

ON ENDOSCOPY AND THE REFINED GROSS-PRASAD CONJECTURE FOR $(\mathrm{SO}_5, \mathrm{SO}_4)$

WEE TECK GAN AND ATSUSHI ICHINO

1. Introduction

In [11], Gross and Prasad gave a remarkable conjecture on the non-vanishing of the periods of automorphic forms on $\mathrm{SO}_n \times \mathrm{SO}_{n-1}$ along the diagonal subgroup SO_{n-1} in terms of the non-vanishing of certain automorphic L -functions at the center of the critical strip. Their conjecture was refined in [21], where a conjectural formula relating the periods to the central critical L -values was given. This refined conjecture follows from a result of Waldspurger [55] for $n = 3$ and that of the second author [20] for $n = 4$. The purpose of this paper is to establish the refined Gross-Prasad conjecture for certain L -packets of automorphic representations in the case $n = 5$. The (candidate) L -packets we consider were constructed by B. Roberts [49] and include all endoscopic L -packets of SO_5 as well as some stable ones. Because of this, our result gives strong evidence that the Gross-Prasad conjecture is related to the theory of endoscopy (cf. Remark 1.3). Special cases of our result have been obtained earlier by Böcherer, Furusawa, and Schulze-Pillot [5] for the so-called Yoshida lifts [57] (which are certain instances of endoscopic representations).

To state our main theorem, we need to introduce quite a number of notations. Let F be a totally real number field with ring of adèles $\mathbb{A} = \mathbb{A}_F$ and let E be a totally real étale quadratic algebra over F . Let W_0 be a two dimensional symplectic space over E , which we may regard as a four dimensional symplectic space $W = \mathrm{R}_{E/F}(W_0)$ over F . Set

$$G = \mathrm{GSp}(W) \cong \mathrm{GSp}_4, \quad \tilde{G}' = \mathrm{R}_{E/F}(\mathrm{GSp}(W_0)) \cong \mathrm{R}_{E/F}(\mathrm{GL}_2),$$

and

$$G' = \{g' \in \tilde{G}' \mid \nu(g') \in \mathbb{G}_m\},$$

where $\nu : \tilde{G}' \rightarrow \mathrm{R}_{E/F}(\mathbb{G}_m)$ is the similitude character. Then we have a natural embedding

$$G' \hookrightarrow G.$$

Let V be a four dimensional quadratic space over F and set

$$H = \mathrm{GO}(V).$$

The discriminant algebra of V is the étale quadratic algebra K over F defined by

$$K = \begin{cases} F \times F & \text{if } \mathrm{disc}(V) \in F^{\times,2}, \\ F(\sqrt{\mathrm{disc}(V)}) & \text{if } \mathrm{disc}(V) \notin F^{\times,2}, \end{cases}$$

and we let $\omega_{K/F}$ be the quadratic character of $\mathbb{A}^\times/F^\times$ associated to K/F by class field theory. Then there exists a quaternion algebra D over F such that H is described by the short exact sequence:

$$1 \longrightarrow \mathrm{R}_{K/F}(\mathbb{G}_m) \xrightarrow{i} (\mathrm{R}_{K/F}(D^\times) \times \mathbb{G}_m) \rtimes \langle \mathbf{t} \rangle \longrightarrow H \longrightarrow 1$$

Here $i(z) = (z, \mathrm{N}_{K/F}(z)^{-1})$ for $z \in \mathrm{R}_{K/F}(\mathbb{G}_m)$ and \mathbf{t} is an involution on $\mathrm{R}_{K/F}(D^\times) \times \mathbb{G}_m$ given by $(g, \lambda) \mapsto (g^c, \lambda)$, where c is the non-trivial automorphism of K over F .

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Now let $\sigma \cong \otimes_v \sigma_v$ be an irreducible unitary cuspidal automorphic representation of $H(\mathbb{A})$ on the space V_σ with central character ω_σ . We assume the following:

- The Jacquet-Langlands transfer of $\sigma|_{D^\times(\mathbb{A}_K)}$ to $\mathrm{GL}_2(\mathbb{A}_K)$ is cuspidal.
- $\sigma_v \otimes \mathrm{sgn} \cong \sigma_v$ for some place v of F .
- If $\sigma_v \otimes \mathrm{sgn} \not\cong \sigma_v$, then $\sigma_v \not\cong \sigma_{0,v}^-$ for any distinguished representation $\sigma_{0,v}$ of $\mathrm{GSO}(V)(F_v)$ (cf. Definition 5.4).

Let π be the theta lift of σ to $G(\mathbb{A})$ on the space V_π . In §7, we will show that π is a non-zero irreducible unitary cuspidal automorphic representation of $G(\mathbb{A})$ with central character ω_σ . The representations π 's constructed in this way are precisely the ones which occur in the L -packets of GSp_4 defined in the paper of Roberts [49] (though he assumed that σ and hence π is tempered). The automorphic representations of $\mathrm{SO}_5(\mathbb{A})$ considered in this paper are precisely those π 's with trivial central characters.

On the other hand, let π' be an irreducible unitary cuspidal automorphic representation of $G'(\mathbb{A})$ on the space $V_{\pi'}$ with central character $\omega_{\sigma'}^{-1}$. By [17, Theorem 4.13], there exists an irreducible unitary cuspidal automorphic representation τ of $\tilde{G}'(\mathbb{A})$ on the space V_τ such that $V_{\pi'} \subset V_\tau^1|_{G'(\mathbb{A})}$. Here V_τ^1 is the subspace of V_τ on which the group

$$\mathfrak{X}_\tau = \{\omega \in (Z_{\tilde{G}'}(\mathbb{A})G'(\mathbb{A})\tilde{G}'(F)\backslash\tilde{G}'(\mathbb{A}))^D \mid \tau \otimes \omega \cong \tau\}$$

acts trivially, and $V_\tau^1|_{G'(\mathbb{A})}$ is the restriction of V_τ^1 to $G'(\mathbb{A})$ as functions. We remark that the cardinality of \mathfrak{X}_τ is finite and does not depend on the choice of τ , i.e., it depends only on π' . We assume the following:

- The base change τ_K of τ to $\tilde{G}'(\mathbb{A}_K) \cong \mathrm{GL}_2(\mathbb{A}_{E \otimes K})$ is cuspidal.
- The Jacquet-Langlands transfer τ_K^D of τ_K to $D^\times(\mathbb{A}_{E \otimes K})$ exists.

Let $\theta(\tau)$ be the theta lift of τ to $H(\mathbb{A}_E)$. In §6, we will show that $\theta(\tau)$ is a non-zero irreducible unitary cuspidal automorphic representation of $H(\mathbb{A}_E)$.

We may now introduce certain automorphic L -functions which appear in the refined Gross-Prasad conjecture under consideration:

$$\begin{aligned} L(s, \pi \times \pi') &= L(s, \sigma \times \theta(\tau)), \\ L(s, \pi, \mathrm{Ad}) &= L(s, \sigma, \mathrm{std})L(s, \sigma, \mathrm{Ad}), \\ L(s, \pi', \mathrm{Ad}) &= L(s, \tau, \mathrm{Ad}). \end{aligned}$$

Here $L(s, \sigma \times \theta(\tau))$ is the triple product L -function of degree eight over K associated to σ and $\theta(\tau)$ defined in §3, $L(s, \sigma, \mathrm{std})$ is the standard L -function of σ , and $L(s, \sigma, \mathrm{Ad})$ (resp. $L(s, \tau, \mathrm{Ad})$) is the adjoint L -function of σ (resp. τ).

Now let $\mathcal{B}_\pi : V_\pi \otimes \bar{V}_\pi \rightarrow \mathbb{C}$ and $\mathcal{B}_{\pi'} : V_{\pi'} \otimes \bar{V}_{\pi'} \rightarrow \mathbb{C}$ be the Petersson pairings given by

$$\begin{aligned} \mathcal{B}_\pi(\phi_1, \phi_2) &= \int_{Z_G(\mathbb{A})G(F)\backslash G(\mathbb{A})} \phi_1(g)\overline{\phi_2(g)} dg, \\ \mathcal{B}_{\pi'}(f_1, f_2) &= \int_{Z_{G'}(\mathbb{A})G'(F)\backslash G'(\mathbb{A})} f_1(g')\overline{f_2(g')} dg', \end{aligned}$$

for $\phi_1, \phi_2 \in V_\pi$ and $f_1, f_2 \in V_{\pi'}$. Here \bar{V}_π and $\bar{V}_{\pi'}$ are the complex conjugate representations of V_π and $V_{\pi'}$, Z_G and $Z_{G'}$ are the identity components of the centers of G and G' , and dg and dg' are the Tamagawa measures on $Z_G(\mathbb{A})\backslash G(\mathbb{A})$ and $Z_{G'}(\mathbb{A})\backslash G'(\mathbb{A})$, respectively. We fix isomorphisms

$$\pi \cong \otimes_v \pi_v \quad \text{and} \quad \pi' \cong \otimes_v \pi'_v,$$

and decompositions

$$\mathcal{B}_\pi = \prod_v \mathcal{B}_{\pi_v} \quad \text{and} \quad \mathcal{B}_{\pi'} = \prod_v \mathcal{B}_{\pi'_v},$$

where $\mathcal{B}_{\pi_v} : \pi_v \otimes \bar{\pi}_v \rightarrow \mathbb{C}$ and $\mathcal{B}_{\pi'_v} : \pi'_v \otimes \bar{\pi}'_v \rightarrow \mathbb{C}$ are local pairings. Moreover, we fix a decomposition $dg' = \prod_v dg'_v$, where dg'_v is a Haar measure on $Z_{G',v} \backslash G'_v$.

Now define a $G'(\mathbb{A}) \times G'(\mathbb{A})$ -invariant functional

$$\mathcal{P} : (V_\pi \boxtimes \bar{V}_\pi) \otimes (V_{\pi'} \boxtimes \bar{V}_{\pi'}) \longrightarrow \mathbb{C}$$

by

$$\begin{aligned} & \mathcal{P}(\phi_1, \phi_2; f_1, f_2) \\ &= \left(\int_{Z_{G'}(\mathbb{A})G'(F) \backslash G'(\mathbb{A})} \phi_1(g') f_1(g') dg' \right) \cdot \left(\int_{Z_{G'}(\mathbb{A})G'(F) \backslash G'(\mathbb{A})} \overline{\phi_2(g') f_2(g')} dg' \right) \end{aligned}$$

for $\phi_1, \phi_2 \in V_\pi$ and $f_1, f_2 \in V_{\pi'}$. We call \mathcal{P} the global period integral. On the other hand, for each place v of F , we define a $G'_v \times G'_v$ -invariant functional

$$\mathcal{P}_v^{\natural} : (\pi_v \boxtimes \bar{\pi}_v) \otimes (\pi'_v \boxtimes \bar{\pi}'_v) \longrightarrow \mathbb{C}$$

by

$$\begin{aligned} & \mathcal{P}_v^{\natural}(\phi_{1,v}, \phi_{2,v}; f_{1,v}, f_{2,v}) \\ &= \int_{Z_{G',v} \backslash G'_v} \mathcal{B}_{\pi_v}(\pi_v(g'_v) \phi_{1,v}, \phi_{2,v}) \mathcal{B}_{\pi'_v}(\pi'_v(g'_v) f_{1,v}, f_{2,v}) dg'_v \end{aligned}$$

for $\phi_{1,v}, \phi_{2,v} \in \pi_v$ and $f_{1,v}, f_{2,v} \in \pi'_v$. In §9, we will show that this integral is absolutely convergent. It was shown in [21, Theorem 1.2] that one has

$$\mathcal{P}_v^{\natural}(\phi_{1,v}, \phi_{2,v}; f_{1,v}, f_{2,v}) = \zeta_v(2)\zeta_v(4) \cdot \frac{L_v(\frac{1}{2}, \pi_v \times \pi'_v)}{L_v(1, \pi_v, \mathrm{Ad})L_v(1, \pi'_v, \mathrm{Ad})}$$

for unramified data satisfying the conditions (U1)–(U6) in [21, §1]. This suggests that one normalizes the functional \mathcal{P}_v^{\natural} by setting

$$\mathcal{P}_v = \frac{1}{\zeta_v(2)\zeta_v(4)} \cdot \frac{L_v(1, \pi_v, \mathrm{Ad})L_v(1, \pi'_v, \mathrm{Ad})}{L_v(\frac{1}{2}, \pi_v \times \pi'_v)} \cdot \mathcal{P}_v^{\natural}.$$

Then the product $\prod_v \mathcal{P}_v$ is another $G'(\mathbb{A}) \times G'(\mathbb{A})$ -invariant functional on $(V_\pi \boxtimes \bar{V}_\pi) \otimes (V_{\pi'} \boxtimes \bar{V}_{\pi'})$. Note that $\prod_v \mathcal{P}_v$ does not depend on the choices of the decompositions of $\mathcal{B}_\pi, \mathcal{B}_{\pi'}$, and dg' .

After this preparation, here is our main theorem:

Theorem 1.1. *We have*

$$\mathcal{P} = \frac{\zeta(2)\zeta(4)}{2^\alpha |\mathfrak{X}_T|} \cdot \frac{L(\frac{1}{2}, \pi \times \pi')}{L(1, \pi, \mathrm{Ad})L(1, \pi', \mathrm{Ad})} \cdot \prod_v \mathcal{P}_v$$

as functionals on $(V_\pi \boxtimes \bar{V}_\pi) \otimes (V_{\pi'} \boxtimes \bar{V}_{\pi'})$. Here

$$\alpha = \begin{cases} 3 & \text{if } \mathrm{disc}(V) \in F^{\times,2}, \\ 2 & \text{if } \mathrm{disc}(V) \notin F^{\times,2}. \end{cases}$$

Remark 1.2. The refined Gross-Prasad conjecture [21, Conjecture 1.5] follows from Theorem 1.1 in these cases.

Remark 1.3. Assume that ω_σ is trivial. We regard π (resp. π') as an automorphic representation of $\mathrm{SO}_5(\mathbb{A})$ (resp. $\mathrm{SO}_4(\mathbb{A})$). Let \mathcal{L}_F be the hypothetical Langlands group of F and W_F the Weil group of F . Let

$$\phi : \mathcal{L}_F \longrightarrow {}^L\mathrm{SO}_5 = \mathrm{Sp}_4(\mathbb{C}) \times W_F$$

and

$$\phi' : \mathcal{L}_F \longrightarrow {}^L\mathrm{SO}_4 = \mathrm{SO}_4(\mathbb{C}) \rtimes W_F$$

be the conjectural Arthur parameters of π and π' , respectively (cf. [49]). Let \mathcal{S}_ϕ (resp. $\mathcal{S}_{\phi'}$) be the centralizer of the image of ϕ (resp. ϕ') in $\mathrm{Sp}_4(\mathbb{C})$ (resp. $\mathrm{SO}_4(\mathbb{C})$). Then the Arthur conjecture [2] asserts that

$$|\mathcal{S}_\phi| = \begin{cases} 4 & \text{if } \mathrm{disc}(V) \in F^{\times,2}, \\ 2 & \text{if } \mathrm{disc}(V) \notin F^{\times,2}, \end{cases}$$

and $|\mathcal{S}_{\phi'}| = 2|\mathfrak{X}_\tau|$.

For the π 's and π' 's considered in this paper, the above expectations of the Arthur conjecture are essentially verified in [49] for SO_5 and in [17] for SO_4 . Hence Theorem 1.1 is compatible with [21, Conjecture 2.1], in the sense that we have

$$2^\alpha |\mathfrak{X}_\tau| = |\mathcal{S}_\phi| \cdot |\mathcal{S}_{\phi'}|.$$

This power of 2 is the most subtle part of Theorem 1.1. It gives strong evidence that the Gross-Prasad conjecture is related to the theory of endoscopy.

Remark 1.4. In the theorem, we have assumed that F and E are totally real, so as to use the Siegel-Weil formula by Kudla, Rallis, and Soudry [35]. This is the only place where this assumption is necessary.

Let us describe the main ideas and inputs in the proof of Theorem 1.1. We have a seesaw diagram of reductive dual pairs:

$$\begin{array}{ccc} G = \mathrm{GSp}(W) & & H' = \mathrm{R}_{E/F}(\mathrm{GO}(V_E))' \\ & \searrow & \swarrow \\ G' = \mathrm{R}_{E/F}(\mathrm{GSp}(W_0))' & & H = \mathrm{GO}(V) \end{array}$$

Here

$$\begin{aligned} \mathrm{R}_{E/F}(\mathrm{GSp}(W_0))' &= \{g' \in \mathrm{R}_{E/F}(\mathrm{GSp}(W_0)) \mid \nu(g') \in \mathbb{G}_m\}, \\ \mathrm{R}_{E/F}(\mathrm{GO}(V_E))' &= \{h' \in \mathrm{R}_{E/F}(\mathrm{GO}(V_E)) \mid \nu(h') \in \mathbb{G}_m\}, \end{aligned}$$

and $V_E = V \otimes_F E$. This gives rise to a global seesaw identity, which can be described as a commutative diagram of equivariant maps:

$$\begin{array}{ccc} & (\omega \boxtimes \bar{\omega}) \otimes (\sigma \boxtimes \bar{\sigma}) \otimes (\pi' \boxtimes \bar{\pi}') & \\ & \swarrow \mathcal{T} \quad \quad \quad \searrow \mathcal{T}' & \\ (\theta(\sigma) \boxtimes \overline{\theta(\sigma)}) \otimes (\pi' \boxtimes \bar{\pi}') & & (\sigma \boxtimes \bar{\sigma}) \otimes (\theta(\pi') \boxtimes \overline{\theta(\pi')}) \\ & \searrow \mathcal{P} \quad \quad \quad \swarrow \mathcal{I} & \\ & \mathbb{C} & \end{array}$$

Here

- ω is the Weil representation,
- \mathcal{T} and \mathcal{T}' are equivariant surjective maps induced by the global theta lifts,
- \mathcal{I} is an invariant functional induced by the triple product period integral.

On the other hand, by integrating matrix coefficients, one has a local analogue (the explicit local seesaw identity) of the above commutative diagram for each place v of F . Because of some local multiplicity one theorems, we may compare the product of the local diagrams with the global diagram. Indeed, one has

$$(1.1) \quad \mathcal{T} \approx \otimes_v \mathcal{T}_v, \quad \mathcal{T}' \approx \otimes_v \mathcal{T}'_v, \quad \mathcal{I} \approx \prod_v \mathcal{I}_v,$$

so that

$$(1.2) \quad \mathcal{P} \approx \prod_v \mathcal{P}_v.$$

Here \approx denotes equality up to a scalar. The main theorem amounts to an explicit determination of the constant of proportionality in (1.2). But by the commutativity of the local and global diagrams above, it suffices to determine the three constants of proportionality in (1.1). To determine the constants of proportionality for \mathcal{T} and \mathcal{T}' , we use the Rallis inner product formula, whereas for \mathcal{I} , we use a formula for triple product period integrals by the second author [20] (or rather its extension from $\mathrm{GSO}(V)$ to $\mathrm{GO}(V)$).

This paper is organized as follows. In §2, we study the restriction of automorphic forms on $\mathrm{GO}(V)(\mathbb{A})$ to $\mathrm{GSO}(V)(\mathbb{A})$. This is needed in §3, where we extend the result of the second author [20] and prove a formula for triple product period integrals for $\mathrm{GO}(V)$. In §4 and §5, we study local theta lifts from GL_2 to $\mathrm{GO}(V)$ and those from $\mathrm{GO}(V)$ to GSp_4 , respectively. In §6 and §7, we study global theta lifts from GL_2 to $\mathrm{GO}(V)$ and those from $\mathrm{GO}(V)$ to GSp_4 , respectively. In particular, we construct explicit pairings on the local theta lifts, and using the Rallis inner product formula, we compare the product of the local pairings with the Petersson pairing on the global theta lift. To prove the Rallis inner product formula, we use the Siegel-Weil formula (the second term identity) for $(\mathrm{O}(V), \mathrm{Sp}_4)$ by Kudla and Rallis [31] and Kudla, Rallis, and Soudry [35], and prove a certain spherical second term identity for $(\mathrm{GSp}_4, \mathrm{GO}_8)$. After choosing Haar measures on various groups in §8, we prove in §9 the explicit local seesaw identity with respect to these measures. Finally, using the local and global seesaw identities and the formula for triple product period integrals, we prove Theorem 1.1 in §10. We also include two appendices: the first one determines completely the local theta correspondence for $\mathrm{GO}(V) \times \mathrm{GSp}_4$ and establishes certain properties of the correspondence we need, while the second one proves the spherical second term identity for $(\mathrm{GSp}_4, \mathrm{GO}_8)$.

2. Automorphic forms on $\mathrm{GO}(V)$

Let F be a number field and V a four dimensional quadratic space over F . Set

$$H = \mathrm{GO}(V), \quad H^0 = \mathrm{GSO}(V), \quad \mu_2 = \langle \mathfrak{t} \rangle.$$

Then we have a short exact sequence:

$$1 \longrightarrow H^0 \longrightarrow H \longrightarrow \mu_2 \longrightarrow 1$$

Let Z_H be the identity component of the center of H . We identify μ_2 with $\{+, -\}$. For each place v of F , let \mathfrak{t}_v be the image of \mathfrak{t} in H_v .

Let dh and dh_0 be the Tamagawa measures on $Z_H(\mathbb{A}) \backslash H(\mathbb{A})$ and $Z_H(\mathbb{A}) \backslash H^0(\mathbb{A})$, respectively. Let $d\epsilon_v$ be the Haar measure on $\mu_2(F_v)$ such that $\mathrm{vol}(\mu_2(F_v)) = 1$. Then the product measure $d\epsilon = \prod_v d\epsilon_v$ is the Tamagawa measure on $\mu_2(\mathbb{A})$. Moreover, we have

$$\int_{Z_H(\mathbb{A})H(F) \backslash H(\mathbb{A})} f(h) dh = \int_{\mu_2(F) \backslash \mu_2(\mathbb{A})} \int_{Z_H(\mathbb{A})H^0(F) \backslash H^0(\mathbb{A})} f(h_0\epsilon) dh_0 d\epsilon$$

for $f \in L^1(Z_H(\mathbb{A})H(F) \backslash H(\mathbb{A}))$.

Let $\Pi \cong \otimes_v \Pi_v$ be an irreducible unitary cuspidal automorphic representation of $H(\mathbb{A})$ on the space V_Π . Let \mathfrak{S} be the set of places v of F such that $\Pi_v \otimes \mathrm{sgn} \cong \Pi_v$. Let $\mathcal{B}_\Pi : V_\Pi \otimes \bar{V}_\Pi \rightarrow \mathbb{C}$ be the Petersson pairing

given by

$$\mathcal{B}_\Pi(\phi_1, \phi_2) = \int_{Z_H(\mathbb{A})H(F)\backslash H(\mathbb{A})} \phi_1(h)\overline{\phi_2(h)} dh$$

for $\phi_1, \phi_2 \in V_\Pi$. We have an $H^0(\mathbb{A})$ -equivariant surjective map

$$V_\Pi \longrightarrow V_\Pi|_{H^0(\mathbb{A})},$$

where $V_\Pi|_{H^0(\mathbb{A})}$ is the restriction of V_Π to $H^0(\mathbb{A})$ as functions.

The case $\mathfrak{S} = \emptyset$.

Let π be the automorphic representation of $H^0(\mathbb{A})$ on the space $V_\pi = V_\Pi|_{H^0(\mathbb{A})}$. Then π is irreducible. The restriction to $H^0(\mathbb{A})$ as functions induces an isomorphism

$$V_\Pi \cong V_\pi$$

as representations of $H^0(\mathbb{A})$. Let $\mathcal{B}_\pi : V_\pi \otimes \bar{V}_\pi \rightarrow \mathbb{C}$ be the Petersson pairing.

Lemma 2.1. *We have*

$$\mathcal{B}_\pi(\phi_1|_{H^0(\mathbb{A})}, \phi_2|_{H^0(\mathbb{A})}) = 2 \cdot \mathcal{B}_\Pi(\phi_1, \phi_2)$$

for $\phi_1, \phi_2 \in V_\Pi$.

Proof. For each $\epsilon \in \mu_2(\mathbb{A})$, we define an $H^0(\mathbb{A})$ -invariant pairing $\mathcal{B}_\pi^\epsilon : V_\pi \otimes \bar{V}_\pi \rightarrow \mathbb{C}$ by

$$\mathcal{B}_\pi^\epsilon(\phi_1|_{H^0(\mathbb{A})}, \phi_2|_{H^0(\mathbb{A})}) = \mathcal{B}_\pi(\Pi(\epsilon)\phi_1|_{H^0(\mathbb{A})}, \Pi(\epsilon)\phi_2|_{H^0(\mathbb{A})})$$

for $\phi_1, \phi_2 \in V_\Pi$. Then we have $\mathcal{B}_\pi^\epsilon = C_\epsilon \cdot \mathcal{B}_\pi$ with some constant C_ϵ . Hence we have

$$\begin{aligned} \mathcal{B}_\pi^\epsilon(\phi_1|_{H^0(\mathbb{A})}, \phi_2|_{H^0(\mathbb{A})}) &= C_\epsilon \cdot \mathcal{B}_\pi(\phi_1|_{H^0(\mathbb{A})}, \phi_2|_{H^0(\mathbb{A})}) \\ &= C_\epsilon \cdot \mathcal{B}_\pi^\epsilon(\Pi(\epsilon)\phi_1|_{H^0(\mathbb{A})}, \Pi(\epsilon)\phi_2|_{H^0(\mathbb{A})}) \\ &= C_\epsilon^2 \cdot \mathcal{B}_\pi(\Pi(\epsilon)\phi_1|_{H^0(\mathbb{A})}, \Pi(\epsilon)\phi_2|_{H^0(\mathbb{A})}) \\ &= C_\epsilon^2 \cdot \mathcal{B}_\pi^\epsilon(\phi_1|_{H^0(\mathbb{A})}, \phi_2|_{H^0(\mathbb{A})}), \end{aligned}$$

so that $C_\epsilon^2 = 1$. Since \mathcal{B}_π^ϵ is positive definite, we have $C_\epsilon = 1$. Hence we have

$$\begin{aligned} \mathcal{B}_\Pi(\phi_1, \phi_2) &= \int_{\mu_2(F)\backslash\mu_2(\mathbb{A})} \int_{Z_H(\mathbb{A})H^0(F)\backslash H^0(\mathbb{A})} \phi_1(h_0\epsilon)\overline{\phi_2(h_0\epsilon)} dh_0 d\epsilon \\ &= \int_{\mu_2(F)\backslash\mu_2(\mathbb{A})} \mathcal{B}_\pi^\epsilon(\phi_1|_{H^0(\mathbb{A})}, \phi_2|_{H^0(\mathbb{A})}) d\epsilon \\ &= \text{vol}(\mu_2(F)\backslash\mu_2(\mathbb{A})) \cdot \mathcal{B}_\pi(\phi_1|_{H^0(\mathbb{A})}, \phi_2|_{H^0(\mathbb{A})}). \end{aligned}$$

□

The case $\mathfrak{S} \neq \emptyset$.

We fix an isomorphism

$$(2.1) \quad V_\Pi \cong \otimes_v \mathcal{V}_v = \varinjlim_S (\otimes_{v \in S} \mathcal{V}_v) \otimes (\otimes_{v \notin S} \phi_v)$$

as representations of $H(\mathbb{A})$, where \mathcal{V}_v is the space of Π_v , S is a sufficiently large finite set of places of F , and ϕ_v is an $H(\mathfrak{o}_v)$ -invariant element of \mathcal{V}_v for $v \notin S$.

If $v \in \mathfrak{S}$, then we can write $\Pi_v|_{H_v^0} = \pi_v^+ \oplus \pi_v^-$, where π_v^\pm is an irreducible admissible representation of H_v^0 . Note that $\pi_v^+ \not\cong \pi_v^-$ and $\pi_v^+ \circ \text{Ad}(\mathfrak{t}_v) \cong \pi_v^-$. We have $\mathcal{V}_v = \mathcal{V}_v^+ \oplus \mathcal{V}_v^-$, where \mathcal{V}_v^\pm is the space of π_v^\pm and $\mathcal{V}_v^- = \Pi_v(\mathfrak{t}_v)(\mathcal{V}_v^+)$. We have $\phi_v = \phi_v^+ + \phi_v^-$ for almost all $v \in \mathfrak{S}$, where ϕ_v^\pm is an $H^0(\mathfrak{o}_v)$ -invariant element of \mathcal{V}_v^\pm and $\phi_v^- = \Pi_v(\mathfrak{t}_v)(\phi_v^+)$. If $v \notin \mathfrak{S}$, then $\pi_v = \Pi_v|_{H_v^0}$ is an irreducible admissible representation of H_v^0 on the space \mathcal{V}_v .

