* if the homogenization of the partition function $w_0^n Z_\mu(w_1/w_0, \dots, w_n/w_0)$ is a Lorentzian polynomial.

For example, if A is an M -matrix of size n , the probability measure on $\{0, 1\}^n$ given by

$$\mu(\{S\}) \propto (\text{the principal minor of } A \text{ corresponding to } S), \quad S \subseteq [n],$$

is Lorentzian by Theorem 12.1. Results from the previous sections reveal basic features of Lorentzian measures, some of which may be interpreted as negative dependence properties.

Proposition 13.2. If μ is Lorentzian, then μ is 2-Rayleigh.

Proof. Lemma 3.2 and Corollary 4.5 show that Z_μ is a $2\left(1 - \frac{1}{n}\right)$ -Rayleigh polynomial. \square

Proposition 13.3. If μ is Lorentzian, then μ is ULC.

Proof. Since any probability measure obtained from a Lorentzian probability measure by applying an external field is Lorentzian, it suffices to prove that μ is ULC. By Theorem 2.10, the bivariate homogeneous polynomial $w_0^n Z_\mu(w_1/w_0, \dots, w_1/w_0)$ is Lorentzian. Therefore, by Example 5.2, its sequence of coefficients must be ultra log-concave. \square

Proposition 13.4. The class of Lorentzian measures is preserved under the symmetric exclusion process.

Proof. The statement is Theorem 6.7 for homogenized partition functions of Lorentzian probability measures. \square

Proposition 13.5. If μ is strongly Rayleigh, then μ is Lorentzian.

Proof. A multi-affine polynomial is stable if and only if it is strongly Rayleigh [Brä07, Theorem 5.6], and a polynomial with nonnegative coefficients is stable if and only if its homogenization is stable [BBL09, Theorem 4.5]. By Proposition 2.2, homogeneous stable polynomials with nonnegative coefficients are Lorentzian. \square

For a matroid M on $[n]$, we define probability measures μ_M and ν_M on $\{0, 1\}^n$ by

$$\begin{aligned} \mu_M &= \text{the uniform measure on } \{0, 1\}^n \text{ concentrated on the independent sets of } M, \\ \nu_M &= \text{the uniform measure on } \{0, 1\}^n \text{ concentrated on the bases of } M. \end{aligned}$$

Proposition 13.6. For any matroid M on $[n]$, the measures μ_M and ν_M are Lorentzian.

Proof. The homogenized partition function f_M of μ_M is Lorentzian by Theorem 11.1 and

$$f_M(w_0, w_1, \dots, w_n) = \lim_{q \rightarrow 0} Z_{q,M}(w_0, qw_1, \dots, qw_n).$$

The partition function of ν_M is Lorentzian by Theorem 7.1. Since the product of Lorentzian polynomials is Lorentzian, it follows that ν_M is Lorentzian. \square

Let G be an arbitrary finite graph and let i and j be any distinct edges of G . A conjecture of Kahn [Kah00] and Grimmett-Winkler [GW04] states that, if F is a forest in G chosen uniformly at random, then

$$\Pr(F \text{ contains } i \text{ and } j) \leq \Pr(F \text{ contains } i) \Pr(F \text{ contains } j).$$

Clearly, the conjecture is equivalent to the statement that μ_M is 1-Rayleigh for any graphic matroid M . Propositions 13.2 and 13.6 show that μ_M is 2-Rayleigh for any matroid M .

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