Let S be a sufficiently large finite set of places of F . For $\epsilon = (\epsilon_v) \in \mu_2(\mathbb{A}_{S \cap \mathfrak{S}})$, let $V_{\Pi, S}^\epsilon$ be the inverse image of

$$(\otimes_{v \in S \cap \mathfrak{S}} \mathcal{V}_v^{\epsilon_v}) \otimes (\otimes_{v \in S, v \notin \mathfrak{S}} \mathcal{V}_v) \otimes (\otimes_{v \notin S} \phi_v)$$

in V_Π by (2.1). Then $H^0(\mathbb{A}_S) \mu_2(\mathbb{A}^{S \cap \mathfrak{S}})$ acts on $V_{\Pi, S}^\epsilon$ and the representation of $H^0(\mathbb{A}_S)$ on $V_{\Pi, S}^\epsilon$ is given by

$$\pi_S^\epsilon = (\otimes_{v \in S \cap \mathfrak{S}} \pi_v^{\epsilon_v}) \otimes (\otimes_{v \in S, v \notin \mathfrak{S}} \pi_v).$$

Hence we have

$$\Pi|_{H^0(\mathbb{A})} \cong \varinjlim_S \oplus_{\epsilon \in \mu_2(\mathbb{A}_{S \cap \mathfrak{S}})} \pi_S^\epsilon, \quad V_\Pi = \varinjlim_S \oplus_{\epsilon \in \mu_2(\mathbb{A}_{S \cap \mathfrak{S}})} V_{\Pi, S}^\epsilon,$$

as representations of $H^0(\mathbb{A})$.

By [15, §1], there exists an irreducible unitary cuspidal automorphic representation π of $H^0(\mathbb{A})$ on the space V_π such that

$$(2.2) \quad V_\Pi|_{H^0(\mathbb{A})} = V_\pi \oplus V_{\pi \circ \mathrm{Ad}(\mathfrak{t})}.$$

We may assume that

$$\begin{aligned} \pi &\cong (\otimes_{v \in \mathfrak{S}} \pi_v^+) \otimes (\otimes_{v \notin \mathfrak{S}} \pi_v), \\ V_\pi &\cong \varinjlim_S (\otimes_{v \in S \cap \mathfrak{S}} \mathcal{V}_v^+) \otimes (\otimes_{v \in S, v \notin \mathfrak{S}} \mathcal{V}_v) \otimes (\otimes_{v \notin S, v \in \mathfrak{S}} \phi_v^+) \otimes (\otimes_{v \notin S, v \notin \mathfrak{S}} \phi_v). \end{aligned}$$

Then we have

$$\begin{aligned} \pi \circ \mathrm{Ad}(\mathfrak{t}) &\cong (\otimes_{v \in \mathfrak{S}} \pi_v^-) \otimes (\otimes_{v \notin \mathfrak{S}} \pi_v), \\ V_{\pi \circ \mathrm{Ad}(\mathfrak{t})} &\cong \varinjlim_S (\otimes_{v \in S \cap \mathfrak{S}} \mathcal{V}_v^-) \otimes (\otimes_{v \in S, v \notin \mathfrak{S}} \mathcal{V}_v) \otimes (\otimes_{v \notin S, v \in \mathfrak{S}} \phi_v^-) \otimes (\otimes_{v \notin S, v \notin \mathfrak{S}} \phi_v). \end{aligned}$$

Lemma 2.2. *For $\phi \in V_{\Pi, S}^1$, the support of ϕ is contained in*

$$H^0(\mathbb{A}) \mu_2(\mathbb{A}^{S \cap \mathfrak{S}}) \cup H^0(\mathbb{A}) \mu_2(\mathbb{A}^{S \cap \mathfrak{S}}) \mathfrak{t}.$$

Proof. Let $\epsilon \in \mu_2(\mathbb{A}_{S \cap \mathfrak{S}})$. By (2.2), we have $V_{\Pi, S}^\epsilon|_{H^0(\mathbb{A})} = 0$ unless $\epsilon \in \mu_2(F)$. Since $V_{\Pi, S}^\epsilon = \{\Pi(\epsilon)\phi \mid \phi \in V_{\Pi, S}^1\}$ and

$$H(\mathbb{A}) = \bigcup_{\epsilon \in \mu_2(\mathbb{A}_{S \cap \mathfrak{S}})} H^0(\mathbb{A}) \mu_2(\mathbb{A}^{S \cap \mathfrak{S}}) \epsilon,$$

the assertion follows. \square

Let $\mathcal{B}_\pi : V_\pi \otimes \bar{V}_\pi \rightarrow \mathbb{C}$ be the Petersson pairing. We fix a decomposition $\mathcal{B}_\pi = \prod_{v \in \mathfrak{S}} \mathcal{B}_v^+ \prod_{v \notin \mathfrak{S}} \mathcal{B}_v$, where

- $\mathcal{B}_v^+ : \mathcal{V}_v^+ \otimes \bar{\mathcal{V}}_v^+ \rightarrow \mathbb{C}$ is an H_v^0 -invariant pairing if $v \in \mathfrak{S}$,
- $\mathcal{B}_v : \mathcal{V}_v \otimes \bar{\mathcal{V}}_v \rightarrow \mathbb{C}$ is an H_v -invariant pairing if $v \notin \mathfrak{S}$,
- $\mathcal{B}_v^+(\phi_v^+, \phi_v^+) = \mathcal{B}_v(\phi_v, \phi_v) = 1$ for almost all v .

For each $v \in \mathfrak{S}$, we define an H_v^0 -invariant pairing $\mathcal{B}_v^- : \mathcal{V}_v^- \otimes \bar{\mathcal{V}}_v^- \rightarrow \mathbb{C}$ by

$$\mathcal{B}_v^-(\phi_1, \phi_2) = \mathcal{B}_v^+(\Pi_v(\mathfrak{t}_v)\phi_1, \Pi_v(\mathfrak{t}_v)\phi_2)$$

for $\phi_1, \phi_2 \in \mathcal{V}_v^-$. Then $\mathcal{B}_v^-(\phi_v^-, \phi_v^-) = 1$ for almost all $v \in \mathfrak{S}$. For each place v of F , we define an H_v -invariant pairing $\mathcal{B}_{\Pi_v}^\natural : \mathcal{V}_v \otimes \bar{\mathcal{V}}_v \rightarrow \mathbb{C}$ as follows:

- If $v \in \mathfrak{S}$, let $\mathcal{B}_{\Pi_v}^\natural(\phi_1^+ + \phi_1^-, \phi_2^+ + \phi_2^-) = \frac{1}{2}(\mathcal{B}_v^+(\phi_1^+, \phi_2^+) + \mathcal{B}_v^-(\phi_1^-, \phi_2^-))$ for $\phi_1^+, \phi_2^+ \in \mathcal{V}_v^+$ and $\phi_1^-, \phi_2^- \in \mathcal{V}_v^-$.

- If $v \notin \mathfrak{S}$, let $\mathcal{B}_{\Pi_v}^{\mathfrak{h}} = \mathcal{B}_v$.

Then $\mathcal{B}_{\Pi_v}^{\mathfrak{h}}(\phi_v, \phi_v) = 1$ for almost all v .

Lemma 2.3. *We have*

$$\mathcal{B}_{\Pi} = \prod_v \mathcal{B}_{\Pi_v}^{\mathfrak{h}}.$$

Proof. Fix a sufficiently large finite set S of places of F . Put

$$S' = S \setminus (S \cap \mathfrak{S}), \quad s = |S \cap \mathfrak{S}|, \quad s' = |S'|.$$

Let $\phi_1 = (\otimes_{v \in S} \phi_{1,v}) \otimes (\otimes_{v \notin S} \phi_v)$, $\phi_2 = (\otimes_{v \in S} \phi_{2,v}) \otimes (\otimes_{v \notin S} \phi_v) \in V_{\Pi, S}^1$, where $\phi_{1,v}, \phi_{2,v} \in \mathcal{V}_v^+$ (resp. $\phi_{1,v}, \phi_{2,v} \in \mathcal{V}_v$) if $v \in S \cap \mathfrak{S}$ (resp. if $v \in S'$). Then $\mathcal{B}_{\Pi}(\phi_1, \phi_2)$ is equal to

$$\begin{aligned} & \int_{\mu_2(F) \backslash \mu_2(\mathbb{A})} \int_{Z_H(\mathbb{A})H^0(F) \backslash H^0(\mathbb{A})} \phi_1(h_0\epsilon) \overline{\phi_2(h_0\epsilon)} dh_0 d\epsilon \\ &= \frac{1}{2} \int_{\mu_2(\mathbb{A})} \int_{Z_H(\mathbb{A})H^0(F) \backslash H^0(\mathbb{A})} \phi_1(h_0\epsilon) \overline{\phi_2(h_0\epsilon)} dh_0 d\epsilon \\ &= \frac{1}{2^{s+s'+1}} \sum_{\epsilon \in \mu_2(\mathbb{A}_{S'})} \int_{Z_H(\mathbb{A})H^0(F) \backslash H^0(\mathbb{A})} \phi_1(h_0\epsilon) \overline{\phi_2(h_0\epsilon)} dh_0. \end{aligned}$$

By Lemma 2.2, this integral is equal to

$$\frac{1}{2^{s+s'+1}} \sum_{\epsilon \in \mu_2(\mathbb{A}_{S'})} \int_{Z_H(\mathbb{A})H^0(F) \backslash H^0(\mathbb{A})} (\phi_1(h_0\epsilon) \overline{\phi_2(h_0\epsilon)} + \phi_1(h_0\epsilon\mathfrak{t}) \overline{\phi_2(h_0\epsilon\mathfrak{t})}) dh_0.$$

We have

$$\begin{aligned} & \int_{Z_H(\mathbb{A})H^0(F) \backslash H^0(\mathbb{A})} \phi_1(h_0\epsilon\mathfrak{t}) \overline{\phi_2(h_0\epsilon\mathfrak{t})} dh_0 \\ &= \int_{Z_H(\mathbb{A})H^0(F) \backslash H^0(\mathbb{A})} \phi_1(\text{Ad}(\mathfrak{t})(h_0)\epsilon) \overline{\phi_2(\text{Ad}(\mathfrak{t})(h_0)\epsilon)} dh_0 \\ &= \int_{Z_H(\mathbb{A})H^0(F) \backslash H^0(\mathbb{A})} \phi_1(h_0\epsilon) \overline{\phi_2(h_0\epsilon)} dh_0. \end{aligned}$$

Hence we have

$$\mathcal{B}_{\Pi}(\phi_1, \phi_2) = \frac{1}{2^{s+s'}} \sum_{\epsilon \in \mu_2(\mathbb{A}_{S'})} \int_{Z_H(\mathbb{A})H^0(F) \backslash H^0(\mathbb{A})} \phi_1(h_0\epsilon) \overline{\phi_2(h_0\epsilon)} dh_0.$$

Since $\Pi(\epsilon)\phi_1, \Pi(\epsilon)\phi_2 \in V_{\Pi, S}^1$ for $\epsilon \in \mu_2(\mathbb{A}_{S'})$, $\mathcal{B}_{\Pi}(\phi_1, \phi_2)$ is equal to

$$\begin{aligned} & \frac{1}{2^{s+s'}} \sum_{\epsilon \in \mu_2(\mathbb{A}_{S'})} \mathcal{B}_{\pi}(\Pi(\epsilon)\phi_1|_{H^0(\mathbb{A})}, \Pi(\epsilon)\phi_2|_{H^0(\mathbb{A})}) \\ &= \frac{1}{2^{s+s'}} \sum_{\epsilon \in \mu_2(\mathbb{A}_{S'})} \prod_{v \in S \cap \mathfrak{S}} \mathcal{B}_v^+(\phi_{1,v}, \phi_{2,v}) \prod_{v \in S'} \mathcal{B}_v(\Pi_v(\epsilon_v)\phi_{1,v}, \Pi_v(\epsilon_v)\phi_{2,v}) \\ &= \frac{1}{2^s} \prod_{v \in S \cap \mathfrak{S}} \mathcal{B}_v^+(\phi_{1,v}, \phi_{2,v}) \prod_{v \in S'} \mathcal{B}_v(\phi_{1,v}, \phi_{2,v}) \\ &= \prod_{v \in S \cap \mathfrak{S}} \mathcal{B}_{\Pi_v}^{\mathfrak{h}}(\phi_{1,v}, \phi_{2,v}) \prod_{v \in S'} \mathcal{B}_{\Pi_v}^{\mathfrak{h}}(\phi_{1,v}, \phi_{2,v}). \end{aligned}$$

This completes the proof. \square

3. Triple product period integrals for $\mathrm{GO}(V)$

Let F be a number field and E an étale quadratic algebra over F . Let V be a four dimensional quadratic space over F . Set

$$H = \mathrm{GO}(V), \quad H^0 = \mathrm{GSO}(V).$$

Let K be the discriminant algebra of V and D a quaternion algebra over F associated to V as in §1.

Let $\Pi \cong \otimes_v \Pi_v$ (resp. $\Pi' \cong \otimes_v \Pi'_v$) be an irreducible unitary cuspidal automorphic representation of $H(\mathbb{A})$ (resp. $H(\mathbb{A}_E)$) on the space V_Π (resp. $V_{\Pi'}$) with central character ω_Π (resp. $\omega_{\Pi'}$). We assume the following:

- $\omega_\Pi \omega_{\Pi'}$ is trivial on $Z_H(\mathbb{A})$.
- $\Pi_v \otimes \mathrm{sgn} \cong \Pi_v$ for some place v of F .
- $\Pi'_v \otimes \mathrm{sgn} \not\cong \Pi'_v$ for all places v of F .

Let π (resp. π') be an irreducible unitary cuspidal automorphic representation of $H^0(\mathbb{A})$ (resp. $H^0(\mathbb{A}_E)$) on the space V_π (resp. $V_{\pi'}$) such that $V_\Pi|_{H^0(\mathbb{A})} = V_\pi \oplus V_{\pi \circ \mathrm{Ad}(\mathfrak{t})}$ (resp. $V_{\Pi'}|_{H^0(\mathbb{A}_E)} = V_{\pi'}$). Let $\dot{\pi}$ (resp. $\dot{\pi}'$) be the Jacquet-Langlands transfer of $\pi|_{D^\times(\mathbb{A}_K)}$ (resp. $\pi'|_{D^\times(\mathbb{A}_{E \otimes K})}$) to $\mathrm{GL}_2(\mathbb{A}_K)$ (resp. $\mathrm{GL}_2(\mathbb{A}_{E \otimes K})$). We define the adjoint L -functions of Π and Π' by

$$L(s, \Pi, \mathrm{Ad}) = L(s, \dot{\pi}, \mathrm{Ad}) \quad \text{and} \quad L(s, \Pi', \mathrm{Ad}) = L(s, \dot{\pi}', \mathrm{Ad}),$$

respectively. Note that $L(s, \Pi, \mathrm{Ad})$ does not depend on the choice of π . We define an L -function $L(s, \Pi \times \Pi')$ of degree eight over K by

$$L(s, \Pi \times \Pi') = \prod_v L_v(s, \dot{\pi}_v \times \dot{\pi}'_v),$$

where $L_v(s, \dot{\pi}_v \times \dot{\pi}'_v)$ is the triple product L -factor associated to the Langlands parameters of $\dot{\pi}_v$ and $\dot{\pi}'_v$ and the eight dimensional representation of ${}^L\mathrm{R}_{(K \times E \otimes K)/K}(\mathrm{GL}_2)$ defined in [45, §0]. We remark that there is another definition of this L -factor à la Garrett [10], Piatetski-Shapiro and Rallis [45], and Ikeda [22] using local zeta integrals and these two definitions agree if v is non-archimedean and $\dot{\pi}_v$ and $\dot{\pi}'_v$ are unramified, but we do not assume that they agree for all v in this paper. The following lemma asserts that $L(s, \Pi \times \Pi')$ does not depend on the choice of π .

Lemma 3.1. *We have*

$$L_v(s, (\dot{\pi}_v \circ c) \times \dot{\pi}'_v) = L_v(s, \dot{\pi}_v \times \dot{\pi}'_v),$$

where c is the non-trivial automorphism of K over F .

Proof. We fix a place v of F and suppress it from the notation. Let W_F be the Weil group of F and \mathcal{L}_F the Langlands group of F given by

$$\mathcal{L}_F = \begin{cases} W_F \times \mathrm{SL}_2(\mathbb{C}) & \text{if } F \text{ is non-archimedean,} \\ W_F & \text{if } F \text{ is archimedean.} \end{cases}$$

We only consider the case where E and K are quadratic extensions of F and $E \neq K$; the other cases are similar. Then $EK \cong E \otimes K$ is a quartic extension of F . Let

$$\mathrm{BC}_{K/F} : {}^L\mathrm{GL}_4 \longrightarrow {}^L\mathrm{R}_{K/F}(\mathrm{GL}_4) \quad \text{and} \quad \mathrm{BC}_{EK/E} : {}^L\mathrm{R}_{E/F}(\mathrm{GL}_2) \longrightarrow {}^L\mathrm{R}_{EK/F}(\mathrm{GL}_2)$$

be the base change L -homomorphisms. We define an L -homomorphism

$$\mathrm{Asai}_{E/F} : {}^L\mathrm{R}_{E/F}(\mathrm{GL}_2) \longrightarrow {}^L\mathrm{GL}_4$$

by

$$\mathrm{Asai}_{E/F}((g_1, g_2), 1) = (g_1 \otimes g_2, 1),$$

$$\text{Asai}_{E/F}((1, 1), w) = \begin{cases} (\text{id}, w) & \text{if } w \in W_E, \\ (\text{sw}, w) & \text{if } w \notin W_E, \end{cases}$$

for $g_1, g_2 \in \text{GL}_2(\mathbb{C})$ and $w \in W_F$, where $\text{sw} : \mathbb{C}^2 \otimes \mathbb{C}^2 \rightarrow \mathbb{C}^2 \otimes \mathbb{C}^2$ is an isomorphism given by $\text{sw}(x \otimes y) = y \otimes x$. Similarly, we define an L -homomorphism $\text{Asai}_{EK/K} : {}^L\text{R}_{EK/F}(\text{GL}_2) \rightarrow {}^L\text{R}_{K/F}(\text{GL}_4)$. Then we have

$$\text{Asai}_{EK/K} \circ \text{BC}_{EK/E} = \text{BC}_{K/F} \circ \text{Asai}_{E/F}.$$

Let $\phi : \mathcal{L}_F \rightarrow {}^L\text{R}_{K/F}(\text{GL}_2)$ and $\phi' : \mathcal{L}_F \rightarrow {}^L\text{R}_{EK/F}(\text{GL}_2)$ be the Langlands parameters of $\dot{\pi}$ and $\dot{\pi}'$, respectively. We identify ϕ and $\text{Asai}_{EK/K} \circ \phi'$ with homomorphisms $\phi : \mathcal{L}_K \rightarrow \text{GL}_2(\mathbb{C})$ and $\text{Asai}_{EK/K} \circ \phi' : \mathcal{L}_K \rightarrow \text{GL}_4(\mathbb{C})$, respectively. By definition, we have

$$L(s, \dot{\pi} \times \dot{\pi}') = L(s, \phi \otimes (\text{Asai}_{EK/K} \circ \phi')).$$

By assumption on Π' , there exists a Langlands parameter $\phi'' : \mathcal{L}_F \rightarrow {}^L\text{R}_{E/F}(\text{GL}_2)$ such that $\phi' = \text{BC}_{EK/E} \circ \phi''$. Hence we have

$$\begin{aligned} L(s, \phi \otimes (\text{Asai}_{EK/K} \circ \phi')) &= L(s, \phi \otimes (\text{Asai}_{EK/K} \circ \text{BC}_{EK/E} \circ \phi'')) \\ &= L(s, \phi \otimes (\text{BC}_{K/F} \circ \text{Asai}_{E/F} \circ \phi'')). \end{aligned}$$

This completes the proof. \square

Let $\mathcal{B}_\Pi : V_\Pi \otimes \bar{V}_\Pi \rightarrow \mathbb{C}$ and $\mathcal{B}_{\Pi'} : V_{\Pi'} \otimes \bar{V}_{\Pi'} \rightarrow \mathbb{C}$ be the Petersson pairings. We fix decompositions $\mathcal{B}_\Pi = \prod_v \mathcal{B}_{\Pi_v}$ and $\mathcal{B}_{\Pi'} = \prod_v \mathcal{B}_{\Pi'_v}$, where $\mathcal{B}_{\Pi_v} : \Pi_v \otimes \bar{\Pi}_v \rightarrow \mathbb{C}$ and $\mathcal{B}_{\Pi'_v} : \Pi'_v \otimes \bar{\Pi}'_v \rightarrow \mathbb{C}$ are pairings. Let dh be the Tamagawa measure on $Z_H(\mathbb{A}) \backslash H(\mathbb{A})$. We fix a decomposition $dh = \prod_v dh_v$, where dh_v is a Haar measure on $Z_{H,v} \backslash H_v$. We define an $H(\mathbb{A}) \times H(\mathbb{A})$ -invariant functional

$$\mathcal{I} : (V_\Pi \boxtimes \bar{V}_\Pi) \otimes (V_{\Pi'} \boxtimes \bar{V}_{\Pi'}) \longrightarrow \mathbb{C}$$

by

$$\begin{aligned} &\mathcal{I}(\phi_1, \phi_2; \phi'_1, \phi'_2) \\ &= \left(\int_{Z_H(\mathbb{A})H(F) \backslash H(\mathbb{A})} \phi_1(h) \phi'_1(h) dh \right) \cdot \left(\int_{Z_H(\mathbb{A})H(F) \backslash H(\mathbb{A})} \overline{\phi_2(h) \phi'_2(h)} dh \right) \end{aligned}$$

for $\phi_1, \phi_2 \in V_\Pi$ and $\phi'_1, \phi'_2 \in V_{\Pi'}$. For each place v of F , we define an $H_v \times H_v$ -invariant functional

$$\mathcal{I}_v^\natural : (\Pi_v \boxtimes \bar{\Pi}_v) \otimes (\Pi'_v \boxtimes \bar{\Pi}'_v) \longrightarrow \mathbb{C}$$

by

$$\begin{aligned} &\mathcal{I}_v^\natural(\phi_{1,v}, \phi_{2,v}; \phi'_{1,v}, \phi'_{2,v}) \\ &= \int_{Z_{H,v} \backslash H_v} \mathcal{B}_{\Pi_v}(\Pi_v(h_v) \phi_{1,v}, \phi_{2,v}) \mathcal{B}_{\Pi'_v}(\Pi'_v(h_v) \phi'_{1,v}, \phi'_{2,v}) dh_v \end{aligned}$$

for $\phi_{1,v}, \phi_{2,v} \in \Pi_v$ and $\phi'_{1,v}, \phi'_{2,v} \in \Pi'_v$. By [20, Lemma 2.1], this integral is absolutely convergent.

Proposition 3.2. *We have*

$$\mathcal{I} = 2^c \cdot \zeta_{E \otimes K}(2) \cdot \frac{L(\frac{1}{2}, \Pi \times \Pi')}{L(1, \Pi, \text{Ad})L(1, \Pi', \text{Ad})} \cdot \prod_v \mathcal{I}_v$$

as functionals on $(V_{\Pi} \boxtimes \bar{V}_{\Pi}) \otimes (V_{\Pi'} \boxtimes \bar{V}_{\Pi'})$. Here

$$c = \begin{cases} -4 & \text{if } E = K = F \times F, \\ -1 & \text{if } E = F \times F \text{ and } K \text{ is a quadratic extension of } F, \\ -3 & \text{if } E \text{ is a quadratic extension of } F \text{ and } K = F \times F, \\ -2 & \text{if } E \text{ and } K \text{ are quadratic extensions of } F \text{ and } E = K, \\ -1 & \text{if } E \text{ and } K \text{ are quadratic extensions of } F \text{ and } E \neq K, \end{cases}$$

and

$$\mathcal{I}_v = \frac{1}{\zeta_{E_v \otimes K_v}(2)} \cdot \frac{L_v(1, \Pi_v, \mathrm{Ad}) L_v(1, \Pi'_v, \mathrm{Ad})}{L_v(\frac{1}{2}, \Pi_v \times \Pi'_v)} \cdot \mathcal{I}_v^{\natural}.$$

The rest of this section is devoted to the proof of Proposition 3.2. Let \mathfrak{S} be the set of places v of F such that $\Pi_v \otimes \mathrm{sgn} \cong \Pi_v$. Fix a sufficiently large finite set S of places of F . Put

$$S' = S \setminus (S \cap \mathfrak{S}), \quad s = |S \cap \mathfrak{S}|, \quad s' = |S'|.$$

We may assume that $\phi_1, \phi_2 \in V_{\Pi, S}^1$ and $\phi'_1, \phi'_2 \in V_{\Pi', S}$. Here $V_{\Pi, S}^1$ is the subspace of V_{Π} given in §2 and $V_{\Pi', S}$ is the subspace of $V_{\Pi'}$ consisting of $\prod_{v \notin S} H(\mathfrak{o}_{E_v})$ -invariant elements.

Lemma 3.3. *We have*

$$\begin{aligned} & \int_{Z_H(\mathbb{A})H(F)\backslash H(\mathbb{A})} \phi(h)\phi'(h) dh \\ &= \frac{1}{2^{s+s'}} \sum_{\epsilon \in \mu_2(\mathbb{A}_{S'})} \int_{Z_H(\mathbb{A})H^0(F)\backslash H^0(\mathbb{A})} \phi(h_0\epsilon)\phi'(h_0\epsilon) dh_0 \end{aligned}$$

for $\phi \in V_{\Pi, S}^1$ and $\phi' \in V_{\Pi', S}$, where dh_0 is the Tamagawa measure on $Z_H(\mathbb{A})\backslash H^0(\mathbb{A})$.

Proof. As in the proof of Lemma 2.3, we have

$$\begin{aligned} & \int_{Z_H(\mathbb{A})H(F)\backslash H(\mathbb{A})} \phi(h)\phi'(h) dh \\ &= \frac{1}{2^{s+s'+1}} \sum_{\epsilon \in \mu_2(\mathbb{A}_S)} \int_{Z_H(\mathbb{A})H^0(F)\backslash H^0(\mathbb{A})} \phi(h_0\epsilon)\phi'(h_0\epsilon) dh_0. \end{aligned}$$

By Lemma 2.2, this integral is equal to

$$\frac{1}{2^{s+s'+1}} \sum_{\epsilon \in \mu_2(\mathbb{A}_{S'})} \int_{Z_H(\mathbb{A})H^0(F)\backslash H^0(\mathbb{A})} (\phi(h_0\epsilon)\phi'(h_0\epsilon) + \phi(h_0\epsilon\mathfrak{t})\phi'(h_0\epsilon\mathfrak{t})) dh_0.$$

This completes the proof. \square

We fix an isomorphism $\pi \cong (\otimes_{v \in \mathfrak{S}} \pi_v^+) \otimes (\otimes_{v \notin \mathfrak{S}} \pi_v)$ and a decomposition $\mathcal{B}_{\pi} = \prod_{v \in \mathfrak{S}} \mathcal{B}_v^+ \prod_{v \notin \mathfrak{S}} \mathcal{B}_v$ as in §2. By Lemma 2.3, we may assume that $\mathcal{B}_{\Pi_v} = \mathcal{B}_{\Pi_v}^{\natural}$, where $\mathcal{B}_{\Pi_v}^{\natural} : \Pi_v \otimes \bar{\Pi}_v \rightarrow \mathbb{C}$ is the pairing given in §2. We fix an isomorphism $\pi' \cong \otimes_v \pi'_v$. Let $\mathcal{B}_{\pi'_v}^b : \pi'_v \otimes \bar{\pi}'_v \rightarrow \mathbb{C}$ be the pairing given by $\mathcal{B}_{\pi'_v}^b = \mathcal{B}_{\Pi'_v} |_{\pi'_v \otimes \bar{\pi}'_v}$. By Lemma 2.1, we have

$$\mathcal{B}_{\pi'} = 2^{\beta} \prod_v \mathcal{B}_{\pi'_v}^b.$$

Here

$$\beta = \begin{cases} 2 & \text{if } E = F \times F, \\ 1 & \text{if } E \text{ is a quadratic extension of } F. \end{cases}$$

Let $dh_{0,v}$ be the Haar measure on $Z_{H,v} \backslash H_v^0$ such that

$$\int_{Z_{H,v} \backslash H_v} f(h_v) dh_v = \frac{1}{2} \sum_{\epsilon_v \in \mu_2(F_v)} \int_{Z_{H,v} \backslash H_v^0} f(h_{0,v} \epsilon_v) dh_{0,v}$$

for $f \in L^1(Z_{H,v} \backslash H_v)$. Then the product measure $\prod_v dh_{0,v}$ is the Tamagawa measure on $Z_H(\mathbb{A}) \backslash H^0(\mathbb{A})$. We define an $H_v^0 \times H_v^0$ -invariant functional

$$\mathcal{J}_v : (\pi_v^\bullet \boxtimes \bar{\pi}_v^\bullet) \otimes (\pi'_v \boxtimes \bar{\pi}'_v) \longrightarrow \mathbb{C}$$

by

$$\begin{aligned} & \mathcal{J}_v(\phi_{1,v}, \phi_{2,v}; \phi'_{1,v}, \phi'_{2,v}) \\ &= \int_{Z_{H,v} \backslash H_v^0} \mathcal{B}_v^\bullet(\pi_v^\bullet(h_{0,v})\phi_{1,v}, \phi_{2,v}) \mathcal{B}_{\pi'_v}^\flat(\pi'_v(h_{0,v})\phi'_{1,v}, \phi'_{2,v}) dh_{0,v} \end{aligned}$$

for $\phi_{1,v}, \phi_{2,v} \in \pi_v^\bullet$ and $\phi'_{1,v}, \phi'_{2,v} \in \pi'_v$, where

$$\bullet = \begin{cases} + & \text{if } v \in \mathfrak{S}, \\ \emptyset & \text{if } v \notin \mathfrak{S}. \end{cases}$$

By [20, Theorem 1.1] and Lemma 3.3, $\mathcal{I}(\phi_1, \phi_2; \phi'_1, \phi'_2)$ is equal to

$$\begin{aligned} & 2^{\beta+c_0} \cdot \zeta_{E \otimes K}^S(2) \cdot \frac{L^S(\frac{1}{2}, \dot{\pi} \times \dot{\pi}')}{L^S(1, \dot{\pi}, \text{Ad}) L^S(1, \dot{\pi}', \text{Ad})} \\ & \times \frac{1}{2^{2s+2s'}} \sum_{\epsilon \in \mu_2(\mathbb{A}_{S'})} \sum_{\epsilon' \in \mu_2(\mathbb{A}_{S'})} \prod_{v \in S} \mathcal{J}_v(\Pi_v(\epsilon_v)\phi_{1,v}, \Pi_v(\epsilon'_v)\phi_{2,v}; \Pi'_v(\epsilon_v)\phi'_{1,v}, \Pi'_v(\epsilon'_v)\phi'_{2,v}) \end{aligned}$$

for $\phi_1 = \otimes_v \phi_{1,v}, \phi_2 = \otimes_v \phi_{2,v} \in V_{\Pi,S}^1$ and $\phi'_1 = \otimes_v \phi'_{1,v}, \phi'_2 = \otimes_v \phi'_{2,v} \in V_{\Pi',S}$. Here

$$c_0 = \begin{cases} -6 & \text{if } E = K = F \times F, \\ -3 & \text{if } E = F \times F \text{ and } K \text{ is a quadratic extension of } F, \\ -4 & \text{if } E \text{ is a quadratic extension of } F \text{ and } K = F \times F, \\ -3 & \text{if } E \text{ and } K \text{ are quadratic extensions of } F \text{ and } E = K, \\ -2 & \text{if } E \text{ and } K \text{ are quadratic extensions of } F \text{ and } E \neq K. \end{cases}$$

To finish the proof of Proposition 3.2, it remains to show the following lemma.

Lemma 3.4. *We have*

$$\begin{aligned} & \frac{1}{2^{2s+2s'}} \sum_{\epsilon \in \mu_2(\mathbb{A}_{S'})} \sum_{\epsilon' \in \mu_2(\mathbb{A}_{S'})} \prod_{v \in S} \mathcal{J}_v(\Pi_v(\epsilon_v)\phi_{1,v}, \Pi_v(\epsilon'_v)\phi_{2,v}; \Pi'_v(\epsilon_v)\phi'_{1,v}, \Pi'_v(\epsilon'_v)\phi'_{2,v}) \\ &= \prod_{v \in S} \mathcal{I}_v^\natural(\phi_{1,v}, \phi_{2,v}; \phi'_{1,v}, \phi'_{2,v}) \end{aligned}$$

for $\phi_1 = \otimes_v \phi_{1,v}, \phi_2 = \otimes_v \phi_{2,v} \in V_{\Pi,S}^1$ and $\phi'_1 = \otimes_v \phi'_{1,v}, \phi'_2 = \otimes_v \phi'_{2,v} \in V_{\Pi',S}$.

Proof. If $v \in \mathfrak{S}$, then

$$\begin{aligned} & \frac{1}{2^2} \cdot \mathcal{J}_v(\phi_{1,v}, \phi_{2,v}; \phi'_{1,v}, \phi'_{2,v}) \\ &= \frac{1}{2} \int_{Z_{H,v} \backslash H_v^0} \mathcal{B}_{\Pi_v}^\natural(\Pi_v(h_{0,v})\phi_{1,v}, \phi_{2,v}) \mathcal{B}_{\Pi'_v}(\Pi'_v(h_{0,v})\phi'_{1,v}, \phi'_{2,v}) dh_{0,v} \\ &= \frac{1}{2} \sum_{\epsilon_v \in \mu_2(F_v)} \int_{Z_{H,v} \backslash H_v^0} \mathcal{B}_{\Pi_v}^\natural(\Pi_v(h_{0,v}\epsilon_v)\phi_{1,v}, \phi_{2,v}) \mathcal{B}_{\Pi'_v}(\Pi'_v(h_{0,v}\epsilon_v)\phi'_{1,v}, \phi'_{2,v}) dh_{0,v} \end{aligned}$$

$$= \mathcal{I}_v^{\natural}(\phi_{1,v}, \phi_{2,v}; \phi'_{1,v}, \phi'_{2,v}).$$

If $v \notin \mathfrak{S}$, then

$$\begin{aligned} & \frac{1}{2^2} \sum_{\epsilon_v \in \mu_2(F_v)} \sum_{\epsilon'_v \in \mu_2(F_v)} \mathcal{J}_v(\Pi_v(\epsilon_v)\phi_{1,v}, \Pi_v(\epsilon'_v)\phi_{2,v}; \Pi'_v(\epsilon_v)\phi'_{1,v}, \Pi'_v(\epsilon'_v)\phi'_{2,v}) \\ &= \frac{1}{2^2} \sum_{\epsilon_v \in \mu_2(F_v)} \sum_{\epsilon'_v \in \mu_2(F_v)} \\ & \quad \times \int_{Z_{H,v} \backslash H_v^0} \mathcal{B}_{\Pi_v}^{\natural}(\Pi_v(h_{0,v}\epsilon_v)\phi_{1,v}, \Pi_v(\epsilon'_v)\phi_{2,v}) \mathcal{B}_{\Pi'_v}(\Pi'_v(h_{0,v}\epsilon_v)\phi'_{1,v}, \Pi'_v(\epsilon'_v)\phi'_{2,v}) dh_{0,v} \\ &= \frac{1}{2} \sum_{\epsilon_v \in \mu_2(F_v)} \int_{Z_{H,v} \backslash H_v^0} \mathcal{B}_{\Pi_v}^{\natural}(\Pi_v(h_{0,v}\epsilon_v)\phi_{1,v}, \phi_{2,v}) \mathcal{B}_{\Pi'_v}(\Pi'_v(h_{0,v}\epsilon_v)\phi'_{1,v}, \phi'_{2,v}) dh_{0,v} \\ &= \mathcal{I}_v^{\natural}(\phi_{1,v}, \phi_{2,v}; \phi'_{1,v}, \phi'_{2,v}). \end{aligned}$$

This completes the proof. \square

This completes the proof of Proposition 3.2.

4. Local theta lifts from GL_2 to $\mathrm{GO}(V)$

Let F be a local field of characteristic zero. Let W be a two dimensional symplectic space over F and V a four dimensional quadratic space over F . Set

$$\begin{aligned} G &= \mathrm{GSp}(W)(F) \cong \mathrm{GL}_2(F), & G_1 &= \mathrm{Sp}(W)(F) \cong \mathrm{SL}_2(F), \\ H &= \mathrm{GO}(V)(F), & H_1 &= \mathrm{O}(V)(F). \end{aligned}$$

Let

$$R = \mathrm{G}(\mathrm{Sp}(W) \times \mathrm{O}(V))(F) = \{(g, h) \in G \times H \mid \nu(g) = \nu(h)\},$$

where $\nu : G \rightarrow F^\times$ and $\nu : H \rightarrow F^\times$ are the similitude characters.

Fix a non-trivial additive character ψ of F . Let ω denote the Weil representation of $G_1 \times H_1$ with respect to ψ . As in [13, §5.1], [47], we extend ω to a representation of R . Let

$$\Omega = \mathrm{c}\text{-ind}_R^{G^+ \times H}(\omega),$$

where $G^+ = \{g \in G \mid \nu(g) \in \nu(H)\}$. The induced Weil representation Ω depends only on the orbit of ψ under the natural action of $\nu(H) \subset F^\times$. Let π^+ be an infinite dimensional irreducible admissible representation of G^+ . Then the maximal $(\pi^+)^{\vee}$ -isotypic quotient of Ω is of the form

$$(\pi^+)^{\vee} \boxtimes \Theta(\pi^+),$$

where $(\pi^+)^{\vee}$ is the contragredient representation of π^+ and $\Theta(\pi^+)$ is a smooth representation of H . If the residual characteristic of F is not two, then the Howe duality conjecture, which is a theorem of Howe [18] and Waldspurger [56], and a result of Roberts [47] assert that $\Theta(\pi^+)$ has a unique irreducible quotient $\theta(\pi^+)$. Even if the residual characteristic of F is two, the same assertion follows from [8, Lemmas 4.1, 5.4]. Thus, we obtain a unique (up to a scalar) R -equivariant surjective map

$$\theta : \omega \otimes \pi^+ \longrightarrow \theta(\pi^+).$$

Lemma 4.1. *Assume that $G^+ \neq G$. Let π be an infinite dimensional irreducible admissible representation of G such that $\pi|_{G^+}$ is reducible. Then we can write $\pi|_{G^+} = \pi^+ \oplus \pi^-$, where π^{\pm} is an irreducible admissible representation of G^+ such that*

$$\theta(\pi^+) \neq 0, \quad \theta(\pi^-) = 0.$$

Proof. If F is a non-archimedean local field, then the assertion follows from [8, §5]. If $F = \mathbb{R}$, see [43]. \square

Let π be an infinite dimensional irreducible admissible representation of G . If $G^+ \neq G$, let

$$\theta(\pi) = \begin{cases} \theta(\pi|_{G^+}) & \text{if } \pi|_{G^+} \text{ is irreducible,} \\ \theta(\pi^+) & \text{if } \pi|_{G^+} \text{ is reducible,} \end{cases}$$

where π^+ is an irreducible subrepresentation of $\pi|_{G^+}$ as in Lemma 4.1. Thus, we obtain a unique (up to a scalar) R -equivariant surjective map

$$\theta : \omega \otimes \pi \longrightarrow \theta(\pi).$$

5. Local theta lifts from $\mathrm{GO}(V)$ to GSp_4

Let F be a local field of characteristic zero. Let V be a four dimensional quadratic space over F and W a four dimensional symplectic space over F . Set

$$\begin{aligned} H &= \mathrm{GO}(V)(F), & H_1 &= \mathrm{O}(V)(F), \\ G &= \mathrm{GSp}(W)(F) \cong \mathrm{GSp}_4(F), & G_1 &= \mathrm{Sp}(W)(F) \cong \mathrm{Sp}_4(F). \end{aligned}$$

Let

$$R = \mathrm{G}(\mathrm{O}(V) \times \mathrm{Sp}(W))(F) = \{(h, g) \in H \times G \mid \nu(h) = \nu(g)\},$$

where $\nu : H \rightarrow F^\times$ and $\nu : G \rightarrow F^\times$ are the similitude characters. Let K be the discriminant algebra of V and D a quaternion algebra over F associated to V as in §1.

Fix a non-trivial additive character ψ of F . Let ω denote the Weil representation of $H_1 \times G_1$ with respect to ψ . As in [13, §5.1], [47], we extend ω to a representation of R . Let

$$\Omega = \mathrm{c}\text{-ind}_R^{H \times G^+}(\omega),$$

where $G^+ = \{g \in G \mid \nu(g) \in \nu(H)\}$. The induced Weil representation Ω depends only on the orbit of ψ under the natural action of $\nu(H) \subset F^\times$.

Lemma 5.1. *Assume that $G^+ \neq G$. Let $g_0 \in G \setminus G^+$. Let σ and π^+ be irreducible admissible representations of H and G^+ , respectively. If $\mathrm{Hom}_{H \times G^+}(\Omega, \sigma \boxtimes \pi^+) \neq 0$, then $\pi^+ \circ \mathrm{Ad}(g_0) \not\cong \pi^+$.*

Proof. The assertion follows from [49, Lemmas 1.4, 1.5] and the proof of [49, Theorem 1.8]. We remark that [49, Lemma 1.5] follows from [43] even if $\mathrm{disc}(V) \notin F^{\times, 2}$. \square

Let σ be an irreducible unitary admissible representation of H . Then the maximal $\bar{\sigma}$ -isotypic quotient of Ω is of the form

$$\bar{\sigma} \boxtimes \Theta(\sigma),$$

where $\bar{\sigma}$ is the complex conjugate representation of σ and $\Theta(\sigma)$ is a smooth representation of G^+ . We call $\Theta(\sigma)$ the big theta lift of σ to G^+ . If the residual characteristic of F is not two, then the Howe duality conjecture, which is a theorem of Howe [18] and Waldspurger [56], and a result of Roberts [47] assert that $\Theta(\sigma)$ has a unique irreducible quotient $\theta(\sigma)$. Even if the residual characteristic of F is two, the same assertion follows from Theorem A.1 in Appendix A. We call $\theta(\sigma)$ the theta lift of σ to G^+ . Thus, we obtain a unique (up to a scalar) R -equivariant surjective map

$$\theta : \omega \otimes \sigma \longrightarrow \theta(\sigma).$$

By Lemma 5.1, we obtain the following lemma.

Lemma 5.2. *Assume that $\theta(\sigma)$ is non-zero and unitary. Let $\pi = \mathrm{ind}_{G^+}^G(\theta(\sigma))$. Then π is irreducible. Moreover, we have*

$$\mathcal{B}_\pi(\pi(g)\phi_1, \phi_2) = 0$$

for $g \in G \setminus G^+$ and $\phi_1, \phi_2 \in \theta(\sigma)$. Here $\mathcal{B}_\pi : \pi \otimes \bar{\pi} \rightarrow \mathbb{C}$ is a pairing and we regard $\theta(\sigma)$ as a subrepresentation of $\pi|_{G^+}$.

Lemma 5.3. *Assume that $\sigma \otimes \mathrm{sgn} \not\cong \sigma$ and the theta lift of σ to $\mathrm{GL}_2(F)^+$ is non-zero. Then we have*

$$\theta(\sigma \otimes \mathrm{sgn}) = 0.$$

Proof. By [46, p. 399], the theta lift of the sign character of $\mathrm{O}(V)(F)$ to $\mathrm{Sp}_{2n}(F)$ is zero unless $n \geq 4$. As in [1, Proposition 1.7], this yields the lemma. \square

Definition 5.4. Set $H^0 = \mathrm{GSO}(V)(F)$. Let σ_0 be an irreducible admissible representation of H^0 . We say that σ_0 is distinguished if

$$\sigma_0 \cong \varsigma_K^D \boxtimes \omega_\varsigma \omega_{K/F}$$

as representations of $D^\times(K) \times F^\times$ for some irreducible admissible representation ς of $\mathrm{GL}_2(F)$ with central character ω_ς . Here ς_K is the base change of ς to $\mathrm{GL}_2(K)$ and ς_K^D is the Jacquet-Langlands transfer of ς_K to $D^\times(K)$. Then we can write $\mathrm{ind}_{H^0}^H(\sigma_0) = \sigma_0^+ \oplus \sigma_0^-$, where σ_0^\pm is an irreducible admissible representation of H such that the theta lift of σ_0^+ to $\mathrm{GL}_2(F)^+$ is non-zero (and hence $\theta(\sigma_0^-) = 0$ by Lemma 5.3).

Let σ be an irreducible unitary admissible representation of H . We assume the following:

- σ is a local component of an irreducible unitary cuspidal automorphic representation.
- If $\sigma \otimes \mathrm{sgn} \not\cong \sigma$, then $\sigma \not\cong \sigma_0^-$ for any distinguished representation σ_0 of $\mathrm{GSO}(V)(F)$.

In §7, we will show that $\theta(\sigma)$ is non-zero and unitary. By Theorem A.1 in Appendix A, we obtain the following proposition.

Proposition 5.5. *If F is a non-archimedean local field, then the multiplicity of $\theta(\sigma)$ in $\Theta(\sigma)$ is one.*

Let $\bar{\Omega}$ be the complex conjugate representation of Ω . Then we have

$$\bar{\Theta}(\bar{\sigma}) \cong \overline{\Theta(\sigma)}, \quad \bar{\theta}(\bar{\sigma}) \cong \overline{\theta(\sigma)},$$

where $\bar{\Theta}(\bar{\sigma})$ (resp. $\bar{\theta}(\bar{\sigma})$) is the big theta lift (resp. the theta lift) of $\bar{\sigma}$ to G^+ with respect to $\bar{\Omega}$. Let

$$\theta : \omega \otimes \sigma \longrightarrow \theta(\sigma), \quad \bar{\theta} : \bar{\omega} \otimes \bar{\sigma} \longrightarrow \bar{\theta}(\sigma),$$

be the unique (up to a scalar) R -equivariant surjective maps.

Let

$$\begin{aligned} \mathbf{H} &= \{\mathbf{h} = (h_1, h_2) \in H \times H \mid \nu(h_1) = \nu(h_2)\}, \\ \mathbf{R} &= \{(\mathbf{h}, g) \in \mathbf{H} \times G \mid \nu(\mathbf{h}) = \nu(g)\}. \end{aligned}$$

We define an \mathbf{R} -equivariant map

$$\mathcal{Z} : (\omega \boxtimes \bar{\omega}) \otimes (\sigma \boxtimes \bar{\sigma}) \longrightarrow \mathbb{C}$$

by

$$\mathcal{Z}(\varphi_1, \varphi_2; f_1, f_2) = \int_{H_1} \mathcal{B}_\omega(\omega(h_1)\varphi_1, \varphi_2) \mathcal{B}_\sigma(\sigma(h_1)f_1, f_2) dh_1$$

for $\varphi_1, \varphi_2 \in \omega$ and $f_1, f_2 \in \sigma$. Here $\mathcal{B}_\omega : \omega \otimes \bar{\omega} \rightarrow \mathbb{C}$ and $\mathcal{B}_\sigma : \sigma \otimes \bar{\sigma} \rightarrow \mathbb{C}$ are pairings. In §7, we will show that this integral is absolutely convergent and $\mathcal{Z} \neq 0$.

Lemma 5.6. *If F is a non-archimedean local field, then there exists a pairing $\mathcal{B}_{\theta(\sigma)} : \theta(\sigma) \otimes \overline{\theta(\sigma)} \rightarrow \mathbb{C}$ such that*

$$\mathcal{Z} = \mathcal{B}_{\theta(\sigma)} \circ (\theta \otimes \bar{\theta}).$$

Proof. It suffices to show that

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\mathbf{R}}((\omega \boxtimes \bar{\omega}) \otimes (\sigma \boxtimes \bar{\sigma}), \mathbb{C}) = 1.$$

We have

$$\begin{aligned} \operatorname{Hom}_{\mathbf{R}}((\omega \boxtimes \bar{\omega}) \otimes (\sigma \boxtimes \bar{\sigma}), \mathbb{C}) &\cong \operatorname{Hom}_{H \times H \times G^+}((\Omega \boxtimes \bar{\Omega}) \otimes (\sigma \boxtimes \bar{\sigma}), \mathbb{C}) \\ &\cong \operatorname{Hom}_{G^+}(\Theta(\sigma) \otimes \overline{\Theta(\sigma)}, \mathbb{C}) \\ &\cong \operatorname{Hom}_{G^+}(\Theta(\sigma), \overline{\Theta(\sigma)}^{\vee}). \end{aligned}$$

Let $l : \Theta(\sigma) \rightarrow \overline{\Theta(\sigma)}^{\vee}$ be a non-zero G^+ -equivariant map. Then the image of l contains the unique irreducible subrepresentation $\overline{\theta(\sigma)}^{\vee} \cong \theta(\sigma)$ of $\overline{\Theta(\sigma)}^{\vee}$. By Proposition 5.5, l factors through the quotient $\Theta(\sigma) \rightarrow \theta(\sigma)$. This yields the lemma. \square

Lemma 5.7. *If F is an archimedean local field, then there exists a pairing $\mathcal{B}_{\theta(\sigma)} : \theta(\sigma) \otimes \overline{\theta(\sigma)} \rightarrow \mathbb{C}$ such that*

$$\mathcal{Z} = \mathcal{B}_{\theta(\sigma)} \circ (\theta \otimes \bar{\theta}).$$

Proof. We can write $\sigma|_{H_1} = \oplus_{i=1}^n \sigma_i$, where $n \leq 2$ and σ_i is an irreducible unitary admissible representation of H_1 . As in [8, Lemma 3.1], we have $\Theta(\sigma)|_{G_1} = \oplus_{i=1}^n \Theta(\sigma_i)$ and $\theta(\sigma)|_{G_1} = \oplus_{i=1}^n \theta(\sigma_i)$. Here $\Theta(\sigma_i)$ (resp. $\theta(\sigma_i)$) is the big theta lift (resp. the theta lift) of σ_i to G_1 . If $n = 2$, then $\sigma_1 \not\cong \sigma_2$ and hence $\theta(\sigma_1) \not\cong \theta(\sigma_2)$. We have

$$\begin{aligned} (\omega \boxtimes \bar{\omega}) \otimes (\sigma \boxtimes \bar{\sigma})|_{H_1 \times H_1 \times G_1 \times G_1} &= \oplus_{i=1}^n \oplus_{j=1}^n (\omega \boxtimes \bar{\omega}) \otimes (\sigma_i \boxtimes \bar{\sigma}_j), \\ \Theta(\sigma) \boxtimes \overline{\Theta(\sigma)}|_{G_1 \times G_1} &= \oplus_{i=1}^n \oplus_{j=1}^n \Theta(\sigma_i) \boxtimes \overline{\Theta(\sigma_j)}, \\ \theta(\sigma) \boxtimes \overline{\theta(\sigma)}|_{G_1 \times G_1} &= \oplus_{i=1}^n \oplus_{j=1}^n \theta(\sigma_i) \boxtimes \overline{\theta(\sigma_j)}. \end{aligned}$$

Let

$$\begin{aligned} t : (\omega \boxtimes \bar{\omega}) \otimes (\sigma \boxtimes \bar{\sigma}) &\longrightarrow \Theta(\sigma) \boxtimes \overline{\Theta(\sigma)}, & p : \Theta(\sigma) \boxtimes \overline{\Theta(\sigma)} &\longrightarrow \theta(\sigma) \boxtimes \overline{\theta(\sigma)}, \\ t_{ij} : (\omega \boxtimes \bar{\omega}) \otimes (\sigma_i \boxtimes \bar{\sigma}_j) &\longrightarrow \Theta(\sigma_i) \boxtimes \overline{\Theta(\sigma_j)}, & p_{ij} : \Theta(\sigma_i) \boxtimes \overline{\Theta(\sigma_j)} &\longrightarrow \theta(\sigma_i) \boxtimes \overline{\theta(\sigma_j)}, \end{aligned}$$

be equivariant surjective maps. We may assume that

$$t = \oplus_{i=1}^n \oplus_{j=1}^n t_{ij} \quad \text{and} \quad p = \oplus_{i=1}^n \oplus_{j=1}^n p_{ij}.$$

In particular, we have

$$\ker(p) = \oplus_{i=1}^n \oplus_{j=1}^n \ker(p_{ij}).$$

Since \mathcal{Z} is an \mathbf{R} -equivariant map, there exists a G^+ -invariant functional $l : \Theta(\sigma) \otimes \overline{\Theta(\sigma)} \rightarrow \mathbb{C}$ such that $\mathcal{Z} = l \circ t$. It remains to show that $\ker(p) \subset \ker(l)$. Let \mathcal{Z}_{ij} (resp. l_{ij}) be the restriction of \mathcal{Z} (resp. l) to $(\omega \boxtimes \bar{\omega}) \otimes (\sigma_i \boxtimes \bar{\sigma}_j)$ (resp. $\Theta(\sigma_i) \otimes \overline{\Theta(\sigma_j)}$). It follows from the definition of \mathcal{Z} that

$$\mathcal{Z}_{ij} = 0 \quad \text{if} \quad i \neq j,$$

so that $l_{ij} = 0$ if $i \neq j$ and

$$\mathcal{Z} = \sum_{i=1}^n \mathcal{Z}_{ii} = \sum_{i=1}^n l_{ii} \circ t_{ii}.$$

By a result of He [16], the G_1 -invariant functional $l_{ii} : \Theta(\sigma_i) \otimes \overline{\Theta(\sigma_i)} \rightarrow \mathbb{C}$ factors through p_{ii} , so that $\ker(p_{ii}) \subset \ker(l_{ii})$. Hence we have

$$(\oplus_{i=1}^n \ker(p_{ii})) \oplus (\oplus_{i \neq j} \Theta(\sigma_i) \otimes \overline{\Theta(\sigma_j)}) \subset \ker(l).$$

This yields the lemma. \square

6. Global theta lifts from GL_2 to $\mathrm{GO}(V)$

Let F be a totally real number field. Let W be a two dimensional symplectic space over F and V a four dimensional quadratic space over F . Let $\tilde{W} = W \oplus (-W)$. Set

$$\begin{aligned} G &= \mathrm{GSp}(W) \cong \mathrm{GL}_2, & G_1 &= \mathrm{Sp}(W) \cong \mathrm{SL}_2, \\ \tilde{G} &= \mathrm{GSp}(\tilde{W}) \cong \mathrm{GSp}_4, & \tilde{G}_1 &= \mathrm{Sp}(\tilde{W}) \cong \mathrm{Sp}_4, \\ H &= \mathrm{GO}(V), & H_1 &= \mathrm{O}(V). \end{aligned}$$

Let

$$\mathbf{G} = \{\mathfrak{g} = (g_1, g_2) \in G \times G \mid \nu(g_1) = \nu(g_2)\},$$

where $\nu : G \rightarrow \mathbb{G}_m$ is the similitude character. Let $\iota : \mathbf{G} \hookrightarrow \tilde{G}$ be the natural embedding. Let K be the discriminant algebra of V and D a quaternion algebra over F associated to V as in §1.

6.1. Weil representations. Fix a non-trivial additive character $\psi = \otimes_v \psi_v$ of \mathbb{A}/F . Let $W = X \oplus Y$ be a complete polarization and set

$$\mathbb{W} = V \otimes W, \quad \mathbb{X} = V \otimes X, \quad \mathbb{Y} = V \otimes Y.$$

Then \mathbb{W} is a symplectic space over F and $\mathbb{W} = \mathbb{X} \oplus \mathbb{Y}$ is a complete polarization. Let $\mathrm{Mp}(\mathbb{W}(\mathbb{A}))$ denote the metaplectic extension of $\mathrm{Sp}(\mathbb{W})(\mathbb{A})$. Let ω be the Weil representation of $\mathrm{Mp}(\mathbb{W}(\mathbb{A}))$ on the space $V_\omega = S(\mathbb{X}(\mathbb{A}))$ with respect to ψ and $\mathcal{B}_\omega : V_\omega \otimes \bar{V}_\omega \rightarrow \mathbb{C}$ the canonical pairing given by

$$\mathcal{B}_\omega(\varphi_1, \varphi_2) = \int_{\mathbb{X}(\mathbb{A})} \varphi_1(x) \overline{\varphi_2(x)} dx$$

for $\varphi_1, \varphi_2 \in V_\omega$. Here dx is the Tamagawa measure on $\mathbb{X}(\mathbb{A})$. For each place v of F , let $\mathrm{Mp}(\mathbb{W}_v)$ denote the metaplectic extension of $\mathrm{Sp}(\mathbb{W})(F_v)$. Let ω_v be the Weil representation of $\mathrm{Mp}(\mathbb{W}_v)$ on the space $S(\mathbb{X}_v)$ with respect to ψ_v and $\mathcal{B}_{\omega_v} : \omega_v \otimes \bar{\omega}_v \rightarrow \mathbb{C}$ the canonical pairing given by

$$\mathcal{B}_{\omega_v}(\varphi_{1,v}, \varphi_{2,v}) = \int_{\mathbb{X}_v} \varphi_{1,v}(x_v) \overline{\varphi_{2,v}(x_v)} dx_v$$

for $\varphi_{1,v}, \varphi_{2,v} \in S(\mathbb{X}_v)$. Here dx_v is the self-dual measure on \mathbb{X}_v with respect to the Fourier transform determined by ψ_v . Then we have $\omega = \otimes_v \omega_v$ and $\mathcal{B}_\omega = \prod_v \mathcal{B}_{\omega_v}$. By [30], there exists a splitting

$$G_1(\mathbb{A}) \times H_1(\mathbb{A}) \longrightarrow \mathrm{Mp}(\mathbb{W}(\mathbb{A})).$$

By [13, §5.1], [47], we can extend it to a splitting

$$G(\mathrm{Sp}(W) \times \mathrm{O}(V))(\mathbb{A}) \longrightarrow \mathrm{Mp}(\mathbb{W}(\mathbb{A})).$$

We regard ω as a representation of $G(\mathrm{Sp}(W) \times \mathrm{O}(V))(\mathbb{A})$ via this splitting. Similarly, we may regard ω_v as a representation of $G(\mathrm{Sp}(W) \times \mathrm{O}(V))(F_v)$.

Let

$$\tilde{\mathbb{W}} = V \otimes \tilde{W}, \quad \tilde{\mathbb{X}} = V \otimes (X \oplus (-X)), \quad \tilde{\mathbb{Y}} = V \otimes (Y \oplus (-Y)).$$

Then $\tilde{\mathbb{W}}$ is a symplectic space over F and $\tilde{\mathbb{W}} = \tilde{\mathbb{X}} \oplus \tilde{\mathbb{Y}}$ is a complete polarization. Let $\tilde{\omega}$ be the Weil representation of $\mathrm{Mp}(\tilde{\mathbb{W}}(\mathbb{A}))$ on $S(\tilde{\mathbb{X}}(\mathbb{A}))$ with respect to ψ . We may regard $\tilde{\omega}$ as a representation of $G(\mathrm{Sp}(\tilde{W}) \times \mathrm{O}(V))(\mathbb{A})$. We have a natural isomorphism

$$S(\tilde{\mathbb{X}}(\mathbb{A})) \cong V_\omega \otimes \bar{V}_\omega$$

as representations of $\mathrm{Mp}(\mathbb{W}(\mathbb{A})) \times \mathrm{Mp}(\mathbb{W}(\mathbb{A}))$. Let

$$\begin{aligned} W^\Delta &= \{(x, x) \mid x \in W\}, & \mathbb{W}^\Delta &= V \otimes W^\Delta, \\ W^\nabla &= \{(x, -x) \mid x \in W\}, & \mathbb{W}^\nabla &= V \otimes W^\nabla. \end{aligned}$$

Then $\tilde{\mathbb{W}} = \mathbb{W}^\nabla \oplus \mathbb{W}^\Delta$ is a complete polarization. Hence we can realize the Weil representation $\tilde{\omega}$ on $S(\mathbb{W}^\nabla(\mathbb{A}))$. By [40, §2], there exists an isomorphism

$$\delta : S(\tilde{\mathbb{X}}(\mathbb{A})) \longrightarrow S(\mathbb{W}^\nabla(\mathbb{A}))$$

as representations of $\mathrm{Mp}(\tilde{\mathbb{W}}(\mathbb{A}))$ such that

$$\delta(\varphi_1 \otimes \bar{\varphi}_2)(0) = \mathcal{B}_\omega(\varphi_1, \varphi_2)$$

for $\varphi_1, \varphi_2 \in V_\omega$.

6.2. Theta lifts. Let $\pi \cong \otimes_v \pi_v$ be an irreducible unitary cuspidal automorphic representation of $G(\mathbb{A})$ on the space V_π with central character ω_π . We assume the following:

- The base change π_K of π to $G(\mathbb{A}_K) \cong \mathrm{GL}_2(\mathbb{A}_K)$ is cuspidal.
- The Jacquet-Langlands transfer π_K^D of π_K to $D^\times(\mathbb{A}_K)$ exists.

Lemma 6.1. *The partial L-function $L^S(s, \pi, \mathrm{Ad} \otimes \omega_{K/F})$ is holomorphic and non-zero at $s = 1$.*

Proof. It is well-known that $L^S(s, \pi, \mathrm{Ad})$ is holomorphic and non-zero at $s = 1$. If K is a quadratic extension of F , then

$$L^S(s, \pi_K, \mathrm{Ad}) = L^S(s, \pi, \mathrm{Ad})L^S(s, \pi, \mathrm{Ad} \otimes \omega_{K/F})$$

is also holomorphic and non-zero at $s = 1$ since π_K is cuspidal. This yields the lemma. \square

Let $\varphi \in V_\omega$. The theta function associated to φ is given by

$$\theta(g, h; \varphi) = \sum_{x \in \mathbb{X}(F)} \omega(g, h)\varphi(x)$$

for $(g, h) \in G(\mathrm{Sp}(W) \times \mathrm{O}(V))(\mathbb{A})$. Let $f \in V_\pi$. For $h \in H(\mathbb{A})$, choose $g \in G(\mathbb{A})$ such that $\nu(g) = \nu(h)$, and put

$$\theta(h; \varphi, f) = \int_{G_1(F) \backslash G_1(\mathbb{A})} \theta(g_1 g, h; \varphi) f(g_1 g) dg_1.$$

Here $dg_1 = \prod_v dg_{1,v}$ is the Tamagawa measure on $G_1(\mathbb{A})$. Note that $\mathrm{vol}(G_1(F) \backslash G_1(\mathbb{A})) = 1$ and we may assume that the volume of a hyperspecial maximal compact subgroup of $G_{1,v}$ with respect to $dg_{1,v}$ is 1 for almost all v . This integral defines an automorphic form $\theta(\varphi, f)$ on $H(\mathbb{A})$. Let $\theta(\pi)$ be the automorphic representation of $H(\mathbb{A})$ on the space $V_{\theta(\pi)}$ generated by $\theta(\varphi, f)$ for all $\varphi \in V_\omega$ and $f \in V_\pi$. By assumption on π , $\theta(\pi)$ is cuspidal. In Lemma 6.9 below, we will show that $V_{\theta(\pi)} \neq 0$. In particular, $\theta(\pi_v) \neq 0$ for all v . Hence $\theta(\pi)$ is irreducible,

$$\theta(\pi) \cong \otimes_v \theta(\pi_v),$$

and $\theta(\pi_v)$ is unitary for all v . Thus, we obtain a $G(\mathrm{Sp}(W) \times \mathrm{O}(V))(\mathbb{A})$ -equivariant surjective map

$$\theta : V_\omega \otimes V_\pi \longrightarrow V_{\theta(\pi)}$$

and $G(\mathrm{Sp}(W) \times \mathrm{O}(V))(F_v)$ -equivariant surjective maps

$$\theta_v : \omega_v \otimes \pi_v \longrightarrow \theta(\pi_v)$$

such that $\theta = \otimes_v \theta_v$. As in [52], we have

$$\theta(\pi)|_{D^\times(\mathbb{A}_K) \times \mathbb{A}^\times} \cong \pi_K^D \boxtimes \omega_\pi \omega_{K/F}$$

by the local unramified theta correspondence and the strong multiplicity one theorem. We should remark that the local theta correspondence for $\mathrm{GL}_2 \times \mathrm{GO}(V)$ has also been studied by Cognet [6], [7] and Roberts [48].

6.3. Eisenstein series. Let P be the parabolic subgroup of \tilde{G} stabilizing W^Δ with modulus character δ_P . We regard $\omega_{K/F}$ as a character of $P(\mathbb{A})$ via the natural homomorphism

$$P \longrightarrow \mathrm{GL}(W^\nabla) \xrightarrow{\det} \mathbb{G}_m.$$

For $\nu \in \mathbb{G}_m$, we define an element $d(\nu)$ of P by

$$d(\nu)|_{W^\nabla} = \mathrm{id}, \quad d(\nu)|_{W^\Delta} = \nu \cdot \mathrm{id}.$$

We fix a maximal compact subgroup \mathbf{K} of $\tilde{G}(\mathbb{A})$ such that $\tilde{G}(\mathbb{A}) = P(\mathbb{A})\mathbf{K}$.

Let $\mathbf{I}(s)$ denote the degenerate principal series representation of $\tilde{G}(\mathbb{A})$ given by

$$\mathbf{I}(s) = \mathrm{Ind}_{P(\mathbb{A})}^{\tilde{G}(\mathbb{A})}(\omega_{K/F} \cdot \delta_P^{s/3}),$$

where Ind denotes the normalized induction. Given a holomorphic section Φ of $\mathbf{I}(s)$, we define an Eisenstein series $E(s, \Phi)$ on $\tilde{G}(\mathbb{A})$ by

$$E(g; s, \Phi) = \sum_{\gamma \in P(F) \backslash \tilde{G}(F)} \Phi(\gamma g, s)$$

for $\mathrm{Re}(s) \gg 0$. By [34, Theorem 1.1], $E(s, \Phi)$ has at most a simple pole at $s = \frac{1}{2}$. Let $P_1 = P \cap \tilde{G}_1$. Let $\mathbf{I}_1(s)$ denote the degenerate principal series representation of $\tilde{G}_1(\mathbb{A})$ given by

$$\mathbf{I}_1(s) = \mathrm{Ind}_{P_1(\mathbb{A})}^{\tilde{G}_1(\mathbb{A})}(\omega_{K/F} \cdot \delta_{P_1}^{s/3}).$$

If Φ_1 is a holomorphic section of $\mathbf{I}_1(s)$, we similarly define an Eisenstein series $E(s, \Phi_1)$ on $\tilde{G}_1(\mathbb{A})$. If Φ is a holomorphic section of $\mathbf{I}(s)$, then $\Phi|_{\tilde{G}_1(\mathbb{A})}$ is a holomorphic section of $\mathbf{I}_1(s)$ and $E(s, \Phi)|_{\tilde{G}_1(\mathbb{A})} = E(s, \Phi|_{\tilde{G}_1(\mathbb{A})})$.

We define a $\mathrm{G}(\mathrm{Sp}(\tilde{W}) \times \mathrm{O}(V))(\mathbb{A})$ -equivariant map

$$[-] : S(\mathbb{W}^\nabla(\mathbb{A})) \longrightarrow \mathbf{I}(\tfrac{1}{2})$$

by

$$[\varphi](g, \tfrac{1}{2}) = |\nu(g)|^{-2} \cdot \tilde{\omega}(d(\nu(g)^{-1})g)\varphi(0)$$

for $g \in \tilde{G}(\mathbb{A})$. Here $\mathrm{G}(\mathrm{Sp}(\tilde{W}) \times \mathrm{O}(V))(\mathbb{A})$ acts on $\mathbf{I}(\frac{1}{2})$ via the projection $\mathrm{G}(\mathrm{Sp}(\tilde{W}) \times \mathrm{O}(V))(\mathbb{A}) \rightarrow \tilde{G}(\mathbb{A})^+$. We extend $[\varphi]$ to a holomorphic section of $\mathbf{I}(s)$ such that its restriction to \mathbf{K} is independent of s . Let

$$E(s, [\varphi]) = \sum_{d=-1}^{\infty} (s - \tfrac{1}{2})^d A_d(\varphi)$$

be the Laurent expansion of $E(s, [\varphi])$ at $s = \frac{1}{2}$.

6.4. Theta integrals. Let r be the Witt index of V and $V = X' \oplus V_0 \oplus Y'$ a Witt decomposition, where V_0 is an anisotropic quadratic space over F of dimension $4 - 2r$. Let dh_1 be the Tamagawa measure on $H_1(\mathbb{A})$ and note that $\mathrm{vol}(H_1(F) \backslash H_1(\mathbb{A})) = 1$. Let $\varphi \in S(\mathbb{W}^\nabla(\mathbb{A}))$. The theta function associated to φ is given by

$$\theta(g, h; \varphi) = \sum_{x \in \mathbb{W}^\nabla(F)} \tilde{\omega}(g, h)\varphi(x)$$

for $(g, h) \in \mathrm{G}(\mathrm{Sp}(\tilde{W}) \times \mathrm{O}(V))(\mathbb{A})$. If $r = 0$, then the theta integral $I(\varphi)$ is given by

$$I(g_1; \varphi) = \int_{H_1(F) \backslash H_1(\mathbb{A})} \theta(g_1, h_1; \varphi) dh_1$$

for $g_1 \in \tilde{G}_1(\mathbb{A})$.

Assume that $r > 0$. Let P' be the parabolic subgroup of H_1 stabilizing Y' with modulus character $\delta_{P'}$. We fix a maximal compact subgroup \mathbf{K}' of $H_1(\mathbb{A})$ such that $H_1(\mathbb{A}) = P'(\mathbb{A})\mathbf{K}'$. Let $d_{ip'}$ be the left-invariant Tamagawa measure on $P'(\mathbb{A})$ and dk' the Haar measure on \mathbf{K}' such that $\text{vol}(\mathbf{K}') = 1$. There exists a constant κ such that

$$\int_{H_1(\mathbb{A})} f(h_1) dh_1 = \kappa \int_{P'(\mathbb{A})} \int_{\mathbf{K}'} f(p'k') d_{ip'} dk'$$

for $f \in L^1(H_1(\mathbb{A}))$.

Put $\varrho' = \frac{3-r}{2}$. Let Φ' be the holomorphic section of $\text{Ind}_{P'(\mathbb{A})}^{H_1(\mathbb{A})}(\delta_{P'}^{s/(3-r)})$ such that $\Phi'(k', s) = 1$ for all $k' \in \mathbf{K}'$. We define an Eisenstein series $\mathcal{E}(s)$ on $H_1(\mathbb{A})$ by

$$\mathcal{E}(h_1; s) = \sum_{\gamma \in P'(F) \backslash H_1(F)} \Phi'(\gamma h_1, s)$$

for $\text{Re}(s) > \varrho'$. By [36, §5], [23, §9], we have

$$\text{Res}_{s=\varrho'} \mathcal{E}(h_1; s) = \kappa$$

for $h_1 \in H_1(\mathbb{A})$.

Let $z \in \mathfrak{z}(\tilde{\mathfrak{g}}_{1,v})$ be the regularizing differential operator as in [35, §3.2], [34, §5], where v is a real place of F . There exists a self-adjoint differential operator $z' \in \mathfrak{z}(\mathfrak{h}_{1,v})$ such that $\tilde{\omega}(z) = \tilde{\omega}(z')$. Then we have $z' \cdot \mathcal{E}(s) = p(s) \cdot \mathcal{E}(s)$ with some $p(s) \in \mathbb{C}[s]$. Following Kudla and Rallis [34, §5], we define the regularized theta integral $I(s, \varphi)$ by

$$I(g_1; s, \varphi) = \frac{1}{\kappa \cdot p(s)} \int_{H_1(F) \backslash H_1(\mathbb{A})} \theta(g_1, h_1; z \cdot \varphi) \mathcal{E}(h_1; s) dh_1$$

for $g_1 \in \tilde{G}_1(\mathbb{A})$. By [34, Lemma 5.5.6], $I(s, \varphi)$ has at most a double pole at $s = \varrho'$. Let

$$I(s, \varphi) = \sum_{d=-2}^{\infty} (s - \varrho')^d B_d(\varphi)$$

be the Laurent expansion of $I(s, \varphi)$ at $s = \varrho'$.

6.5. The Siegel-Weil formula. Let $\mathcal{A}(\tilde{G}_1)$ denote the space of automorphic forms on $\tilde{G}_1(\mathbb{A})$ and $\mathcal{R}(\tilde{G}_1)$ the subspace of $\mathcal{A}(\tilde{G}_1)$ generated by $\text{Res}_{s=1/2} E(s, \Phi_1)$ for all holomorphic sections Φ_1 of $\mathbf{I}_1(s)$.

Let $\varphi \in S(\mathbb{W}^\nabla(\mathbb{A}))$. If $r = 0$, then the Siegel-Weil formula by Kudla and Rallis [31] asserts that

$$I(\varphi) = A_0(\varphi)|_{\tilde{G}_1(\mathbb{A})}.$$

If $r > 0$, then the Siegel-Weil formula (the second term identity) by Kudla, Rallis, and Soudry [35, §6] asserts that

$$(6.1) \quad B_{-1}(\varphi) \equiv A_0(\varphi)|_{\tilde{G}_1(\mathbb{A})} \pmod{\mathcal{R}(\tilde{G}_1)}.$$

Remark 6.2. In [35, §6], Kudla, Rallis, and Soudry proved (6.1) up to a scalar. Computing Fourier coefficients as in [35, Proposition 6.2], [54, Proposition 5.1.1], [19, Proposition 6.2], we can determine the constant of proportionality.

6.6. The doubling method. Let $\mathcal{A}(\tilde{G})$ denote the space of automorphic forms on $\tilde{G}(\mathbb{A})$ and $\mathcal{R}(\tilde{G})$ the subspace of $\mathcal{A}(\tilde{G})$ generated by $\mathrm{Res}_{s=1/2} E(s, \Phi)$ for all holomorphic sections Φ of $\mathbf{I}(s)$. If $\mathcal{F} \in \mathcal{R}(\tilde{G})$, then $\mathcal{F}|_{\tilde{G}_1(\mathbb{A})} \in \mathcal{R}(\tilde{G}_1)$.

Let $\mathcal{B}_\pi : V_\pi \otimes \bar{V}_\pi \rightarrow \mathbb{C}$ be the Petersson pairing given by

$$\mathcal{B}_\pi(f_1, f_2) = \int_{Z_G(\mathbb{A})G(F)\backslash G(\mathbb{A})} f_1(g)\overline{f_2(g)} dg$$

for $f_1, f_2 \in V_\pi$. Here Z_G is the identity component of the center of G and dg is the Tamagawa measure on $Z_G(\mathbb{A})G(F)\backslash G(\mathbb{A})$. Note that $\mathrm{vol}(Z_G(\mathbb{A})G(F)\backslash G(\mathbb{A})) = 2$. We fix a decomposition $\mathcal{B}_\pi = \prod_v \mathcal{B}_{\pi_v}$, where $\mathcal{B}_{\pi_v} : \pi_v \otimes \bar{\pi}_v \rightarrow \mathbb{C}$ is a pairing. Let

$$G(\mathbb{A})^+ = \{g \in G(\mathbb{A}) \mid \nu(g) \in \nu(H(\mathbb{A}))\}$$

and $G(F)^+ = G(F) \cap G(\mathbb{A})^+$. Put

$$\mathfrak{v} = \mathrm{vol}(Z_G(\mathbb{A})G(F)^+\backslash G(\mathbb{A})^+) = \begin{cases} 2 & \text{if } K = F \times F, \\ 1 & \text{if } K \text{ is a quadratic extension of } F. \end{cases}$$

Lemma 6.3. *We have*

$$\int_{Z_G(\mathbb{A})G(F)^+\backslash G(\mathbb{A})^+} f_1(g)\overline{f_2(g)} dg = \frac{\mathfrak{v}}{2} \cdot \mathcal{B}_\pi(f_1, f_2)$$

for $f_1, f_2 \in V_\pi$.

Proof. We may assume that K is a quadratic extension of F . Let $\mathcal{G} = Z_G(\mathbb{A})G(\mathbb{A})^+G(F)$. Note that $|\mathcal{G}\backslash G(\mathbb{A})| = 2$. By assumption on π , the group

$$\{\omega \in (\mathcal{G}\backslash G(\mathbb{A}))^D \mid \pi \otimes \omega \cong \pi\}$$

is trivial and hence $\pi|_{\mathcal{G}}$ is irreducible. The restriction to \mathcal{G} as functions induces an isomorphism

$$V_\pi \cong V_\pi|_{\mathcal{G}}$$

as representations of \mathcal{G} .

We define a \mathcal{G} -invariant pairing $\mathcal{B}_\pi^+ : V_\pi|_{\mathcal{G}} \otimes \bar{V}_\pi|_{\mathcal{G}} \rightarrow \mathbb{C}$ by

$$\mathcal{B}_\pi^+(f_1|_{\mathcal{G}}, f_2|_{\mathcal{G}}) = \int_{Z_G(\mathbb{A})G(F)^+\backslash G(\mathbb{A})^+} f_1(g)\overline{f_2(g)} dg$$

for $f_1, f_2 \in V_\pi$. As in the proof of Lemma 2.1, we have

$$\mathcal{B}_\pi^+(\pi(g_0)f_1|_{\mathcal{G}}, \pi(g_0)f_2|_{\mathcal{G}}) = \mathcal{B}_\pi^+(f_1|_{\mathcal{G}}, f_2|_{\mathcal{G}})$$

for $g_0 \in G(\mathbb{A}) \setminus \mathcal{G}$. Hence we have

$$\mathcal{B}_\pi(f_1, f_2) = \sum_{g_0 \in \mathcal{G}\backslash G(\mathbb{A})} \mathcal{B}_\pi^+(\pi(g_0)f_1|_{\mathcal{G}}, \pi(g_0)f_2|_{\mathcal{G}}) = 2 \cdot \mathcal{B}_\pi^+(f_1|_{\mathcal{G}}, f_2|_{\mathcal{G}}).$$

□

Let

$$\mathbf{G}(\mathbb{A})^+ = \{\mathbf{g} \in \mathbf{G}(\mathbb{A}) \mid \nu(\mathbf{g}) \in \nu(H(\mathbb{A}))\}$$

and $\mathbf{G}(F)^+ = \mathbf{G}(F) \cap \mathbf{G}(\mathbb{A})^+$. For a holomorphic section Φ of $\mathbf{I}(s)$ and $f_1, f_2 \in V_\pi$, the zeta integral of Piatetski-Shapiro and Rallis [44], [12, §6.2] is given by

$$Z(s, \Phi, f_1, f_2) = \int_{Z_{\tilde{G}}(\mathbb{A})\mathbf{G}(F)^+\backslash \mathbf{G}(\mathbb{A})^+} E(\iota(g_1, g_2); s, \Phi) f_1(g_1)\overline{f_2(g_2)} dg.$$

Here $Z_{\tilde{G}}$ is the identity component of the center of \tilde{G} and dg is the Tamagawa measure on $Z_{\tilde{G}}(\mathbb{A}) \backslash \mathbf{G}(\mathbb{A})$. Note that $\text{vol}(Z_{\tilde{G}}(\mathbb{A})\mathbf{G}(F)^+ \backslash \mathbf{G}(\mathbb{A})^+) = \mathfrak{v}$. For each place v of F , let

$$Z_v(s, \Phi_v, f_{1,v}, f_{2,v}) = \int_{G_{1,v}} \Phi_v(\iota(g_{1,v}, 1), s) \mathcal{B}_{\pi_v}(\pi_v(g_{1,v})f_{1,v}, f_{2,v}) dg_{1,v}.$$

Lemma 6.4. *For a holomorphic section $\Phi = \otimes_v \Phi_v$ of $\mathbf{I}(s)$ and $f_1 = \otimes_v f_{1,v}, f_2 = \otimes_v f_{2,v} \in V_\pi$, we have*

$$Z(s, \Phi, f_1, f_2) = \frac{\mathfrak{v}}{2} \cdot \frac{L^S(s + \frac{1}{2}, \pi, \text{Ad} \otimes \omega_{K/F})}{L^S(s + \frac{3}{2}, \omega_{K/F}) \zeta^S(2s + 1)} \cdot \prod_{v \in S} Z_v(s, \Phi_v, f_{1,v}, f_{2,v}).$$

Proof. The assertion follows from the doubling method [44], [12, §6.2] and Lemma 6.3. \square

6.7. Local zeta integrals. Let $\mathbf{I}_v(s) = \text{Ind}_{P_v}^{\tilde{G}_v}(\omega_{K_v/F_v} \cdot \delta_{P_v}^{s/3})$ denote the degenerate principal series representation of \tilde{G}_v .

Lemma 6.5. *For a holomorphic section Φ_v of $\mathbf{I}_v(s)$ and $f_{1,v}, f_{2,v} \in \pi_v$, the integral $Z_v(s, \Phi_v, f_{1,v}, f_{2,v})$ is absolutely convergent at $s = \frac{1}{2}$.*

Proof. By [44, Proposition 6.4], the function $g_{1,v} \mapsto \Phi_v(\iota(g_{1,v}, 1), \frac{1}{2})$ belongs to $L^{1+\varepsilon}(G_{1,v})$ for any $\varepsilon > 0$. This yields the lemma. \square

For $\varphi_{1,v}, \varphi_{2,v} \in S(\mathbb{X}_v)$, we have

$$\begin{aligned} & Z_v(\tfrac{1}{2}, [\delta(\varphi_{1,v} \otimes \bar{\varphi}_{2,v})], f_{1,v}, f_{2,v}) \\ &= \int_{G_{1,v}} \mathcal{B}_{\omega_v}(\omega_v(g_{1,v})\varphi_{1,v}, \varphi_{2,v}) \mathcal{B}_{\pi_v}(\pi_v(g_{1,v})f_{1,v}, f_{2,v}) dg_{1,v}. \end{aligned}$$

Lemma 6.6. *There exist $\varphi_v \in S(\mathbb{W}_v^\nabla)$ and $f_{1,v}, f_{2,v} \in \pi_v$ such that*

$$Z_v(\tfrac{1}{2}, [\varphi_v], f_{1,v}, f_{2,v}) \neq 0.$$

Proof. We fix a place v of F and suppress it from the notation. By [34, Proposition 7.2.1], there exist $\Phi \in \mathbf{I}(\frac{1}{2})$ and $f_1, f_2 \in \pi$ such that $Z(\frac{1}{2}, \Phi, f_1, f_2) \neq 0$. Let R be the image of the equivariant map $S(\mathbb{W}^\nabla) \rightarrow \mathbf{I}(\frac{1}{2})$, where $\mathbb{W}^\nabla = V \otimes W^\nabla$. It suffices to show that there exist $\Phi \in R$ and $f_1, f_2 \in \pi$ such that $Z(\frac{1}{2}, \Phi, f_1, f_2) \neq 0$.

We first consider the case $K = F \times F$. If D is split, then $\mathbf{I}(\frac{1}{2}) = R$ by [33], [37] and the assertion is obvious. We assume that D is division and $Z(\frac{1}{2}, \Phi, f_1, f_2) = 0$ for all $\Phi \in R$ and $f_1, f_2 \in \pi$. If F is archimedean, let R_- be the image of the equivariant map $S(\mathbb{W}_-^\nabla) \rightarrow \mathbf{I}(\frac{1}{2})$, where $\mathbb{W}_-^\nabla = (-V) \otimes W^\nabla$. Let V_0 be the two dimensional split quadratic space over F . Let R_0 be the image of the equivariant map $S(\mathbb{W}_0^\nabla) \rightarrow \mathbf{I}(-\frac{1}{2})$, where $\mathbb{W}_0^\nabla = V_0 \otimes W^\nabla$. By [33], [37], we have

$$\begin{cases} \mathbf{I}(\frac{1}{2})/R \cong R_0 & \text{if } F \text{ is non-archimedean,} \\ \mathbf{I}(\frac{1}{2})/(R + R_-) \cong R_0 & \text{if } F \text{ is archimedean.} \end{cases}$$

Since $\pi \circ \text{Ad}(g_0) \cong \pi$ for $g_0 \in G \setminus G^+$, we have $Z(\frac{1}{2}, \Phi, f_1, f_2) = 0$ for all $\Phi \in R_-$ and $f_1, f_2 \in \pi$ if F is archimedean. Hence $Z(\frac{1}{2}, \Phi, f_1, f_2)$ defines a non-zero equivariant map

$$R_0 \otimes (\pi \boxtimes \bar{\pi}) \longrightarrow \mathbb{C}.$$

As in [14, Proposition 3.1], this shows that the theta lift of π to $\text{GO}(V_0)(F)$ is non-zero. Hence π is a principal series representation of G . This contradicts the assumption that the Jacquet-Langlands transfer π^D of π to D^\times exists.

We next consider the case where K is a quadratic extension of F . We assume that $Z(\frac{1}{2}, \Phi, f_1, f_2) = 0$ for all $\Phi \in R$ and $f_1, f_2 \in \pi$. Let V_- be the four dimensional quadratic space over F such that $\mathrm{disc}(V_-) = \mathrm{disc}(V)$ and $V_- \not\cong V$. Let R_- be the image of the equivariant map $S(\mathbb{W}_-^\nabla) \rightarrow \mathbf{I}(\frac{1}{2})$, where $\mathbb{W}_-^\nabla = V_- \otimes W^\nabla$. By [33], [37], we have

$$\mathbf{I}(\frac{1}{2}) = R + R_-.$$

Since $\pi \circ \mathrm{Ad}(g_0) \cong \pi$ for $g_0 \in G \setminus G^+$, we have $Z(\frac{1}{2}, \Phi, f_1, f_2) = 0$ for all $\Phi \in R_-$ and $f_1, f_2 \in \pi$ and hence a contradiction. \square

6.8. The Rallis inner product formula.

Lemma 6.7. *For $\mathcal{F} \in \mathcal{R}(\tilde{G}_1)$ and $f_1, f_2 \in V_\pi$, we have*

$$\int_{G_1(F) \backslash G_1(\mathbb{A})} \int_{G_1(F) \backslash G_1(\mathbb{A})} \mathcal{F}(\iota(g_1, g_2)) f_1(g_1) \overline{f_2(g_2)} dg_1 dg_2 = 0.$$

Proof. The assertion follows from Lemmas 6.1, 6.4, and 6.5. Here we have used the version of Lemma 6.4 for isometry groups. \square

Let $\mathbb{A}^{\times,+} = \nu(H(\mathbb{A}))$, $F^{\times,+} = F^\times \cap \mathbb{A}^{\times,+}$, and $\mathcal{C} = \mathbb{A}^{\times,2} F^{\times,+} \backslash \mathbb{A}^{\times,+}$. The similitude characters induce isomorphisms

$$Z_G(\mathbb{A})G_1(\mathbb{A})G(F)^+ \backslash G(\mathbb{A})^+ \cong \mathcal{C}, \quad Z_H(\mathbb{A})H_1(\mathbb{A})H(F) \backslash H(\mathbb{A}) \cong \mathcal{C}.$$

Fix cross sections $c \mapsto g_c$ and $c \mapsto h_c$ of $G(\mathbb{A})^+ \rightarrow \mathcal{C}$ and $H(\mathbb{A}) \rightarrow \mathcal{C}$, respectively. Let dh be the Tamagawa measure on $Z_H(\mathbb{A}) \backslash H(\mathbb{A})$ and note that $\mathrm{vol}(Z_H(\mathbb{A})H(F) \backslash H(\mathbb{A})) = \mathfrak{v}$.

Lemma 6.8. *Let $\varphi_1 = \otimes_v \varphi_{1,v}$, $\varphi_2 = \otimes_v \varphi_{2,v} \in V_w$ and $f_1 = \otimes_v f_{1,v}$, $f_2 = \otimes_v f_{2,v} \in V_\pi$. Then we have*

$$\begin{aligned} & \int_{Z_H(\mathbb{A})H(F) \backslash H(\mathbb{A})} \theta(h; \varphi_1, f_1) \overline{\theta(h; \varphi_2, f_2)} dh \\ &= \frac{\mathfrak{v}}{2} \cdot \frac{L^S(1, \pi, \mathrm{Ad} \otimes \omega_{K/F})}{\zeta_K^S(2)} \cdot \prod_{v \in S} Z_v(\frac{1}{2}, [\delta(\varphi_{1,v} \otimes \bar{\varphi}_{2,v}), f_{1,v}, f_{2,v}]). \end{aligned}$$

Proof. Let $\Phi \in \mathbf{I}(\frac{1}{2})$. We extend Φ to a holomorphic section of $\mathbf{I}(s)$ such that its restriction to \mathbf{K} is independent of s . Let

$$E(s, \Phi) = \sum_{d=-1}^{\infty} (s - \frac{1}{2})^d E_d(\frac{1}{2}, \Phi)$$

be the Laurent expansion of $E(s, \Phi)$ at $s = \frac{1}{2}$. Then the map $\Phi \mapsto E_0(\frac{1}{2}, \Phi)$ induces a $\tilde{G}(\mathbb{A})$ -equivariant map

$$\mathbf{I}(\frac{1}{2}) \longrightarrow \mathcal{A}(\tilde{G})/\mathcal{R}(\tilde{G}).$$

We only consider the case $r > 0$. We have

$$\begin{aligned} Z(s, [\varphi], f_1, f_2) &= \mathfrak{v} \int_{\mathcal{C}} \int_{G_1(F) \backslash G_1(\mathbb{A})} \int_{G_1(F) \backslash G_1(\mathbb{A})} E(\iota(g_1, g_2) \iota(g_c, g_c); s, [\varphi]) \\ &\quad \times f_1(g_1 g_c) \overline{f_2(g_2 g_c)} dg_1 dg_2 dc. \end{aligned}$$

Here dc is the Haar measure on \mathcal{C} such that $\mathrm{vol}(\mathcal{C}) = 1$ and dg_1, dg_2 are the Tamagawa measures on $G_1(\mathbb{A})$. By Lemma 6.7, $Z(s, [\varphi], f_1, f_2)$ is holomorphic at $s = \frac{1}{2}$. We have

$$[\varphi](g\iota(g_c, g_c), \frac{1}{2}) = [\tilde{\omega}(\iota(g_c, g_c), h_c)\varphi](g, \frac{1}{2})$$

for $g \in \tilde{G}(\mathbb{A})$ and $c \in \mathcal{C}$. For each $c \in \mathcal{C}$, there exists $\mathcal{F}_c \in \mathcal{R}(\tilde{G})$ such that

$$A_0(g\iota(g_c, g_c); \varphi) = A_0(g; \tilde{\omega}(\iota(g_c, g_c), h_c)\varphi) + \mathcal{F}_c(g)$$

for $g \in \tilde{G}(\mathbb{A})$. By Lemma 6.7, $Z(\frac{1}{2}, [\varphi], f_1, f_2)$ is equal to

$$\begin{aligned} & \mathfrak{v} \int_{\mathcal{C}} \int_{G_1(F) \backslash G_1(\mathbb{A})} \int_{G_1(F) \backslash G_1(\mathbb{A})} A_0(\iota(g_1, g_2) \iota(g_c, g_c); \varphi) f_1(g_1 g_c) \overline{f_2(g_2 g_c)} dg_1 dg_2 dc \\ &= \mathfrak{v} \int_{\mathcal{C}} \int_{G_1(F) \backslash G_1(\mathbb{A})} \int_{G_1(F) \backslash G_1(\mathbb{A})} A_0(\iota(g_1, g_2); \tilde{\omega}(\iota(g_c, g_c), h_c) \varphi) f_1(g_1 g_c) \overline{f_2(g_2 g_c)} dg_1 dg_2 dc. \end{aligned}$$

For each $c \in \mathcal{C}$, there exists $\mathcal{F}'_c \in \mathcal{R}(\tilde{G}_1)$ such that

$$A_0(g_1; \tilde{\omega}(\iota(g_c, g_c), h_c) \varphi) = B_{-1}(g_1; \tilde{\omega}(\iota(g_c, g_c), h_c) \varphi) + \mathcal{F}'_c(g_1)$$

for $g_1 \in \tilde{G}_1(\mathbb{A})$ by (6.1). By Lemma 6.7, $Z(\frac{1}{2}, [\varphi], f_1, f_2)$ is equal to

$$\mathfrak{v} \int_{\mathcal{C}} \int_{G_1(F) \backslash G_1(\mathbb{A})} \int_{G_1(F) \backslash G_1(\mathbb{A})} B_{-1}(\iota(g_1, g_2); \tilde{\omega}(\iota(g_c, g_c), h_c) \varphi) f_1(g_1 g_c) \overline{f_2(g_2 g_c)} dg_1 dg_2 dc.$$

This integral is equal to the residue at $s = \varrho'$ of

$$\begin{aligned} & \mathfrak{v} \int_{\mathcal{C}} \int_{G_1(F) \backslash G_1(\mathbb{A})} \int_{G_1(F) \backslash G_1(\mathbb{A})} I(\iota(g_1, g_2); s, \tilde{\omega}(\iota(g_c, g_c), h_c) \varphi) f_1(g_1 g_c) \overline{f_2(g_2 g_c)} dg_1 dg_2 dc \\ &= \frac{\mathfrak{v}}{\kappa \cdot p(s)} \int_{\mathcal{C}} \int_{G_1(F) \backslash G_1(\mathbb{A})} \int_{G_1(F) \backslash G_1(\mathbb{A})} \int_{H_1(F) \backslash H_1(\mathbb{A})} \theta(\iota(g_1, g_2), h_1; z \cdot \tilde{\omega}(\iota(g_c, g_c), h_c) \varphi) \\ & \quad \times \mathcal{E}(h_1; s) f_1(g_1 g_c) \overline{f_2(g_2 g_c)} dh_1 dg_1 dg_2 dc \\ &= \frac{\mathfrak{v}}{\kappa} \int_{\mathcal{C}} \int_{H_1(F) \backslash H_1(\mathbb{A})} \int_{G_1(F) \backslash G_1(\mathbb{A})} \int_{G_1(F) \backslash G_1(\mathbb{A})} \theta(\iota(g_1, g_2), h_1; \tilde{\omega}(\iota(g_c, g_c), h_c) \varphi) \\ & \quad \times \mathcal{E}(h_1; s) f_1(g_1 g_c) \overline{f_2(g_2 g_c)} dg_1 dg_2 dh_1 dc. \end{aligned}$$

Hence $Z(\frac{1}{2}, [\varphi], f_1, f_2)$ is equal to

$$\begin{aligned} & \mathfrak{v} \int_{\mathcal{C}} \int_{H_1(F) \backslash H_1(\mathbb{A})} \int_{G_1(F) \backslash G_1(\mathbb{A})} \int_{G_1(F) \backslash G_1(\mathbb{A})} \theta(g_1 g_c, h_1 h_c; \varphi_1) \overline{\theta(g_2 g_c, h_1 h_c; \varphi_2)} \\ & \quad \times f_1(g_1 g_c) \overline{f_2(g_2 g_c)} dg_1 dg_2 dh_1 dc \\ &= \int_{Z_H(\mathbb{A}) H(F) \backslash H(\mathbb{A})} \theta(h; \varphi_1, f_1) \overline{\theta(h; \varphi_2, f_2)} dh. \end{aligned}$$

□

By Lemmas 6.1, 6.6, and 6.8, we obtain the following lemma.

Lemma 6.9. *We have*

$$V_{\theta(\pi)} \neq 0.$$

Let $\mathcal{B}_{\theta(\pi)} : V_{\theta(\pi)} \otimes \bar{V}_{\theta(\pi)} \rightarrow \mathbb{C}$ be the Petersson pairing given by

$$\mathcal{B}_{\theta(\pi)}(\phi_1, \phi_2) = \int_{Z_H(\mathbb{A}) H(F) \backslash H(\mathbb{A})} \phi_1(h) \overline{\phi_2(h)} dh$$

for $\phi_1, \phi_2 \in V_{\theta(\pi)}$. For each place v of F , we define an equivariant map

$$\mathcal{Z}_v^\sharp : (\omega_v \boxtimes \bar{\omega}_v) \otimes (\pi_v \boxtimes \bar{\pi}_v) \longrightarrow \mathbb{C}$$

by

$$\begin{aligned} & \mathcal{Z}_v^\sharp(\varphi_{1,v}, \varphi_{2,v}; f_{1,v}, f_{2,v}) \\ &= \zeta_{K_v}(2) L_v(1, \pi_v, \text{Ad} \otimes \omega_{K_v/F_v})^{-1} \cdot Z_v(\frac{1}{2}, [\delta(\varphi_{1,v} \otimes \bar{\varphi}_{2,v})], f_{1,v}, f_{2,v}) \end{aligned}$$

for $\varphi_{1,v}, \varphi_{2,v} \in S(\mathbb{X}_v)$ and $f_{1,v}, f_{2,v} \in \pi_v$. By Lemma 6.6, $\mathcal{Z}_v^\sharp \neq 0$. By Lemma 6.8, there exists a pairing $\mathcal{B}_{\theta(\pi_v)}^\sharp : \theta(\pi_v) \otimes \overline{\theta(\pi_v)} \longrightarrow \mathbb{C}$ such that

$$\mathcal{Z}_v^\sharp = \mathcal{B}_{\theta(\pi_v)}^\sharp \circ (\theta_v \otimes \bar{\theta}_v).$$

By Lemma 6.8, we obtain the following proposition.

Proposition 6.10. *We have*

$$\mathcal{B}_{\theta(\pi)} = 2^\beta \cdot \frac{L(1, \pi, \mathrm{Ad} \otimes \omega_{K/F})}{\zeta_K(2)} \cdot \prod_v \mathcal{B}_{\theta(\pi_v)}^\sharp.$$

Here

$$\beta = \begin{cases} 0 & \text{if } K = F \times F, \\ -1 & \text{if } K \text{ is a quadratic extension of } F. \end{cases}$$

7. Global theta lifts from $\mathrm{GO}(V)$ to GSp_4

Let F be a number field. Let V be a four dimensional quadratic space over F and W a four dimensional symplectic space over F . Let $\tilde{V} = V \oplus (-V)$. Set

$$\begin{aligned} H &= \mathrm{GO}(V), & H_1 &= \mathrm{O}(V), \\ \tilde{H} &= \mathrm{GO}(\tilde{V}) \cong \mathrm{GO}_8, & \tilde{H}_1 &= \mathrm{O}(\tilde{V}) \cong \mathrm{O}_8, \\ G &= \mathrm{GSp}(W) \cong \mathrm{GSp}_4, & G_1 &= \mathrm{Sp}(W) \cong \mathrm{Sp}_4. \end{aligned}$$

Let

$$\mathbf{H} = \{\mathbf{h} = (h_1, h_2) \in H \times H \mid \nu(h_1) = \nu(h_2)\},$$

where $\nu : H \rightarrow \mathbb{G}_m$ is the similitude character. Let $\iota : \mathbf{H} \hookrightarrow \tilde{H}$ be the natural embedding. Let K be the discriminant algebra of V and D a quaternion algebra over F associated to V as in §1.

7.1. Weil representations. Fix a non-trivial additive character $\psi = \otimes_v \psi_v$ of \mathbb{A}/F . We may assume that $\psi_v(x) = \exp(2\pi\sqrt{-1} \mathrm{tr}_{F_v/\mathbb{R}}(x))$ for $x \in F_v$ if v is archimedean (cf. [49, Lemma 5.1]). Let $W = X \oplus Y$ be a complete polarization and set

$$\mathbb{W} = W \otimes V, \quad \mathbb{X} = X \otimes V, \quad \mathbb{Y} = Y \otimes V.$$

Then \mathbb{W} is a symplectic space over F and $\mathbb{W} = \mathbb{X} \oplus \mathbb{Y}$ is a complete polarization. Let ω be the Weil representation of $\mathrm{Mp}(\mathbb{W}(\mathbb{A}))$ on the space $V_\omega = S(\mathbb{X}(\mathbb{A}))$ with respect to ψ and $\mathcal{B}_\omega : V_\omega \otimes \bar{V}_\omega \rightarrow \mathbb{C}$ the canonical pairing. For each place v of F , let ω_v be the Weil representation of $\mathrm{Mp}(\mathbb{W}_v)$ on the space $S(\mathbb{X}_v)$ with respect to ψ_v and $\mathcal{B}_{\omega_v} : \omega_v \otimes \bar{\omega}_v \rightarrow \mathbb{C}$ the canonical pairing. Then we have $\omega = \otimes_v \omega_v$ and $\mathcal{B}_\omega = \prod_v \mathcal{B}_{\omega_v}$. By [30], [13, §5.1], [47], we may regard ω (resp. ω_v) as a representation of $\mathrm{G}(\mathrm{O}(V) \times \mathrm{Sp}(W))(\mathbb{A})$ (resp. $\mathrm{G}(\mathrm{O}(V) \times \mathrm{Sp}(W))(F_v)$).

Let

$$\tilde{\mathbb{W}} = W \otimes \tilde{V}, \quad \tilde{\mathbb{X}} = X \otimes \tilde{V}, \quad \tilde{\mathbb{Y}} = Y \otimes \tilde{V}.$$

Then $\tilde{\mathbb{W}}$ is a symplectic space over F and $\tilde{\mathbb{W}} = \tilde{\mathbb{X}} \oplus \tilde{\mathbb{Y}}$ is a complete polarization. Let $\tilde{\omega}$ be the Weil representation of $\mathrm{Mp}(\tilde{\mathbb{W}}(\mathbb{A}))$ on $S(\tilde{\mathbb{X}}(\mathbb{A}))$ with respect to ψ . We may regard $\tilde{\omega}$ as a representation of $\mathrm{G}(\mathrm{O}(\tilde{V}) \times \mathrm{Sp}(W))(\mathbb{A})$. We have a natural isomorphism

$$S(\tilde{\mathbb{X}}(\mathbb{A})) \cong V_\omega \otimes \bar{V}_\omega$$

as representations of $\mathrm{Mp}(\mathbb{W}(\mathbb{A})) \times \mathrm{Mp}(\mathbb{W}(\mathbb{A}))$. Let

$$\begin{aligned} V^\Delta &= \{(x, x) \mid x \in V\}, & \mathbb{W}^\Delta &= W \otimes V^\Delta, \\ V^\nabla &= \{(x, -x) \mid x \in V\}, & \mathbb{W}^\nabla &= W \otimes V^\nabla. \end{aligned}$$

Then $\tilde{\mathbb{W}} = \mathbb{W}^\nabla \oplus \mathbb{W}^\Delta$ is a complete polarization. Hence we can realize the Weil representation $\tilde{\omega}$ on $S(\mathbb{W}^\nabla(\mathbb{A}))$. By [40, §2], there exists an isomorphism

$$\delta : S(\tilde{\mathbb{X}}(\mathbb{A})) \longrightarrow S(\mathbb{W}^\nabla(\mathbb{A}))$$

as representations of $\mathrm{Mp}(\tilde{\mathbb{W}}(\mathbb{A}))$ such that

$$\delta(\varphi_1 \otimes \bar{\varphi}_2)(0) = \mathcal{B}_\omega(\varphi_1, \varphi_2)$$

for $\varphi_1, \varphi_2 \in V_\omega$.

More generally, let $\mathcal{V} = F^{2n}$ be the space of row vectors equipped with a non-degenerate symmetric bilinear form $\langle x, y \rangle = xJ^t y$ for $x, y \in \mathcal{V}$, where

$$J = \begin{pmatrix} 0 & \mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix}.$$

We identify $\mathrm{GO}(\mathcal{V})$ with

$$\mathrm{GO}_{2n} = \{h \in \mathrm{GL}_{2n} \mid hJ^t h = \nu(h)J, \nu(h) \in \mathbb{G}_m\}.$$

Let $\mathcal{W} = F^{2r}$ be the space of column vectors equipped with a non-degenerate anti-symmetric bilinear form $\langle x, y \rangle = {}^t x J' y$ for $x, y \in \mathcal{W}$, where

$$J' = \begin{pmatrix} 0 & \mathbf{1}_r \\ -\mathbf{1}_r & 0 \end{pmatrix}.$$

We identify $\mathrm{GSp}(\mathcal{W})$ with

$$\mathrm{GSp}_{2r} = \{g \in \mathrm{GL}_{2r} \mid {}^t g J' g = \nu(g)J', \nu(g) \in \mathbb{G}_m\}.$$

Let $\mathbf{W} = \mathcal{W} \otimes \mathcal{V}$,

$$\begin{aligned} \mathcal{X} &= \{(x, 0) \in F^{2n} \mid x \in F^n\}, & \mathbf{X} &= \mathcal{W} \otimes \mathcal{X}, \\ \mathcal{Y} &= \{(0, y) \in F^{2n} \mid y \in F^n\}, & \mathbf{Y} &= \mathcal{W} \otimes \mathcal{Y}. \end{aligned}$$

Then \mathbf{W} is a symplectic space over F and $\mathbf{W} = \mathbf{X} \oplus \mathbf{Y}$ is a complete polarization. We identify \mathbf{X} with $\mathrm{M}_{2r,n}(F)$. Let ω be the Weil representation of $\mathrm{Mp}(\mathbf{W}(\mathbb{A}))$ on the space $S(\mathrm{M}_{2r,n}(\mathbb{A}))$ with respect to ψ . We may regard ω as a representation of $\mathrm{G}(\mathrm{O}_{2n} \times \mathrm{Sp}_{2r})(\mathbb{A})$.

Choosing bases, we fix an isomorphism

$$S(\mathbb{W}^\nabla(\mathbb{A})) \cong S(\mathrm{M}_{4,4}(\mathbb{A}))$$

as representations of $\mathrm{G}(\mathrm{O}(\tilde{V}) \times \mathrm{Sp}(W))(\mathbb{A}) \cong \mathrm{G}(\mathrm{O}_8 \times \mathrm{Sp}_4)(\mathbb{A})$.

7.2. Theta lifts. Let $\sigma \cong \otimes_v \sigma_v$ be an irreducible unitary cuspidal automorphic representation of $H(\mathbb{A})$ on the space V_σ . We assume the following:

- The Jacquet-Langlands transfer of $\sigma|_{D^\times(\mathbb{A}_K)}$ to $\mathrm{GL}_2(\mathbb{A}_K)$ is cuspidal.
- $\sigma_v \otimes \mathrm{sgn} \cong \sigma_v$ for some place v of F .
- If $\sigma_v \otimes \mathrm{sgn} \not\cong \sigma_v$, then $\sigma_v \not\cong \sigma_{0,v}^-$ for any distinguished representation $\sigma_{0,v}$ of $\mathrm{GSO}(V)(F_v)$.

Lemma 7.1. *The partial L-function $L^S(s, \sigma, \mathrm{std})$ is holomorphic and non-zero at $s = 1$.*

Proof. We first consider the case $K = F \times F$. We have $\sigma|_{D^\times(\mathbb{A}_K)} \cong \tau_1^D \boxtimes \tau_2^D$ with an irreducible unitary cuspidal automorphic representation τ_i of $\mathrm{GL}_2(\mathbb{A})$ such that $\tau_1 \not\cong \tau_2$ and $\omega_{\tau_1} = \omega_{\tau_2}$. Here τ_i^D is the Jacquet-Langlands transfer of τ_i to $D^\times(\mathbb{A})$. Then we have

$$L^S(s, \sigma, \mathrm{std}) = L^S(s, \tau_1 \times \tau_2^\vee)$$

and the assertion is well-known.

We next consider the case where K is a quadratic extension of F . We have $\sigma|_{D^\times(\mathbb{A}_K) \times \mathbb{A}^\times} \cong \tau^D \boxtimes \chi$ with an irreducible unitary cuspidal automorphic representation τ of $\mathrm{GL}_2(\mathbb{A}_K)$ and a Hecke character χ of \mathbb{A}^\times such that $\tau^c \not\cong \tau$ and $\omega_\tau = \chi \circ \mathrm{N}_{K/F}$. Here τ^D is the Jacquet-Langlands transfer of τ to $D^\times(\mathbb{A}_K)$. Let μ be a Hecke character of \mathbb{A}_K^\times such that $\mu|_{\mathbb{A}^\times} = \chi^{-1}$. Then we have

$$L^S(s, \sigma, \mathrm{std}) = L^S(s, \tau \otimes \mu, \mathrm{Asai}).$$

Since $(\tau^c \otimes \mu^c)^\vee \not\cong \tau \otimes \mu$ and $L_v(s, \tau_v \otimes \mu_v, \mathrm{Asai})$ is holomorphic and non-zero at $s = 1$ for all v , the assertion follows from [27, Proposition 5.3]. \square

Let

$$G(\mathbb{A})^+ = \{g \in G(\mathbb{A}) \mid \nu(g) \in \nu(H(\mathbb{A}))\}$$

and $G(F)^+ = G(F) \cap G(\mathbb{A})^+$. Let $\varphi \in V_\omega$. The theta function associated to φ is given by

$$\theta(h, g; \varphi) = \sum_{x \in \mathbb{X}(F)} \omega(h, g)\varphi(x)$$

for $(h, g) \in \mathrm{G}(\mathrm{O}(V) \times \mathrm{Sp}(W))(\mathbb{A})$. Let $f \in V_\sigma$. For $g \in G(\mathbb{A})^+$, choose $h \in H(\mathbb{A})$ such that $\nu(h) = \nu(g)$, and put

$$\theta(g; \varphi, f) = \int_{H_1(F) \backslash H_1(\mathbb{A})} \theta(h_1 h, g; \varphi) f(h_1 h) dh_1.$$

Here $dh_1 = \prod_v dh_{1,v}$ is the Tamagawa measure on $H_1(\mathbb{A})$. Note that $\mathrm{vol}(H_1(F) \backslash H_1(\mathbb{A})) = 1$ and we may assume that the volume of a hyperspecial maximal compact subgroup of $H_{1,v}$ with respect to $dh_{1,v}$ is 1 for almost all v . This integral defines an automorphic form $\theta(\varphi, f)$ on $G(\mathbb{A})^+$. We extend $\theta(\varphi, f)$ to an automorphic form on $G(\mathbb{A})$ by the natural embedding

$$G(F)^+ \backslash G(\mathbb{A})^+ \hookrightarrow G(F) \backslash G(\mathbb{A})$$

and extension by zero. Let $\theta(\sigma)$ be the automorphic representation of $G(\mathbb{A})^+$ on the space $V_{\theta(\sigma)}$ generated by $\theta(\varphi, f)$ for all $\varphi \in V_\omega$ and $f \in V_\sigma$. By assumption on σ , $\theta(\sigma)$ is cuspidal. In Lemma 7.12 below, we will show that $V_{\theta(\sigma)} \neq 0$. In particular, $\theta(\sigma_v) \neq 0$ for all v . Hence $\theta(\sigma)$ is irreducible,

$$\theta(\sigma) \cong \otimes_v \theta(\sigma_v),$$

and $\theta(\sigma_v)$ is unitary for all v . Thus, we obtain a $\mathrm{G}(\mathrm{O}(V) \times \mathrm{Sp}(W))(\mathbb{A})$ -equivariant surjective map

$$\theta : V_\omega \otimes V_\sigma \longrightarrow V_{\theta(\sigma)}$$

and $\mathrm{G}(\mathrm{O}(V) \times \mathrm{Sp}(W))(F_v)$ -equivariant surjective maps

$$\theta_v : \omega_v \otimes \sigma_v \longrightarrow \theta(\sigma_v)$$

such that $\theta = \otimes_v \theta_v$.

Let π be the automorphic representation of $G(\mathbb{A})$ on the space V_π generated by $V_{\theta(\sigma)}$. For each place v of F , let $\pi_v = \mathrm{ind}_{G_v^+}^{G_v}(\theta(\sigma_v))$. By Lemma 5.2, π_v is irreducible.

Lemma 7.2. *We have*

$$\pi \cong \otimes_v \pi_v.$$

Proof. Since $G(\mathbb{A})^+$ is an open subgroup of $G(\mathbb{A})$, we have a natural $G(\mathbb{A})$ -equivariant map

$$(7.1) \quad \mathrm{c}\text{-ind}_{G(\mathbb{A})^+}^{G(\mathbb{A})}(V_{\theta(\sigma)}) \longrightarrow V_\pi.$$

By definition, (7.1) is surjective. Since $\mathrm{c}\text{-ind}_{G(\mathbb{A})^+}^{G(\mathbb{A})}(\theta(\sigma)) \cong \otimes_v \pi_v$ is irreducible, (7.1) is injective. \square

7.3. Eisenstein series. For each $r \in \mathbb{N}$ with $r \leq n$, we define a parabolic subgroup $P_{n,r}$ of GO_{2n} by

$$P_{n,r} = \left\{ \begin{pmatrix} a & * & * & * \\ 0 & a' & * & b' \\ 0 & 0 & \nu(h')^t a^{-1} & 0 \\ 0 & c' & * & d' \end{pmatrix} \in \mathrm{GO}_{2n} \mid a \in \mathrm{GL}_r, h' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathrm{GO}_{2n-2r} \right\}.$$

Let $\delta_{P_{n,r}}$ be the modulus character of $P_{n,r}(\mathbb{A})$. For $\nu \in \mathbb{G}_m$, let

$$d(\nu) = \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & \nu \mathbf{1}_n \end{pmatrix}.$$

We define a maximal compact subgroup $\mathbf{K} = \prod_v \mathbf{K}_v$ of $\mathrm{GO}_{2n}(\mathbb{A})$ by

$$\mathbf{K}_v = \begin{cases} \mathrm{GO}_{2n}(\mathfrak{o}_v) & \text{if } v \text{ is non-archimedean,} \\ \mathrm{GO}_{2n}(F_v) \cap \mathrm{O}(2n) & \text{if } v \text{ is real,} \\ \mathrm{GO}_{2n}(F_v) \cap \mathrm{U}(2n) & \text{if } v \text{ is complex.} \end{cases}$$

Then we have $\mathrm{GO}_{2n}(\mathbb{A}) = P_{n,r}(\mathbb{A})\mathbf{K}$.

Let $\mathbf{I}^{(n,r)}(s)$ denote the degenerate principal series representation of $\mathrm{GO}_{2n}(\mathbb{A})$ given by

$$\mathbf{I}^{(n,r)}(s) = \mathrm{Ind}_{P_{n,r}(\mathbb{A})}^{\mathrm{GO}_{2n}(\mathbb{A})} (\delta_{P_{n,r}}^{s/(2n-r-1)}),$$

where Ind denotes the normalized induction. Given a holomorphic section Φ of $\mathbf{I}^{(n,r)}(s)$, we define an Eisenstein series $E^{(n,r)}(s, \Phi)$ on $\mathrm{GO}_{2n}(\mathbb{A})$ by

$$E^{(n,r)}(h; s, \Phi) = \sum_{\gamma \in P_{n,r}(F) \backslash \mathrm{GO}_{2n}(F)} \Phi(\gamma h, s)$$

for $\mathrm{Re}(s) \gg 0$. If Φ^o is the holomorphic section of $\mathbf{I}^{(n,r)}(s)$ such that $\Phi^o(k, s) = 1$ for all $k \in \mathbf{K}$, we write $E^{(n,r)}(s) = E^{(n,r)}(s, \Phi^o)$. For each $s_0 \in \mathbb{C}$, let

$$E^{(n,r)}(s) = \sum_{d \gg -\infty} (s - s_0)^d E_d^{(n,r)}(s_0)$$

be the Laurent expansion of $E^{(n,r)}(s)$ at $s = s_0$.

We define a $\mathrm{G}(\mathrm{O}_{2n} \times \mathrm{Sp}_{2r})(\mathbb{A})$ -equivariant map

$$[-] : S(\mathrm{M}_{2r,n}(\mathbb{A})) \longrightarrow \mathbf{I}^{(n,n)}(r - \frac{n-1}{2})$$

by

$$[\varphi](h, r - \frac{n-1}{2}) = |\nu(h)|^{-nr/2} \cdot \omega(d(\nu(h)^{-1})h)\varphi(0)$$

for $h \in \mathrm{GO}_{2n}(\mathbb{A})$. Here $\mathrm{G}(\mathrm{O}_{2n} \times \mathrm{Sp}_{2r})(\mathbb{A})$ acts on $\mathbf{I}^{(n,n)}(r - \frac{n-1}{2})$ via the projection $\mathrm{G}(\mathrm{O}_{2n} \times \mathrm{Sp}_{2r})(\mathbb{A}) \rightarrow \mathrm{GO}_{2n}(\mathbb{A})$. We extend $[\varphi]$ to a holomorphic section of $\mathbf{I}^{(n,n)}(s)$ such that its restriction to \mathbf{K} is independent of s .

7.4. Theta integrals. We define a parabolic subgroup P' of GSp_{2r} by

$$P' = \left\{ \begin{pmatrix} a & * \\ 0 & \nu^t a^{-1} \end{pmatrix} \in \mathrm{GSp}_{2r} \mid a \in \mathrm{GL}_r, \nu \in \mathbb{G}_m \right\}.$$

Let $\delta_{P'}$ be the modulus character of $P'(\mathbb{A})$. For $\nu \in \mathbb{G}_m$, let

$$d(\nu) = \begin{pmatrix} \mathbf{1}_r & 0 \\ 0 & \nu \mathbf{1}_r \end{pmatrix}.$$

We define a maximal compact subgroup $\mathbf{K}' = \prod_v \mathbf{K}'_v$ of $\mathrm{GSp}_{2r}(\mathbb{A})$ by

$$\mathbf{K}'_v = \begin{cases} \mathrm{Ad}(d(\varpi_v^{c_v}))(\mathrm{GSp}_{2r}(\mathfrak{o}_v)) & \text{if } v \text{ is non-archimedean,} \\ \mathrm{GSp}_{2r}(F_v) \cap \mathrm{O}(2r) & \text{if } v \text{ is real,} \\ \mathrm{GSp}_{2r}(F_v) \cap \mathrm{U}(2r) & \text{if } v \text{ is complex.} \end{cases}$$

Here ϖ_v is a uniformizer of F_v and c_v is the order of ψ_v . Then we have $\mathrm{GSp}_{2r}(\mathbb{A}) = P'(\mathbb{A})\mathbf{K}'$. Let $P'_1 = P' \cap \mathrm{Sp}_{2r}$ and $\mathbf{K}'_1 = \mathbf{K}' \cap \mathrm{Sp}_{2r}(\mathbb{A})$ so that $\mathrm{Sp}_{2r}(\mathbb{A}) = P'_1(\mathbb{A})\mathbf{K}'_1$. Let dg_1 be the Tamagawa measure on $\mathrm{Sp}_{2r}(\mathbb{A})$ and note that $\mathrm{vol}(\mathrm{Sp}_{2r}(F) \backslash \mathrm{Sp}_{2r}(\mathbb{A})) = 1$. Let $d_l p'$ be the left-invariant Tamagawa measure on $P'_1(\mathbb{A})$ and dk' the Haar measure on \mathbf{K}'_1 such that $\mathrm{vol}(\mathbf{K}'_1) = 1$. There exists a constant κ such that

$$\int_{\mathrm{Sp}_{2r}(\mathbb{A})} f(g_1) dg_1 = \kappa \int_{P'_1(\mathbb{A})} \int_{\mathbf{K}'_1} f(p'k') d_l p' dk'$$

for $f \in L^1(\mathrm{Sp}_{2r}(\mathbb{A}))$.

Put $\varrho' = \frac{r+1}{2}$. Let Φ' be the holomorphic section of $\mathrm{Ind}_{P'(\mathbb{A})}^{\mathrm{GSp}_{2r}(\mathbb{A})}(\delta_{P'}^{s/(r+1)})$ such that $\Phi'(k', s) = 1$ for all $k' \in \mathbf{K}'$. We define an Eisenstein series $\mathcal{E}(s)$ on $\mathrm{GSp}_{2r}(\mathbb{A})$ by

$$\mathcal{E}(g; s) = \sum_{\gamma \in P'(F) \backslash \mathrm{GSp}_{2r}(F)} \Phi'(\gamma g, s)$$

for $\mathrm{Re}(s) > \varrho'$. By [36, §5], [19, Lemma 9.1], we have

$$\mathrm{Res}_{s=\varrho'} \mathcal{E}(g; s) = \kappa$$

for $g \in \mathrm{GSp}_{2r}(\mathbb{A})$.

Let $\varphi \in S(\mathrm{M}_{2r,n}(\mathbb{A}))$. The theta function associated to φ is given by

$$\theta(h, g; \varphi) = \sum_{x \in \mathrm{M}_{2r,n}(F)} \omega(h, g) \varphi(x)$$

for $(h, g) \in \mathrm{G}(\mathrm{O}_{2n} \times \mathrm{Sp}_{2r})(\mathbb{A})$. Let $z \in C_c^\infty(\mathrm{O}_{2n}(F_v) // \mathrm{O}_{2n}(\mathfrak{o}_v))$ be the regularizing Hecke operator as in [34, §5], [25, §2.3], where v is a certain non-archimedean place of F depending on φ . There exists a self-adjoint Hecke operator $z' \in C_c^\infty(\mathrm{Sp}_{2r}(F_v) // \mathrm{Sp}_{2r}(\mathfrak{o}_v))$ such that $\omega(z) = \omega(z')$. Then we have $z' \cdot \mathcal{E}(s) = p(s) \cdot \mathcal{E}(s)$ with some $p(s) \in \mathbb{C}[q_v^s, q_v^{-s}]$. Here q_v is the cardinality of $\mathfrak{o}_v / \varpi_v \mathfrak{o}_v$. Following Kudla and Rallis [34, §5], we define the regularized theta integral $I^{(n,r)}(s, \varphi)$ by

$$I^{(n,r)}(h; s, \varphi) = \frac{1}{\kappa \cdot p(s)} \int_{\mathrm{Sp}_{2r}(F) \backslash \mathrm{Sp}_{2r}(\mathbb{A})} \theta(h, g_1 g; z \cdot \varphi) \mathcal{E}(g_1 g; s) dg_1$$

for $h \in \mathrm{GO}_{2n}(\mathbb{A})$, where $g \in \mathrm{GSp}_{2r}(\mathbb{A})$ such that $\nu(g) = \nu(h)$. By [34, Lemma 5.5.6], $I^{(n,r)}(s, \varphi)$ has at most a double (resp. simple) pole at $s = \varrho'$ if $r \leq n - 1 < 2r$ (resp. $2r \leq n - 1$).

Let $\hat{\varphi} \in S(\mathrm{M}_{r,2n}(\mathbb{A}))$ be the partial Fourier transform of $\varphi \in S(\mathrm{M}_{2r,n}(\mathbb{A}))$ defined by

$$\hat{\varphi}(u, v) = \int_{\mathrm{M}_{r,n}(\mathbb{A})} \varphi \begin{pmatrix} x \\ u \end{pmatrix} \psi(\mathrm{tr}(v^t x)) dx$$

for $u, v \in \mathrm{M}_{r,n}(\mathbb{A})$, where dx is the Tamagawa measure on $\mathrm{M}_{r,n}(\mathbb{A})$. For $h \in \mathrm{GO}_{2n}(\mathbb{A})$, choose $g \in \mathrm{GSp}_{2r}(\mathbb{A})$ such that $\nu(g) = \nu(h)$, and put

$$\begin{aligned} & \Psi(\varphi)(h, s) \\ &= \int_{\mathrm{GL}_r(\mathbb{A})} \int_{\mathbf{K}'_1} (\omega(h, k'g) \varphi) \widehat{(\mathbf{0}_{r,n}, {}^t a, \mathbf{0}_{r,n-r})} \Phi'(k'g, s) |\det(a)|^{s+n-\varrho'} dk' d^\times a. \end{aligned}$$

Here $d^\times a$ is the Tamagawa measure on $\mathrm{GL}_r(\mathbb{A})$. By [34, Lemma 5.5.2], $\Psi(\varphi)$ is a meromorphic section of $\mathbf{I}^{(n,r)}(s)$ and is holomorphic for $\mathrm{Re}(s) > -\frac{2n-3r-1}{2}$. By [34, §5.5], we have

$$I^{(n,r)}(s, \varphi) = E^{(n,r)}(s, \Psi(\varphi)).$$

We now consider the spherical case. Let $\xi(s) = \mathfrak{D}^{s/2} \zeta(s)$, where \mathfrak{D} is the absolute value of the discriminant of F and $\zeta(s)$ is the zeta function of F including archimedean factors. Put $\rho = \mathrm{Res}_{s=1} \xi(s)$. We define $\varphi_r = \otimes_v \varphi_{r,v} \in S(\mathrm{M}_{r,r}(\mathbb{A}))$ as follows:

- If v is non-archimedean, then $\varphi_{r,v}$ is the characteristic function of $\mathrm{M}_{r,r}(\mathfrak{o}_v)$.
- If v is real, then $\varphi_{r,v}(x) = \exp(-\pi \mathrm{tr}(^t x x))$ for $x \in \mathrm{M}_{r,r}(F_v)$.
- If v is complex, then $\varphi_{r,v}(x) = \exp(-2\pi \mathrm{tr}(^t \bar{x} x))$ for $x \in \mathrm{M}_{r,r}(F_v)$.

Put

$$\Xi_r(s) = \int_{\mathrm{GL}_r(\mathbb{A})} \varphi_r(a) |\det(a)|^s d^\times a,$$

where $d^\times a$ is the Tamagawa measure on $\mathrm{GL}_r(\mathbb{A})$.

Lemma 7.3. *We have*

$$\Xi_r(s) = \mathfrak{D}^{-rs/2} \cdot \rho^{-1} \prod_{i=2}^r \xi(i)^{-1} \cdot \prod_{j=0}^{r-1} \xi(s-j).$$

Here we omit $\prod_{i=2}^r \xi(i)^{-1}$ if $r = 1$.

Proof. If $r = 1$, then we have $\Xi_1(s) = \rho^{-1} \zeta(s) = \mathfrak{D}^{-s/2} \rho^{-1} \xi(s)$. Assume that $r \geq 2$. Let T (resp. U) be the subgroup of GL_r consisting of diagonal matrices (resp. unipotent upper triangular matrices). We define a maximal compact subgroup $\mathcal{K} = \prod_v \mathcal{K}_v$ of $\mathrm{GL}_r(\mathbb{A})$ by

$$\mathcal{K}_v = \begin{cases} \mathrm{GL}_r(\mathfrak{o}_v) & \text{if } v \text{ is non-archimedean,} \\ \mathrm{GL}_r(F_v) \cap \mathrm{O}(r) & \text{if } v \text{ is real,} \\ \mathrm{GL}_r(F_v) \cap \mathrm{U}(r) & \text{if } v \text{ is complex.} \end{cases}$$

Let $d^\times t$ (resp. du) be the Tamagawa measure on $T(\mathbb{A})$ (resp. $U(\mathbb{A})$) and dk the Haar measure on \mathcal{K} such that $\mathrm{vol}(\mathcal{K}) = 1$. By [36, §5], we have

$$\int_{\mathrm{GL}_r(\mathbb{A})} f(a) d^\times a = \varkappa \int_{T(\mathbb{A})} \int_{U(\mathbb{A})} \int_{\mathcal{K}} f(tuk) d^\times t du dk$$

for $f \in L^1(\mathrm{GL}_r(\mathbb{A}))$, where $\varkappa = \rho^{r-1} \prod_{i=2}^r \xi(i)^{-1}$. Hence we have

$$\Xi_r(s) = \varkappa \int_{T(\mathbb{A})} \int_{U(\mathbb{A})} \varphi_r(tu) |\det(t)|^s d^\times t du.$$

Changing variables, we have

$$\Xi_r(s) = \varkappa \cdot \prod_{j=0}^{r-1} \Xi_1(s-j) \cdot \left(\int_{\mathbb{A}} \varphi_1(x) dx \right)^{r(r-1)/2},$$

where dx is the Tamagawa measure on \mathbb{A} . Since

$$\int_{\mathbb{A}} \varphi_1(x) dx = \mathfrak{D}^{-1/2},$$

the assertion follows. \square

We define **the spherical Schwartz function** $\varphi^o = \otimes_v \varphi_v^o \in S(\mathrm{M}_{2r,n}(\mathbb{A}))$ as follows:

- If v is non-archimedean, then

$$\varphi_v^o \begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} q_v^{-c_v nr/2} & \text{if } x \in M_{r,n}(\varpi_v^{-c_v} \mathfrak{o}_v) \text{ and } y \in M_{r,n}(\mathfrak{o}_v), \\ 0 & \text{otherwise.} \end{cases}$$

- If v is real, then $\varphi_v^o(x) = \exp(-\pi \operatorname{tr}({}^t x x))$ for $x \in M_{2r,n}(F_v)$.
- If v is complex, then $\varphi_v^o(x) = \exp(-2\pi \operatorname{tr}({}^t \bar{x} x))$ for $x \in M_{2r,n}(F_v)$.

Here q_v is the cardinality of $\mathfrak{o}_v/\varpi_v \mathfrak{o}_v$ and c_v is the order of ψ_v . Then we have $\hat{\varphi}^o = \otimes_v \hat{\varphi}_v^o$, where

- $\hat{\varphi}_v^o$ is the characteristic function of $M_{r,2n}(\mathfrak{o}_v)$ if v is non-archimedean,
- $\hat{\varphi}_v^o(x) = \exp(-\pi \operatorname{tr}({}^t x x))$ for $x \in M_{r,2n}(F_v)$ if v is real,
- $\hat{\varphi}_v^o(x) = \exp(-2\pi \operatorname{tr}({}^t \bar{x} x))$ for $x \in M_{r,2n}(F_v)$ if v is complex.

Moreover, we have

$$\omega(k, k') \varphi^o = \varphi^o$$

for $(k, k') \in (\mathbf{K} \times \mathbf{K}') \cap G(\mathrm{O}_{2n} \times \mathrm{Sp}_{2r})(\mathbb{A})$. Hence we have

$$(7.2) \quad E^{(n,n)}(s, [\varphi^o]) = \mathfrak{D}^{-nr/2} \cdot E^{(n,n)}(s).$$

Lemma 7.4.

$$I^{(n,r)}(s, \varphi^o) = \mathfrak{D}^{-r(s+n-\varrho')/2} \cdot \rho^{-1} \prod_{i=2}^n \xi(i)^{-1} \cdot \prod_{j=0}^{r-1} \xi(s+n-\varrho'-j) \cdot E^{(n,r)}(s).$$

Proof. By Lemma 7.3, we have

$$\Psi(\varphi^o)(1, s) = \mathfrak{D}^{-r(s+n-\varrho')/2} \cdot \rho^{-1} \prod_{i=2}^n \xi(i)^{-1} \cdot \prod_{j=0}^{r-1} \xi(s+n-\varrho'-j).$$

Since $\Psi(\varphi^o)$ is \mathbf{K} -invariant, the assertion follows. \square

7.5. The Siegel-Weil formula. We identify GO_8 and $P_{4,4}$ with \tilde{H} and the parabolic subgroup of \tilde{H} stabilizing V^Δ , respectively. Let $\mathcal{A}(\tilde{H})$ denote the space of automorphic forms on $\tilde{H}(\mathbb{A})$ and $\mathcal{R}(\tilde{H})$ the subspace of $\mathcal{A}(\tilde{H})$ generated by $\operatorname{Res}_{s=1/2} E^{(4,4)}(s, \Phi)$ for all holomorphic sections Φ of $\mathbf{I}^{(4,4)}(s)$. We remark that $E^{(4,4)}(s, \Phi)$ has at most a simple pole at $s = \frac{1}{2}$ by [32, Theorem 1.0.1].

Let $\varphi^o \in S(M_{4,4}(\mathbb{A}))$ be the spherical Schwartz function as defined in p. 30. Let $S(M_{4,4}(\mathbb{A}))^o$ be the subspace of $S(M_{4,4}(\mathbb{A}))$ generated by $\omega(h_1)\varphi^o$ for all $h_1 \in \tilde{H}_1(\mathbb{A})$. Let

$$\begin{aligned} A_0 &: S(M_{4,4}(\mathbb{A})) \longrightarrow \mathcal{A}(\tilde{H})/\mathcal{R}(\tilde{H}), \\ B_{-1} &: S(M_{4,4}(\mathbb{A})) \longrightarrow \mathcal{A}(\tilde{H}), \\ C_0 &: S(M_{4,4}(\mathbb{A})) \longrightarrow \mathcal{A}(\tilde{H}), \end{aligned}$$

be the $\tilde{H}_1(\mathbb{A})$ -equivariant maps defined by

$$\begin{aligned} E^{(4,4)}(s, [\varphi]) &= \sum_{d=-1}^{\infty} (s - \tfrac{1}{2})^d A_d(\varphi), \\ I^{(4,2)}(s, \varphi) &= \sum_{d=-2}^{\infty} (s - \tfrac{3}{2})^d B_d(\varphi), \end{aligned}$$

$$I^{(4,1)}(s, \text{pr}(\varphi)) = \sum_{d=-1}^{\infty} (s-1)^d C_d(\varphi),$$

for $\varphi \in S(M_{4,4}(\mathbb{A}))$. Here $\text{pr} : S(M_{4,4}(\mathbb{A})) \rightarrow S(M_{2,4}(\mathbb{A}))$ is the $\tilde{H}_1(\mathbb{A})$ -equivariant map defined by

$$\text{pr}(\varphi) \begin{pmatrix} x \\ y \end{pmatrix} = \int_{\mathbb{A}^4} \varphi \begin{pmatrix} u \\ x \\ 0 \\ y \end{pmatrix} du$$

for $x, y \in \mathbb{A}^4$, where du is the Tamagawa measure on \mathbb{A}^4 .

Proposition 7.5. *We have*

$$B_{-1}(\varphi) \equiv A_0(\varphi) + \mathfrak{D}^{-2} \rho \xi(4)^{-1} \cdot C_0(\varphi) \pmod{\mathcal{R}(\tilde{H})}$$

for $\varphi \in S(M_{4,4}(\mathbb{A}))^\circ$.

Proof. Let $\varphi^\circ \in S(M_{4,4}(\mathbb{A}))$ be the spherical Schwartz function. Then $\text{pr}(\varphi^\circ) \in S(M_{2,4}(\mathbb{A}))$ is also the spherical Schwartz function. By (7.2), we have

$$A_0(\varphi^\circ) = \mathfrak{D}^{-4} \cdot E_0^{(4,4)}\left(\frac{1}{2}\right).$$

By Lemma 7.4 and Lemma B.6 in Appendix B, we have

$$B_{-1}(\varphi^\circ) \equiv \mathfrak{D}^{-4} \rho^{-1} \xi(2)^{-1} \xi(3) \xi(4) \cdot E_{-1}^{(4,2)}\left(\frac{3}{2}\right) \pmod{\mathcal{R}(\tilde{H})}.$$

By Lemma 7.4 and Lemma B.7 in Appendix B, we have

$$C_0(\varphi^\circ) \equiv \mathfrak{D}^{-2} \rho^{-1} \xi(4) \cdot E_0^{(4,1)}(1) \pmod{\mathcal{R}(\tilde{H})}.$$

Hence we have

$$\mathfrak{D}^4 \cdot B_{-1}(\varphi^\circ) \equiv \mathfrak{D}^4 \cdot A_0(\varphi^\circ) + \mathfrak{D}^2 \rho \xi(4)^{-1} \cdot C_0(\varphi^\circ) \pmod{\mathcal{R}(\tilde{H})}$$

by Proposition B.8 in Appendix B. This completes the proof. \square

7.6. The doubling method. Let $\mathcal{B}_\sigma : V_\sigma \otimes \bar{V}_\sigma \rightarrow \mathbb{C}$ be the Petersson pairing given by

$$\mathcal{B}_\sigma(f_1, f_2) = \int_{Z_H(\mathbb{A})H(F)\backslash H(\mathbb{A})} f_1(h) \overline{f_2(h)} dh$$

for $f_1, f_2 \in V_\sigma$. Here Z_H is the identity component of the center of H and dh is the Tamagawa measure on $Z_H(\mathbb{A})\backslash H(\mathbb{A})$. Put

$$\mathfrak{v} = \text{vol}(Z_H(\mathbb{A})H(F)\backslash H(\mathbb{A})) = \begin{cases} 2 & \text{if } K = F \times F, \\ 1 & \text{if } K \text{ is a quadratic extension of } F. \end{cases}$$

We fix a decomposition $\mathcal{B}_\sigma = \prod_v \mathcal{B}_{\sigma_v}$, where $\mathcal{B}_{\sigma_v} : \sigma_v \otimes \bar{\sigma}_v \rightarrow \mathbb{C}$ is a pairing.

For a holomorphic section Φ of $\mathbf{I}^{(4,4)}(s)$ and $f_1, f_2 \in V_\sigma$, the zeta integral of Piatetski-Shapiro and Rallis [44], [12, §6.2] is given by

$$Z(s, \Phi, f_1, f_2) = \int_{Z_{\tilde{H}}(\mathbb{A})\mathbf{H}(F)\backslash \mathbf{H}(\mathbb{A})} E^{(4,4)}(\iota(h_1, h_2); s, \Phi) f_1(h_1) \overline{f_2(h_2)} d\mathbf{h}.$$

Here $Z_{\tilde{H}}$ is the identity component of the center of \tilde{H} and $d\mathbf{h}$ is the Tamagawa measure on $Z_{\tilde{H}}(\mathbb{A})\backslash \mathbf{H}(\mathbb{A})$. Note that $\text{vol}(Z_{\tilde{H}}(\mathbb{A})\mathbf{H}(F)\backslash \mathbf{H}(\mathbb{A})) = \mathfrak{v}$. For each place v of F , let

$$Z_v(s, \Phi_v, f_{1,v}, f_{2,v}) = \int_{H_{1,v}} \Phi_v(\iota(h_{1,v}, 1), s) \mathcal{B}_{\sigma_v}(\sigma_v(h_{1,v}) f_{1,v}, f_{2,v}) dh_{1,v}.$$

Lemma 7.6. *For a holomorphic section $\Phi = \otimes_v \Phi_v$ of $\mathbf{I}^{(4,4)}(s)$ and $f_1 = \otimes_v f_{1,v}, f_2 = \otimes_v f_{2,v} \in V_\sigma$, we have*

$$Z(s, \Phi, f_1, f_2) = \frac{L^S(s + \frac{1}{2}, \sigma, \mathrm{std})}{\zeta^S(2s+1)\zeta^S(2s+3)} \cdot \prod_{v \in S} Z_v(s, \Phi_v, f_{1,v}, f_{2,v}).$$

Proof. The assertion follows from the doubling method [44], [12, §6.2]. \square

7.7. Local zeta integrals. Let $\mathbf{I}_v^{(4,4)}(s) = \mathrm{Ind}_{P_{4,4,v}}^{\tilde{H}_v}(\delta_{P_{4,4,v}}^{s/3})$ denote the degenerate principal series representation of \tilde{H}_v .

Lemma 7.7. *For a holomorphic section Φ_v of $\mathbf{I}_v^{(4,4)}(s)$ and $f_{1,v}, f_{2,v} \in \sigma_v$, the integral $Z_v(s, \Phi_v, f_{1,v}, f_{2,v})$ is absolutely convergent at $s = \frac{1}{2}$.*

Proof. By [3, Proposition 3.3], [41, Appendix], [38], there exist $\varphi_v, \varphi'_v \in S(\mathrm{M}_{4,4}(F_v))$ such that

$$\Phi_v(h_v, \frac{1}{2}) = [\varphi_v](h_v, \frac{1}{2}) + [\varphi'_v](h_v, \frac{1}{2}) \cdot \mathrm{sgn}(h_v)$$

for $h_v \in \tilde{H}_v$. Hence we may assume that $\Phi_v = [\delta(\varphi_{1,v} \otimes \bar{\varphi}_{2,v})]$ with $\varphi_{1,v}, \varphi_{2,v} \in S(\mathbb{X}_v)$. But then we have

$$\Phi_v(\iota(h_{1,v}, 1), \frac{1}{2}) = \mathcal{B}_{\omega_v}(\omega_v(h_{1,v})\varphi_{1,v}, \varphi_{2,v})$$

for $h_{1,v} \in H_{1,v}$. By [39, Theorem 3.2], the function $h_{1,v} \mapsto \mathcal{B}_{\omega_v}(\omega_v(h_{1,v})\varphi_{1,v}, \varphi_{2,v})$ belongs to $L^{1+\varepsilon}(H_{1,v})$ for any $\varepsilon > 0$. This yields the lemma. \square

For $\varphi_{1,v}, \varphi_{2,v} \in S(\mathbb{X}_v)$, we have

$$\begin{aligned} & Z_v(\frac{1}{2}, [\delta(\varphi_{1,v} \otimes \bar{\varphi}_{2,v})], f_{1,v}, f_{2,v}) \\ &= \int_{H_{1,v}} \mathcal{B}_{\omega_v}(\omega_v(h_{1,v})\varphi_{1,v}, \varphi_{2,v}) \mathcal{B}_{\sigma_v}(\sigma_v(h_{1,v})f_{1,v}, f_{2,v}) dh_{1,v}. \end{aligned}$$

Let $S(\mathrm{M}_{4,4}(F_v))^\circ$ be the subspace of $S(\mathrm{M}_{4,4}(F_v))$ generated by $\omega(h_{1,v})\varphi_v^\circ$ for all $h_{1,v} \in \tilde{H}_{1,v}$.

Lemma 7.8. *There exist $\varphi_v \in S(\mathrm{M}_{4,4}(F_v))^\circ$ and $f_{1,v}, f_{2,v} \in \sigma_v$ such that*

$$Z_v(\frac{1}{2}, [\varphi_v], f_{1,v}, f_{2,v}) \neq 0.$$

Proof. We fix a place v of F and suppress it from the notation. By [32], there exist $\Phi \in \mathbf{I}^{(4,4)}(\frac{1}{2})$ and $f_1, f_2 \in \sigma$ such that $Z(\frac{1}{2}, \Phi, f_1, f_2) \neq 0$. Let R (resp. R_0) be the image of the equivariant map $S(\mathrm{M}_{4,4}(F)) \rightarrow \mathbf{I}^{(4,4)}(\frac{1}{2})$ (resp. $S(\mathrm{M}_{2,4}(F)) \rightarrow \mathbf{I}^{(4,4)}(-\frac{1}{2})$). By [3, Proposition 3.3], [41, Appendix], [38], we have

$$\mathbf{I}^{(4,4)}(\frac{1}{2}) = R + R \otimes \mathrm{sgn}, \quad \mathbf{I}^{(4,4)}(\frac{1}{2})/R \cong R_0 \otimes \mathrm{sgn}.$$

Moreover, R is generated by a \mathbf{K} -invariant element of $\mathbf{I}^{(4,4)}(\frac{1}{2})$. It suffices to show that there exist $\Phi \in R$ and $f_1, f_2 \in \sigma$ such that $Z(\frac{1}{2}, \Phi, f_1, f_2) \neq 0$.

We assume that $Z(\frac{1}{2}, \Phi, f_1, f_2) = 0$ for all $\Phi \in R$ and $f_1, f_2 \in \sigma$. If $\sigma \otimes \mathrm{sgn} \cong \sigma$, then we have $Z(\frac{1}{2}, \Phi, f_1, f_2) = 0$ for all $\Phi \in R \otimes \mathrm{sgn}$ and $f_1, f_2 \in \sigma$ and hence a contradiction. If $\sigma \otimes \mathrm{sgn} \not\cong \sigma$, then $Z(\frac{1}{2}, \Phi, f_1, f_2)$ defines a non-zero equivariant map

$$(R_0 \otimes \mathrm{sgn}) \otimes (\sigma \boxtimes \bar{\sigma}) \longrightarrow \mathbb{C}.$$

As in [14, Proposition 3.1], this shows that the theta lift of $\sigma \otimes \mathrm{sgn}$ to $\mathrm{GL}_2(F)^+$ is non-zero. By [8, Lemmas 4.1, 5.4], [43], [1], we have

$$\sigma|_{D \times (K) \times F^\times} \cong (\sigma \otimes \mathrm{sgn})|_{D \times (K) \times F^\times} \cong \varsigma_K^D \boxtimes \omega_\varsigma \omega_{K/F}$$

for some irreducible admissible representation ς of $\mathrm{GL}_2(F)$ with central character ω_ς . Thus, $\sigma \cong (\varsigma_K^D \boxtimes \omega_\varsigma \omega_{K/F})^-$ for a distinguished representation $\varsigma_K^D \boxtimes \omega_\varsigma \omega_{K/F}$ of $\mathrm{GSO}(V)(F)$. This contradicts the assumption on σ . \square

7.8. The Rallis inner product formula.

Lemma 7.9. *For $\mathcal{F} \in \mathcal{R}(\tilde{H})$ and $f_1, f_2 \in V_\sigma$, we have*

$$\int_{Z_{\tilde{H}}(\mathbb{A})\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})} \mathcal{F}(\iota(h_1, h_2)) f_1(h_1) \overline{f_2(h_2)} d\mathbf{h} = 0.$$

Proof. The assertion follows from Lemmas 7.1, 7.6, and 7.7. \square

Lemma 7.10. *For $\varphi \in S(\mathrm{M}_{4,4}(\mathbb{A}))$ and $f_1, f_2 \in V_\sigma$, we have*

$$\int_{Z_{\tilde{H}}(\mathbb{A})\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})} C_0(\iota(h_1, h_2); \varphi) f_1(h_1) \overline{f_2(h_2)} d\mathbf{h} = 0.$$

Proof. Let W' be a two dimensional symplectic space over F and set $G' = \mathrm{GSp}(W') \cong \mathrm{GL}_2$. Let $W' = X' \oplus Y'$ be a complete polarization and set

$$\mathbb{W}' = W' \otimes V, \quad \mathbb{X}' = X' \otimes V, \quad \mathbb{Y}' = Y' \otimes V.$$

Then \mathbb{W}' is a symplectic space over F and $\mathbb{W}' = \mathbb{X}' \oplus \mathbb{Y}'$ is a complete polarization. Let ω' be the Weil representation of $\mathrm{Mp}(\mathbb{W}'(\mathbb{A}))$ on the space $V_{\omega'} = S(\mathbb{X}'(\mathbb{A}))$ with respect to ψ . We may regard ω' as a representation of $\mathrm{G}(\mathrm{O}(V) \times \mathrm{Sp}(W'))(\mathbb{A})$. By [40, §2], there exists a natural isomorphism

$$\delta' : V_{\omega'} \otimes \bar{V}_{\omega'} \longrightarrow S(\mathrm{M}_{2,4}(\mathbb{A})).$$

Let $\vartheta(\sigma)$ be the theta lift of σ to $G'(\mathbb{A})$. Then $\vartheta(\sigma) = 0$. Indeed, it is easy to see that $\vartheta(\sigma)$ is cuspidal, and if $\vartheta(\sigma) \neq 0$, then $\sigma_v \otimes \mathrm{sgn} \not\cong \sigma_v$ for all v by the local unramified theta correspondence and the strong multiplicity one theorem. This contradicts the assumption on σ . For $\varphi'_1, \varphi'_2 \in V_{\omega'}$ and $f_1, f_2 \in V_\sigma$, we have

$$\begin{aligned} & \int_{Z_{\tilde{H}}(\mathbb{A})\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})} I^{(4,1)}(\iota(h_1, h_2); s, \delta'(\varphi'_1 \otimes \bar{\varphi}'_2)) f_1(h_1) \overline{f_2(h_2)} d\mathbf{h} \\ &= \frac{1}{\kappa} \int_{Z_{G'}(\mathbb{A})G'(F)^+\backslash G'(\mathbb{A})^+} \theta(g'; \varphi'_1, f_1) \overline{\theta(g'; \varphi'_2, f_2)} \mathcal{E}(g'; s) dg'. \end{aligned}$$

(See also the proof of Lemma 7.11 below.) Since $\vartheta(\sigma) = 0$, this integral is zero. This completes the proof. \square

Let $\mathbb{A}^{\times,+} = \nu(H(\mathbb{A}))$, $F^{\times,+} = F^\times \cap \mathbb{A}^{\times,+}$, and $\mathcal{C} = \mathbb{A}^{\times,2} F^{\times,+} \backslash \mathbb{A}^{\times,+}$. The similitude characters induce isomorphisms

$$Z_G(\mathbb{A})G_1(\mathbb{A})G(F)^+\backslash G(\mathbb{A})^+ \cong \mathcal{C}, \quad Z_H(\mathbb{A})H_1(\mathbb{A})H(F)\backslash H(\mathbb{A}) \cong \mathcal{C}.$$

Fix cross sections $c \mapsto g_c$ and $c \mapsto h_c$ of $G(\mathbb{A})^+ \rightarrow \mathcal{C}$ and $H(\mathbb{A}) \rightarrow \mathcal{C}$, respectively. Let dg be the Tamagawa measure on $Z_G(\mathbb{A})\backslash G(\mathbb{A})$ and note that $\mathrm{vol}(Z_G(\mathbb{A})G(F)\backslash G(\mathbb{A})) = 2$.

Lemma 7.11. *Let $\varphi = \otimes_v \varphi_v \in S(\mathrm{M}_{4,4}(\mathbb{A}))^o$ and $f_1 = \otimes_v f_{1,v}, f_2 = \otimes_v f_{2,v} \in V_\sigma$. We write $\varphi = \sum_i \delta(\varphi_{1,i} \otimes \bar{\varphi}_{2,i})$, where $\varphi_{1,i}, \varphi_{2,i} \in V_\omega$. Then we have*

$$\begin{aligned} & \sum_i \int_{Z_G(\mathbb{A})G(F)\backslash G(\mathbb{A})} \theta(g; \varphi_{1,i}, f_1) \overline{\theta(g; \varphi_{2,i}, f_2)} dg \\ &= \frac{L^S(1, \sigma, \mathrm{std})}{\zeta^S(2)\zeta^S(4)} \cdot \prod_{v \in S} Z_v(\tfrac{1}{2}, [\varphi_v], f_{1,v}, f_{2,v}). \end{aligned}$$

Proof. By Lemma 7.9, $Z(s, [\varphi], f_1, f_2)$ is holomorphic at $s = \frac{1}{2}$. By Proposition 7.5 and Lemmas 7.9 and 7.10, we have

$$Z\left(\frac{1}{2}, [\varphi], f_1, f_2\right) = \int_{Z_{\mathbb{H}(\mathbb{A})}\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})} B_{-1}(\iota(h_1, h_2); \varphi) f_1(h_1) \overline{f_2(h_2)} dh.$$

This integral is equal to the residue at $s = \frac{3}{2}$ of

$$\begin{aligned} & \mathfrak{v} \int_{\mathcal{C}} \int_{H_1(F)\backslash H_1(\mathbb{A})} \int_{H_1(F)\backslash H_1(\mathbb{A})} I^{(4,2)}(\iota(h_1 h_c, h_2 h_c); s, \varphi) f_1(h_1 h_c) \overline{f_2(h_2 h_c)} dh_1 dh_2 dc \\ &= \frac{\mathfrak{v}}{\kappa \cdot p(s)} \int_{\mathcal{C}} \int_{H_1(F)\backslash H_1(\mathbb{A})} \int_{H_1(F)\backslash H_1(\mathbb{A})} \int_{G_1(F)\backslash G_1(\mathbb{A})} \theta(\iota(h_1 h_c, h_2 h_c), g_1 g_c; z \cdot \varphi) \\ & \quad \times \mathcal{E}(g_1 g_c; s) f_1(h_1 h_c) \overline{f_2(h_2 h_c)} dg_1 dh_1 dh_2 dc \\ &= \frac{\mathfrak{v}}{\kappa} \int_{\mathcal{C}} \int_{G_1(F)\backslash G_1(\mathbb{A})} \int_{H_1(F)\backslash H_1(\mathbb{A})} \int_{H_1(F)\backslash H_1(\mathbb{A})} \theta(\iota(h_1 h_c, h_2 h_c), g_1 g_c; \varphi) \\ & \quad \times \mathcal{E}(g_1 g_c; s) f_1(h_1 h_c) \overline{f_2(h_2 h_c)} dh_1 dh_2 dg_1 dc. \end{aligned}$$

Here dc is the Haar measure on \mathcal{C} such that $\mathrm{vol}(\mathcal{C}) = 1$ and dh_1, dh_2 are the Tamagawa measures on $H_1(\mathbb{A})$. Hence $Z\left(\frac{1}{2}, [\varphi], f_1, f_2\right)$ is equal to

$$\begin{aligned} & \mathfrak{v} \sum_i \int_{\mathcal{C}} \int_{G_1(F)\backslash G_1(\mathbb{A})} \int_{H_1(F)\backslash H_1(\mathbb{A})} \int_{H_1(F)\backslash H_1(\mathbb{A})} \theta(h_1 h_c, g_1 g_c; \varphi_{1,i}) \overline{\theta(h_2 h_c, g_1 g_c; \varphi_{2,i})} \\ & \quad \times f_1(h_1 h_c) \overline{f_2(h_2 h_c)} dh_1 dh_2 dg_1 dc \\ &= \sum_i \int_{Z_G(\mathbb{A})G(F)^+\backslash G(\mathbb{A})^+} \theta(g; \varphi_{1,i}, f_1) \overline{\theta(g; \varphi_{2,i}, f_2)} dg. \end{aligned}$$

Note that $\mathrm{vol}(Z_G(\mathbb{A})G(F)^+\backslash G(\mathbb{A})^+) = \mathfrak{v}$. Since the supports of $\theta(\varphi_{1,i}, f_1)$ and $\theta(\varphi_{2,i}, f_2)$ are contained in $G(F)G(\mathbb{A})^+$, we have

$$\begin{aligned} & \int_{Z_G(\mathbb{A})G(F)^+\backslash G(\mathbb{A})^+} \theta(g; \varphi_{1,i}, f_1) \overline{\theta(g; \varphi_{2,i}, f_2)} dg \\ &= \int_{Z_G(\mathbb{A})G(F)\backslash G(\mathbb{A})} \theta(g; \varphi_{1,i}, f_1) \overline{\theta(g; \varphi_{2,i}, f_2)} dg. \end{aligned}$$

This completes the proof. \square

By Lemmas 7.1, 7.8, and 7.11, we obtain the following lemma.

Lemma 7.12. *We have*

$$V_{\theta(\sigma)} \neq 0.$$

Let $\mathcal{B}_\pi : V_\pi \otimes \bar{V}_\pi \rightarrow \mathbb{C}$ be the Petersson pairing given by

$$\mathcal{B}_\pi(\phi_1, \phi_2) = \int_{Z_G(\mathbb{A})G(F)\backslash G(\mathbb{A})} \phi_1(g) \overline{\phi_2(g)} dg$$

for $\phi_1, \phi_2 \in V_\pi$. For each place v of F , we define an equivariant map

$$\mathcal{Z}_v^\sharp : (\omega_v \boxtimes \bar{\omega}_v) \otimes (\sigma_v \boxtimes \bar{\sigma}_v) \longrightarrow \mathbb{C}$$

by

$$\begin{aligned} & \mathcal{Z}_v^\sharp(\varphi_{1,v}, \varphi_{2,v}; f_{1,v}, f_{2,v}) \\ &= \zeta_v(2) \zeta_v(4) L_v(1, \sigma_v, \mathrm{std})^{-1} \cdot Z_v\left(\frac{1}{2}, [\delta(\varphi_{1,v} \otimes \bar{\varphi}_{2,v})], f_{1,v}, f_{2,v}\right) \end{aligned}$$

for $\varphi_{1,v}, \varphi_{2,v} \in S(\mathbb{X}_v)$ and $f_{1,v}, f_{2,v} \in \sigma_v$. By Lemma 7.8, $\mathcal{Z}_v^\sharp \neq 0$. By Lemmas 5.6 and 5.7, there exists a non-zero G_v^+ -invariant pairing $\mathcal{B}_{\theta(\sigma_v)}^\sharp : \theta(\sigma_v) \otimes \overline{\theta(\sigma_v)} \rightarrow \mathbb{C}$ such that

$$\mathcal{Z}_v^\sharp = \mathcal{B}_{\theta(\sigma_v)}^\sharp \circ (\theta_v \otimes \bar{\theta}_v).$$

We extend $\mathcal{B}_{\theta(\sigma_v)}^\sharp$ uniquely to a G_v -invariant pairing $\mathcal{B}_{\pi_v}^\sharp : \pi_v \otimes \bar{\pi}_v \rightarrow \mathbb{C}$ such that $\mathcal{B}_{\pi_v}^\sharp|_{\theta(\sigma_v) \otimes \overline{\theta(\sigma_v)}} = \mathcal{B}_{\theta(\sigma_v)}^\sharp$. By Lemma 7.11, we obtain the following proposition.

Proposition 7.13. *We have*

$$\mathcal{B}_\pi = \frac{L(1, \sigma, \text{std})}{\zeta(2)\zeta(4)} \cdot \prod_v \mathcal{B}_{\pi_v}^\sharp.$$

8. Tamagawa measures

Let F be a totally real number field and E a totally real étale quadratic algebra over F . Let W_0 be a two dimensional symplectic space over E and V a four dimensional quadratic space over F . Set

$$\begin{aligned} G' &= \{g' \in \mathbf{R}_{E/F}(\mathrm{GSp}(W_0)) \mid \nu(g') \in \mathbb{G}_m\}, & G'_1 &= \mathbf{R}_{E/F}(\mathrm{Sp}(W_0)) \cong \mathbf{R}_{E/F}(\mathrm{SL}_2), \\ H &= \mathrm{GO}(V), & H_1 &= \mathrm{O}(V). \end{aligned}$$

Let $Z_{G'}$ and Z_H be the identity component of the center of G' and H , respectively. We have isogenies

$$\mathrm{pr} : G'_1 \longrightarrow Z_{G'} \backslash G', \quad \mathrm{pr} : H_1 \longrightarrow Z_H \backslash H.$$

Let $\omega_{G'}$ and ω_H be non-zero invariant differential forms of top degree on $Z_{G'} \backslash G'$ and $Z_H \backslash H$ over F , respectively. Then $\omega_{G'_1} = \mathrm{pr}^*(\omega_{G'})$ and $\omega_{H_1} = \mathrm{pr}^*(\omega_H)$ are also non-zero invariant differential forms of top degree on G'_1 and H_1 over F , respectively. Fix a non-trivial additive character $\psi = \otimes_v \psi_v$ of \mathbb{A}/F . For each place v of F , let dx_v be the self-dual Haar measure on F_v with respect to ψ_v . Let $dg'_{1,v}, dg'_v, dh_{1,v}^0, dh_v^0$ be the Haar measures on

$$G'_{1,v}, \quad Z_{G',v} \backslash G'_v, \quad H_{1,v}, \quad Z_{H,v} \backslash H_v,$$

determined by dx_v and $\omega_{G'_1}, \omega_{G'}, \omega_{H_1}, \omega_H$, respectively. Let $dh_{1,v} = \frac{1}{2}dh_{1,v}^0$ and $dh_v = \frac{1}{2}dh_v^0$. Let $F_v^{\times,+} = \nu(H_v)$. Then we have

$$\begin{aligned} \int_{Z_{G',v} \backslash G'_v} f(g'_v) dg'_v &= \frac{1}{2} \sum_{c \in F_v^{\times,2} \backslash F_v^{\times}} \int_{G'_{1,v}} f(g'_{1,v} g'_c) dg'_{1,v}, \\ \int_{Z_{H,v} \backslash H_v} \phi(h_v) dh_v &= \frac{1}{2} \sum_{c \in F_v^{\times,2} \backslash F_v^{\times,+}} \int_{H_{1,v}} \phi(h_{1,v} h_c) dh_{1,v}, \end{aligned}$$

for $f \in L^1(Z_{G',v} \backslash G'_v)$ and $\phi \in L^1(Z_{H,v} \backslash H_v)$. Here $g'_c \in Z_{G',v} \backslash G'_v$ with $\nu(g'_c) = c$ and $h_c \in Z_{H,v} \backslash H_v$ with $\nu(h_c) = c$. By definition, the product measures

$$\prod_v dg'_{1,v}, \quad \prod_v dg'_v, \quad \prod_v dh_{1,v}, \quad \prod_v dh_v,$$

are the Tamagawa measures on

$$G'_1(\mathbb{A}), \quad Z_{G'}(\mathbb{A}) \backslash G'(\mathbb{A}), \quad H_1(\mathbb{A}), \quad Z_H(\mathbb{A}) \backslash H(\mathbb{A}),$$

respectively. We remark that the products of these measures are convergent, although the volumes of hyperspecial maximal compact subgroups are not 1 for almost all v .

9. The explicit local seesaw identity

We retain the notation of §1 and §8. We fix a place v of F and suppress it from the notation. Let $\mathbb{W} = V \otimes_F W = \mathrm{R}_{E/F}(W_0 \otimes_E V_E)$, where $V_E = V \otimes_F E$. Let ω be the Weil representation of $\mathrm{Mp}(\mathbb{W})$ with respect to ψ . We may regard ω as a representation of $\mathrm{G}(\mathrm{O}(V) \times \mathrm{Sp}(W))(F)$ or that of $\mathrm{G}(\mathrm{Sp}(W_0) \times \mathrm{O}(V_E))(E)$. In particular, we have a seesaw diagram:

$$\begin{array}{ccc} G = \mathrm{GSp}(W)(F) & & H' = \mathrm{GO}(V_E)(E)' \\ & \searrow & \swarrow \\ & & H = \mathrm{GO}(V)(F) \\ & \swarrow & \searrow \\ G' = \mathrm{GSp}(W_0)(E)' & & \end{array}$$

The goal of this section is to establish an **explicit** seesaw identity for this seesaw diagram.

Recall that σ is an irreducible unitary admissible representation of H and τ is an irreducible unitary admissible representation of $\mathrm{GSp}(W_0)(E)$ containing an irreducible subrepresentation π' of G' . Let $\mathcal{B}_\sigma : \sigma \otimes \bar{\sigma} \rightarrow \mathbb{C}$ and $\mathcal{B}_\tau : \tau \otimes \bar{\tau} \rightarrow \mathbb{C}$ be pairings. Let $\mathcal{B}_{\pi'}^\flat : \pi' \otimes \bar{\pi}' \rightarrow \mathbb{C}$ be the pairing given by $\mathcal{B}_{\pi'}^\flat = \mathcal{B}_\tau|_{\pi' \otimes \bar{\pi}'}$. Let $\mathcal{C} = F^{\times, 2} \setminus F^{\times, +}$. Put

$$\begin{aligned} & \mathcal{Q}(\varphi_1, \varphi_2; \phi_1, \phi_2; f_1, f_2) \\ &= \sum_{c \in \mathcal{C}} \int_{G'_1} \int_{H_1} \mathcal{B}_\omega(\omega(g'_1 g'_c, h_1 h_c) \varphi_1, \varphi_2) \mathcal{B}_\sigma(\sigma(h_1 h_c) \phi_1, \phi_2) \mathcal{B}_\tau(\tau(g'_1 g'_c) f_1, f_2) dh_1 dg'_1 \end{aligned}$$

for $\varphi_1, \varphi_2 \in \omega$, $\phi_1, \phi_2 \in \sigma$, and $f_1, f_2 \in \pi'$. Here dh_1 and dg'_1 are the Haar measures on H_1 and G'_1 given in §8, respectively.

Lemma 9.1. *The integral $\mathcal{Q}(\varphi_1, \varphi_2; \phi_1, \phi_2; f_1, f_2)$ is absolutely convergent.*

Proof. We only consider the case $E = K = F \times F$ and $D = \mathrm{M}_{2,2}(F)$; the other cases are similar. It suffices to show that the integral

$$\begin{aligned} & \int_{(F^+)^4} \mathcal{B}_\omega(\omega(t(a_1, a_2), t(a_3, a_4)) \varphi_1, \varphi_2) \mathcal{B}_\sigma(\sigma(t(a_3, a_4)) \phi_1, \phi_2) \\ & \times \mathcal{B}_\tau(\tau(t(a_1, a_2)) f_1, f_2) |a_1^{-2} a_2^{-2} a_3^{-2} a_4^{-2}| d^\times a_1 d^\times a_2 d^\times a_3 d^\times a_4 \end{aligned}$$

is absolutely convergent. Here $F^+ = \{a \in F^\times \mid |a| \leq 1\}$,

$$t(a, a') = \left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} a' & 0 \\ 0 & a'^{-1} \end{pmatrix} \right),$$

and we regard $t(a, a')$ as an element of G' or that of H . By the Kim-Shahidi estimate [28], [26], there exist $\frac{7}{9} < \lambda_1, \lambda_2, \lambda_3, \lambda_4 \leq 1$ such that

$$\begin{aligned} |\mathcal{B}_\tau(\tau(t(a_1, a_2)) f_1, f_2)| &\leq C \cdot |a_1|^{\lambda_1} |a_2|^{\lambda_2}, \\ |\mathcal{B}_\sigma(\sigma(t(a_3, a_4)) \phi_1, \phi_2)| &\leq C \cdot |a_3|^{\lambda_3} |a_4|^{\lambda_4}, \end{aligned}$$

for $a_1, a_2, a_3, a_4 \in F^+$ with some constant C . We define a function Υ on F^\times by

$$\Upsilon(a) = \begin{cases} 1 & \text{if } a \in F^+, \\ |a|^{-1} & \text{otherwise.} \end{cases}$$

For $\phi, \phi' \in S(F)$, we have

$$\left| \int_F \phi(ax) \overline{\phi'(x)} dx \right| \leq C' \cdot \Upsilon(a)$$

for $a \in F^\times$ with some constant C' . Realizing the Weil representation ω on $S(M_{2,2}(F)^2)$, we have

$$\begin{aligned} & \omega(t(a_1, a_2), t(a_3, a_4))\varphi_1(x, y) \\ &= |a_1^2 a_2^2| \varphi_1 \left(\begin{pmatrix} a_1 a_3^{-1} a_4 x_1 & a_1 a_3^{-1} a_4^{-1} x_2 \\ a_1 a_3 a_4 x_3 & a_1 a_3 a_4^{-1} x_4 \end{pmatrix}, \begin{pmatrix} a_2 a_3^{-1} a_4 y_1 & a_2 a_3^{-1} a_4^{-1} y_2 \\ a_2 a_3 a_4 y_3 & a_2 a_3 a_4^{-1} y_4 \end{pmatrix} \right) \end{aligned}$$

for

$$x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}.$$

Hence we have

$$\begin{aligned} & |\mathcal{B}_\omega(\omega(t(a_1, a_2), t(a_3, a_4))\varphi_1, \varphi_2)| \\ & \leq C'' \cdot |a_1^2 a_2^2| \Upsilon(a_1 a_3^{-1} a_4) \Upsilon(a_1 a_3^{-1} a_4^{-1}) \Upsilon(a_1 a_3 a_4) \Upsilon(a_1 a_3 a_4^{-1}) \\ & \quad \times \Upsilon(a_2 a_3^{-1} a_4) \Upsilon(a_2 a_3^{-1} a_4^{-1}) \Upsilon(a_2 a_3 a_4) \Upsilon(a_2 a_3 a_4^{-1}) \end{aligned}$$

with some constant C'' . By symmetry, it suffices to show that the integral

$$\begin{aligned} & \int_{|a_1| \leq |a_2| \leq 1} \int_{|a_3| \leq |a_4| \leq 1} \Upsilon(a_1 a_3^{-1} a_4) \Upsilon(a_1 a_3^{-1} a_4^{-1}) \Upsilon(a_2 a_3^{-1} a_4) \Upsilon(a_2 a_3^{-1} a_4^{-1}) \\ & \quad \times |a_1^{\lambda_1} a_2^{\lambda_2} a_3^{\lambda_3-2} a_4^{\lambda_4-2}| d^\times a_1 d^\times a_2 d^\times a_3 d^\times a_4 \end{aligned}$$

is convergent. We change variables $b_1 = a_3 a_4$ and $b_2 = a_3 a_4^{-1}$. Let $\mu_1 = \frac{\lambda_3 + \lambda_4}{2}$ and $\mu_2 = \frac{\lambda_3 - \lambda_4}{2}$. Note that $\frac{7}{9} < \mu_1 \leq 1$ and $|\mu_2| < \frac{1}{9}$. It remains to show that the integral

$$\begin{aligned} & \int_{|a_1| \leq |a_2| \leq 1} \int_{|b_1| \leq |b_2| \leq 1} \Upsilon(a_1 b_1^{-1}) \Upsilon(a_1 b_2^{-1}) \Upsilon(a_2 b_1^{-1}) \Upsilon(a_2 b_2^{-1}) \\ & \quad \times |a_1^{\lambda_1} a_2^{\lambda_2} b_1^{\mu_1-2} b_2^{\mu_2}| d^\times a_1 d^\times a_2 d^\times b_1 d^\times b_2 \end{aligned}$$

is convergent. The integrand is equal to

$$\begin{cases} |a_1^{\lambda_1} a_2^{\lambda_2} b_1^{\mu_1-2} b_2^{\mu_2}| & \text{if } |a_1| \leq |a_2| \leq |b_1| \leq |b_2| \leq 1, \\ |a_1^{\lambda_1} a_2^{\lambda_2-1} b_1^{\mu_1-1} b_2^{\mu_2}| & \text{if } |a_1| \leq |b_1| \leq |a_2| \leq |b_2| \leq 1, \\ |a_1^{\lambda_1} a_2^{\lambda_2-2} b_1^{\mu_1-1} b_2^{\mu_2+1}| & \text{if } |a_1| \leq |b_1| \leq |b_2| \leq |a_2| \leq 1, \\ |a_1^{\lambda_1-1} a_2^{\lambda_2-1} b_1^{\mu_1} b_2^{\mu_2}| & \text{if } |b_1| \leq |a_1| \leq |a_2| \leq |b_2| \leq 1, \\ |a_1^{\lambda_1-1} a_2^{\lambda_2-2} b_1^{\mu_1} b_2^{\mu_2+1}| & \text{if } |b_1| \leq |a_1| \leq |b_2| \leq |a_2| \leq 1, \\ |a_1^{\lambda_1-2} a_2^{\lambda_2-2} b_1^{\mu_1} b_2^{\mu_2+2}| & \text{if } |b_1| \leq |b_2| \leq |a_1| \leq |a_2| \leq 1. \end{cases}$$

We change variables

$$\begin{cases} a_1 = t_1 t_2 t_3 t_4, a_2 = t_1 t_2 t_3, b_1 = t_1 t_2, b_2 = t_1 & \text{if } |a_1| \leq |a_2| \leq |b_1| \leq |b_2| \leq 1, \\ a_1 = t_1 t_2 t_3 t_4, b_1 = t_1 t_2 t_3, a_2 = t_1 t_2, b_2 = t_1 & \text{if } |a_1| \leq |b_1| \leq |a_2| \leq |b_2| \leq 1, \\ a_1 = t_1 t_2 t_3 t_4, b_1 = t_1 t_2 t_3, b_2 = t_1 t_2, a_2 = t_1 & \text{if } |a_1| \leq |b_1| \leq |b_2| \leq |a_2| \leq 1, \\ b_1 = t_1 t_2 t_3 t_4, a_1 = t_1 t_2 t_3, a_2 = t_1 t_2, b_2 = t_1 & \text{if } |b_1| \leq |a_1| \leq |a_2| \leq |b_2| \leq 1, \\ b_1 = t_1 t_2 t_3 t_4, a_1 = t_1 t_2 t_3, b_2 = t_1 t_2, a_2 = t_1 & \text{if } |b_1| \leq |a_1| \leq |b_2| \leq |a_2| \leq 1, \\ b_1 = t_1 t_2 t_3 t_4, b_2 = t_1 t_2 t_3, a_1 = t_1 t_2, a_2 = t_1 & \text{if } |b_1| \leq |b_2| \leq |a_1| \leq |a_2| \leq 1, \end{cases}$$

where $t_1, t_2, t_3, t_4 \in F^+$. Then the integrand is equal to

$$\begin{cases} |t_1^{\lambda_1+\lambda_2+\mu_1+\mu_2-2} t_2^{\lambda_1+\lambda_2+\mu_1-2} t_3^{\lambda_1+\lambda_2} t_4^{\lambda_1}| & \text{if } |a_1| \leq |a_2| \leq |b_1| \leq |b_2| \leq 1, \\ |t_1^{\lambda_1+\lambda_2+\mu_1+\mu_2-2} t_2^{\lambda_1+\lambda_2+\mu_1-2} t_3^{\lambda_1+\mu_1-1} t_4^{\lambda_1}| & \text{if } |a_1| \leq |b_1| \leq |a_2| \leq |b_2| \leq 1, \\ |t_1^{\lambda_1+\lambda_2+\mu_1+\mu_2-2} t_2^{\lambda_1+\mu_1+\mu_2} t_3^{\lambda_1+\mu_1-1} t_4^{\lambda_1}| & \text{if } |a_1| \leq |b_1| \leq |b_2| \leq |a_2| \leq 1, \\ |t_1^{\lambda_1+\lambda_2+\mu_1+\mu_2-2} t_2^{\lambda_1+\lambda_2+\mu_1-2} t_3^{\lambda_1+\mu_1-1} t_4^{\mu_1}| & \text{if } |b_1| \leq |a_1| \leq |a_2| \leq |b_2| \leq 1, \\ |t_1^{\lambda_1+\lambda_2+\mu_1+\mu_2-2} t_2^{\lambda_1+\mu_1+\mu_2} t_3^{\lambda_1+\mu_1-1} t_4^{\mu_1}| & \text{if } |b_1| \leq |a_1| \leq |b_2| \leq |a_2| \leq 1, \\ |t_1^{\lambda_1+\lambda_2+\mu_1+\mu_2-2} t_2^{\lambda_1+\mu_1+\mu_2} t_3^{\mu_1+\mu_2+2} t_4^{\mu_1}| & \text{if } |b_1| \leq |b_2| \leq |a_1| \leq |a_2| \leq 1. \end{cases}$$

It is easy to check that the integral in each of the six ranges is convergent. This completes the proof of the lemma. \square

Let $\theta(\tau)$ be the theta lift of τ to $H(E)$. Let $\theta : \omega \otimes \tau \rightarrow \theta(\tau)$ be an equivariant surjective map. Let $\theta(\pi')$ be the image of $\omega \otimes \pi'$ in $\theta(\tau)$. Let

$$\begin{aligned} T &: (\omega \boxtimes \bar{\omega}) \otimes (\sigma \boxtimes \bar{\sigma}) \otimes (\pi' \boxtimes \bar{\pi}') \longrightarrow (\theta(\sigma) \boxtimes \overline{\theta(\sigma)}) \otimes (\pi' \boxtimes \bar{\pi}'), \\ T' &: (\omega \boxtimes \bar{\omega}) \otimes (\sigma \boxtimes \bar{\sigma}) \otimes (\pi' \boxtimes \bar{\pi}') \longrightarrow (\sigma \boxtimes \bar{\sigma}) \otimes (\theta(\pi') \boxtimes \overline{\theta(\pi')}), \end{aligned}$$

be equivariant surjective maps induced by

$$\begin{aligned} \theta \otimes \bar{\theta} &: (\omega \boxtimes \bar{\omega}) \otimes (\sigma \boxtimes \bar{\sigma}) \longrightarrow \theta(\sigma) \boxtimes \overline{\theta(\sigma)}, \\ \theta \otimes \bar{\theta} &: (\omega \boxtimes \bar{\omega}) \otimes (\pi' \boxtimes \bar{\pi}') \longrightarrow \theta(\pi') \boxtimes \overline{\theta(\pi')}, \end{aligned}$$

respectively. Let $\mathcal{B}_\pi^\sharp : \pi \otimes \bar{\pi} \rightarrow \mathbb{C}$ and $\mathcal{B}_{\theta(\tau)}^\sharp : \theta(\tau) \otimes \overline{\theta(\tau)} \rightarrow \mathbb{C}$ be the pairings given by

$$(9.1) \quad \begin{aligned} &\mathcal{B}_\pi^\sharp(\theta(\varphi_1, \phi_1), \theta(\varphi_2, \phi_2)) \\ &= \frac{\zeta_{E \otimes K}(2)}{L(1, \tau, \mathrm{Ad} \otimes \omega_{E \otimes K/E})} \int_{H_1} \mathcal{B}_\omega(\omega(h_1)\varphi_1, \varphi_2) \mathcal{B}_\sigma(\sigma(h_1)\phi_1, \phi_2) dh_1 \end{aligned}$$

and

$$(9.2) \quad \begin{aligned} &\mathcal{B}_{\theta(\tau)}^\sharp(\theta(\varphi_1, f_1), \theta(\varphi_2, f_2)) \\ &= \frac{\zeta(2)\zeta(4)}{L(1, \sigma, \mathrm{std})} \int_{G'_1} \mathcal{B}_\omega(\omega(g'_1)\varphi_1, \varphi_2) \mathcal{B}_\tau(\tau(g'_1)f_1, f_2) dg'_1 \end{aligned}$$

for $\varphi_1, \varphi_2 \in \omega$, $\phi_1, \phi_2 \in \sigma$, and $f_1, f_2 \in \tau$, respectively. Here dh_1 and dg'_1 are the Haar measures on H_1 and G'_1 given in §8, respectively. We remark that the normalizing factors in the front of the integrals are introduced to ensure that (9.1) and (9.2) are 1 for unramified data if we take Haar measures such that the volumes of hyperspecial maximal compact subgroups are 1. (Notice that such measures do not agree with the measures given in §8.) We define a $G' \times G'$ -invariant functional

$$\mathcal{P}^\sharp : (\pi \boxtimes \bar{\pi}) \otimes (\pi' \boxtimes \bar{\pi}') \longrightarrow \mathbb{C}$$

by

$$(9.3) \quad \begin{aligned} \mathcal{P}^\sharp(\phi_1, \phi_2; f_1, f_2) &= \frac{1}{\zeta(2)\zeta(4)} \cdot \frac{L(1, \sigma, \mathrm{std})L(1, \sigma, \mathrm{Ad})L(1, \tau, \mathrm{Ad})}{L(\frac{1}{2}, \sigma \times \theta(\tau))} \\ &\quad \times \int_{Z_{G'} \backslash G'} \mathcal{B}_\pi^\sharp(\pi(g')\phi_1, \phi_2) \mathcal{B}_{\pi'}^\sharp(\pi'(g')f_1, f_2) dg' \end{aligned}$$

for $\phi_1, \phi_2 \in \pi$ and $f_1, f_2 \in \pi'$. Here dg' is the Haar measure on $Z_{G'} \backslash G'$ given in §8. By Lemma 9.1, this integral is absolutely convergent. We define an $H \times H$ -invariant functional

$$\mathcal{I}^\sharp : (\sigma \boxtimes \bar{\sigma}) \otimes (\theta(\tau) \boxtimes \overline{\theta(\tau)}) \longrightarrow \mathbb{C}$$

by

$$(9.4) \quad \begin{aligned} \mathcal{I}^\sharp(\phi_1, \phi_2; f_1, f_2) &= \frac{1}{\zeta_{E \otimes K}(2)} \cdot \frac{L(1, \sigma, \text{Ad})L(1, \tau_K, \text{Ad})}{L(\frac{1}{2}, \sigma \times \theta(\tau))} \\ &\quad \times \int_{Z_H \backslash H} \mathcal{B}_\sigma(\sigma(h)\phi_1, \phi_2) \mathcal{B}_{\theta(\tau)}^\sharp(\theta(\tau)(h)f_1, f_2) dh \end{aligned}$$

for $\phi_1, \phi_2 \in \sigma$ and $f_1, f_2 \in \theta(\tau)$. Here dh is the Haar measure on $Z_H \backslash H$ given in §8.

Now we have the following lemma, which should be thought of as an explicit local seesaw identity.

Lemma 9.2. *We have*

$$\mathcal{P}^\sharp \circ \mathcal{T} = \mathcal{I}^\sharp \circ \mathcal{T}'$$

as functionals on $(\omega \boxtimes \bar{\omega}) \otimes (\sigma \boxtimes \bar{\sigma}) \otimes (\pi' \boxtimes \bar{\pi}')$. Namely, the diagram

$$\begin{array}{ccc} & (\omega \boxtimes \bar{\omega}) \otimes (\sigma \boxtimes \bar{\sigma}) \otimes (\pi' \boxtimes \bar{\pi}') & \\ \mathcal{T} \swarrow & & \searrow \mathcal{T}' \\ (\theta(\sigma) \boxtimes \bar{\theta}(\sigma)) \otimes (\pi' \boxtimes \bar{\pi}') & & (\sigma \boxtimes \bar{\sigma}) \otimes (\theta(\pi') \boxtimes \bar{\theta}(\pi')) \\ \mathcal{P}^\sharp \searrow & & \swarrow \mathcal{I}^\sharp \\ & \mathbb{C} & \end{array}$$

is commutative.

Proof. Given the absolute convergence of Lemma 9.1, the commutativity of the diagram is essentially a consequence of Fubini's theorem. More precisely, we have

$$\begin{aligned} & \zeta_{E \otimes K}(2) L(1, \tau, \text{Ad} \otimes \omega_{E \otimes K/E})^{-1} \cdot \mathcal{Q}(\varphi_1, \varphi_2; \phi_1, \phi_2; f_1, f_2) \\ &= \sum_{c \in \mathbb{C}} \int_{H_1} \mathcal{B}_\sigma(\sigma(h_1 h_c)\phi_1, \phi_2) \mathcal{B}_{\theta(\tau)}^\sharp(\theta(\tau)(h_1 h_c)\theta(\varphi_1, f_1), \theta(\varphi_2, f_2)) dh_1 \\ &= 2 \int_{Z_H \backslash H} \mathcal{B}_\sigma(\sigma(h)\phi_1, \phi_2) \mathcal{B}_{\theta(\tau)}^\sharp(\theta(\tau)(h)\theta(\varphi_1, f_1), \theta(\varphi_2, f_2)) dh. \end{aligned}$$

Also, we have

$$\begin{aligned} & \zeta(2)\zeta(4)L(1, \sigma, \text{std})^{-1} \cdot \mathcal{Q}(\varphi_1, \varphi_2; \phi_1, \phi_2; f_1, f_2) \\ &= \sum_{c \in \mathbb{C}} \int_{G'_1} \mathcal{B}_\pi^\sharp(\pi(g'_1 g'_c)\theta(\varphi_1, \phi_1), \theta(\varphi_2, \phi_2)) \mathcal{B}_{\pi'}^\flat(\pi'(g'_1 g'_c)f_1, f_2) dg'_1 \\ &= 2 \int_{Z_{G'} \backslash G'^+} \mathcal{B}_\pi^\sharp(\pi(g')\theta(\varphi_1, \phi_1), \theta(\varphi_2, \phi_2)) \mathcal{B}_{\pi'}^\flat(\pi'(g')f_1, f_2) dg', \end{aligned}$$

where $G'^+ = \{g' \in G' \mid \nu(g') \in \nu(H)\}$. By Lemma 5.2, this integral is equal to

$$2 \int_{Z_{G'} \backslash G'} \mathcal{B}_\pi^\sharp(\pi(g')\theta(\varphi_1, \phi_1), \theta(\varphi_2, \phi_2)) \mathcal{B}_{\pi'}^\flat(\pi'(g')f_1, f_2) dg'.$$

This completes the proof. □

10. Proof of Theorem 1.1

We retain the notation of §1 and §8. Let $\mathbb{W} = V \otimes_F W = \mathrm{R}_{E/F}(W_0 \otimes_E V_E)$, where $V_E = V \otimes_F E$. Let ω be the Weil representation of $\mathrm{Mp}(\mathbb{W}(\mathbb{A}))$ on the space V_ω with respect to ψ . We may regard ω as a representation of $G(\mathrm{O}(V) \times \mathrm{Sp}(W))(\mathbb{A})$ or that of $G(\mathrm{Sp}(W_0) \times \mathrm{O}(V_E))(\mathbb{A}_E)$.

We fix a subspace $\tilde{V}_{\pi'}$ of V_τ^1 such that the restriction to $G'(\mathbb{A})$ as functions induces an isomorphism

$$\tilde{V}_{\pi'} \cong V_{\pi'}$$

as representations of $G'(\mathbb{A})$. Let $\mathcal{B}_\tau : V_\tau \otimes \bar{V}_\tau \rightarrow \mathbb{C}$ and $\mathcal{B}_{\pi'} : V_{\pi'} \otimes \bar{V}_{\pi'} \rightarrow \mathbb{C}$ be the Petersson pairings. We fix a decomposition $\mathcal{B}_\tau = \prod_v \mathcal{B}_{\tau_v}$, where $\mathcal{B}_{\tau_v} : \tau_v \otimes \bar{\tau}_v \rightarrow \mathbb{C}$ is a pairing. Let $\mathcal{B}_{\pi'_v}^b : \pi'_v \otimes \bar{\pi}'_v \rightarrow \mathbb{C}$ be the pairing given by $\mathcal{B}_{\pi'_v}^b = \mathcal{B}_{\tau_v}|_{\pi'_v \otimes \bar{\pi}'_v}$.

Lemma 10.1. *We have*

$$\mathcal{B}_{\pi'} = 2^{\beta'} |\mathfrak{X}_\tau| \prod_v \mathcal{B}_{\pi'_v}^b.$$

Here

$$\beta' = \begin{cases} -1 & \text{if } E = F \times F, \\ 0 & \text{if } E \text{ is a quadratic extension of } F. \end{cases}$$

Proof. By [17, Remark 4.20], we have

$$\mathcal{B}_{\pi'}(\tilde{f}_1|_{G'(\mathbb{A})}, \tilde{f}_2|_{G'(\mathbb{A})}) = |\mathfrak{X}_\tau| \cdot \frac{\mathrm{vol}(Z_{G'}(\mathbb{A})G'(F)\backslash G'(\mathbb{A}))}{\mathrm{vol}(Z_{\tilde{G}'}(\mathbb{A})\tilde{G}'(F)\backslash \tilde{G}'(\mathbb{A}))} \cdot \mathcal{B}_\tau(\tilde{f}_1, \tilde{f}_2)$$

for $\tilde{f}_1, \tilde{f}_2 \in \tilde{V}_{\pi'}$. Here $Z_{G'}$ and $Z_{\tilde{G}'}$ are the identity components of the centers of G' and \tilde{G}' , respectively. Since $\mathrm{vol}(Z_{G'}(\mathbb{A})G'(F)\backslash G'(\mathbb{A})) = 2$ and

$$\mathrm{vol}(Z_{\tilde{G}'}(\mathbb{A})\tilde{G}'(F)\backslash \tilde{G}'(\mathbb{A})) = \begin{cases} 4 & \text{if } E = F \times F, \\ 2 & \text{if } E \text{ is a quadratic extension of } F, \end{cases}$$

the assertion follows. \square

Let $\theta(\tau)$ be the theta lift of τ to $H(\mathbb{A}_E)$ on the space $V_{\theta(\tau)}$. Let

$$\theta : V_\omega \otimes V_\tau \longrightarrow V_{\theta(\tau)}$$

and

$$\theta_v : \omega_v \otimes \tau_v \longrightarrow \theta(\tau_v)$$

be equivariant surjective maps such that $\theta = \otimes_v \theta_v$. Let $V_{\theta(\pi')}$ and $\theta(\pi'_v)$ be the images of $V_\omega \otimes \tilde{V}_{\pi'}$ and $\omega_v \otimes \pi'_v$ in $V_{\theta(\tau)}$ and $\theta(\tau_v)$, respectively. Let

$$\begin{aligned} \mathcal{T} : (V_\omega \boxtimes \bar{V}_\omega) \otimes (V_\sigma \boxtimes \bar{V}_\sigma) \otimes (\tilde{V}_{\pi'} \boxtimes \bar{\tilde{V}}_{\pi'}) &\longrightarrow (V_{\theta(\sigma)} \boxtimes \bar{V}_{\theta(\sigma)}) \otimes (\tilde{V}_{\pi'} \boxtimes \bar{\tilde{V}}_{\pi'}) \\ &\longrightarrow (V_{\theta(\sigma)} \boxtimes \bar{V}_{\theta(\sigma)}) \otimes (V_{\pi'} \boxtimes \bar{V}_{\pi'}), \end{aligned}$$

$$\mathcal{T}' : (V_\omega \boxtimes \bar{V}_\omega) \otimes (V_\sigma \boxtimes \bar{V}_\sigma) \otimes (\tilde{V}_{\pi'} \boxtimes \bar{\tilde{V}}_{\pi'}) \longrightarrow (V_\sigma \boxtimes \bar{V}_\sigma) \otimes (V_{\theta(\pi')} \boxtimes \bar{V}_{\theta(\pi')}),$$

be equivariant surjective maps induced by

$$\theta \otimes \bar{\theta} : (V_\omega \boxtimes \bar{V}_\omega) \otimes (V_\sigma \boxtimes \bar{V}_\sigma) \longrightarrow V_{\theta(\sigma)} \boxtimes \bar{V}_{\theta(\sigma)},$$

$$\theta \otimes \bar{\theta} : (V_\omega \boxtimes \bar{V}_\omega) \otimes (\tilde{V}_{\pi'} \boxtimes \bar{\tilde{V}}_{\pi'}) \longrightarrow V_{\theta(\pi')} \boxtimes \bar{V}_{\theta(\pi')},$$

respectively. Let

$$\mathcal{T}_v : (\omega_v \boxtimes \bar{\omega}_v) \otimes (\sigma_v \boxtimes \bar{\sigma}_v) \otimes (\pi'_v \boxtimes \bar{\pi}'_v) \longrightarrow (\theta(\sigma_v) \boxtimes \bar{\theta}(\sigma_v)) \otimes (\pi'_v \boxtimes \bar{\pi}'_v),$$

$$\mathcal{T}'_v : (\omega_v \boxtimes \bar{\omega}_v) \otimes (\sigma_v \boxtimes \bar{\sigma}_v) \otimes (\pi'_v \boxtimes \bar{\pi}'_v) \longrightarrow (\sigma_v \boxtimes \bar{\sigma}_v) \otimes (\theta(\pi'_v) \boxtimes \bar{\theta}(\pi'_v)),$$

be equivariant surjective maps such that $\mathcal{T} = \otimes_v \mathcal{T}_v$ and $\mathcal{T}' = \otimes_v \mathcal{T}'_v$. We define an $H(\mathbb{A}) \times H(\mathbb{A})$ -invariant functional

$$\mathcal{I} : (V_\sigma \boxtimes \bar{V}_\sigma) \otimes (V_{\theta(\tau)} \boxtimes \bar{V}_{\theta(\tau)}) \longrightarrow \mathbb{C}$$

by

$$\begin{aligned} & \mathcal{I}(\phi_1, \phi_2; f_1, f_2) \\ &= \left(\int_{Z_H(\mathbb{A})H(F)\backslash H(\mathbb{A})} \phi_1(h) f_1(h) dh \right) \cdot \left(\int_{Z_H(\mathbb{A})H(F)\backslash H(\mathbb{A})} \overline{\phi_2(h) f_2(h)} dh \right) \end{aligned}$$

for $\phi_1, \phi_2 \in V_\sigma$ and $f_1, f_2 \in V_{\theta(\tau)}$. Here $dh = \prod_v dh_v$ is the Tamagawa measure on $Z_H(\mathbb{A})\backslash H(\mathbb{A})$. Put

$$\mathfrak{v} = \text{vol}(Z_H(\mathbb{A})H(F)\backslash H(\mathbb{A})) = \begin{cases} 2 & \text{if } K = F \times F, \\ 1 & \text{if } K \text{ is a quadratic extension of } F. \end{cases}$$

Now we have the global seesaw identity.

Lemma 10.2. *We have*

$$\mathcal{P} \circ \mathcal{T} = \mathcal{I} \circ \mathcal{T}'$$

as functionals on $(V_\omega \boxtimes \bar{V}_\omega) \otimes (V_\sigma \boxtimes \bar{V}_\sigma) \otimes (\tilde{V}_{\pi'} \boxtimes \bar{\tilde{V}}_{\pi'})$.

Proof. Let $\mathcal{C} = \mathbb{A}^{\times, 2} F^{\times, +} \backslash \mathbb{A}^{\times, +}$. Put

$$\mathfrak{Q}(\varphi, \phi, \tilde{f}) = \int_{\mathcal{C}} \int_{G'_1(F)\backslash G'_1(\mathbb{A})} \int_{H_1(F)\backslash H_1(\mathbb{A})} \theta(g'_1 g'_c, h_1 h_c; \varphi) \phi(h_1 h_c) \tilde{f}(g'_1 g'_c) dh_1 dg'_1 dc$$

for $\varphi \in V_\omega$, $\phi \in V_\sigma$, and $\tilde{f} \in \tilde{V}_{\pi'}$. Here dc is the Haar measure on \mathcal{C} such that $\text{vol}(\mathcal{C}) = 1$, dg'_1 is the Tamagawa measure on $G'_1(\mathbb{A})$, and dh_1 is the Tamagawa measure on $H_1(\mathbb{A})$. We have

$$\begin{aligned} \mathfrak{Q}(\varphi, \phi, \tilde{f}) &= \int_{\mathcal{C}} \int_{H_1(F)\backslash H_1(\mathbb{A})} \theta(h_1 h_c; \varphi, \tilde{f}) \phi(h_1 h_c) dh_1 dc \\ &= \frac{1}{\mathfrak{v}} \int_{Z_H(\mathbb{A})H(F)\backslash H(\mathbb{A})} \theta(h; \varphi, \tilde{f}) \phi(h) dh. \end{aligned}$$

Also, we have

$$\begin{aligned} \mathfrak{Q}(\varphi, \phi, \tilde{f}) &= \int_{\mathcal{C}} \int_{G'_1(F)\backslash G'_1(\mathbb{A})} \theta(g'_1 g'_c; \varphi, \phi) f(g'_1 g'_c) dg'_1 dc \\ &= \frac{1}{\mathfrak{v}} \int_{Z_{G'}(\mathbb{A})G'(F)^+\backslash G'(\mathbb{A})^+} \theta(g'; \varphi, \phi) f(g') dg', \end{aligned}$$

where $f = \tilde{f}|_{G'(\mathbb{A})}$. Note that $\text{vol}(Z_{G'}(\mathbb{A})G'(F)^+\backslash G'(\mathbb{A})^+) = \mathfrak{v}$. Since the support of $\theta(\varphi, \phi)$ is contained in $G(F)G(\mathbb{A})^+$, we have

$$\int_{Z_{G'}(\mathbb{A})G'(F)^+\backslash G'(\mathbb{A})^+} \theta(g'; \varphi, \phi) f(g') dg' = \int_{Z_{G'}(\mathbb{A})G'(F)\backslash G'(\mathbb{A})} \theta(g'; \varphi, \phi) f(g') dg'.$$

This completes the proof. \square

Let $\mathcal{B}_\sigma : V_\sigma \otimes \bar{V}_\sigma \rightarrow \mathbb{C}$ and $\mathcal{B}_{\theta(\tau)} : V_{\theta(\tau)} \otimes \bar{V}_{\theta(\tau)} \rightarrow \mathbb{C}$ be the Petersson pairings. We fix decompositions $\mathcal{B}_\sigma = \prod_v \mathcal{B}_{\sigma_v}$ and $\mathcal{B}_{\theta(\tau)} = \prod_v \mathcal{B}_{\theta(\tau_v)}$, where $\mathcal{B}_{\sigma_v} : \sigma_v \otimes \bar{\sigma}_v \rightarrow \mathbb{C}$ and $\mathcal{B}_{\theta(\tau_v)} : \theta(\tau_v) \otimes \overline{\theta(\tau_v)} \rightarrow \mathbb{C}$ are pairings. For each place v of F , we define an $H_v \times H_v$ -invariant functional

$$\mathcal{I}_v^\natural : (\sigma_v \boxtimes \bar{\sigma}_v) \otimes (\theta(\tau_v) \boxtimes \overline{\theta(\tau_v)}) \longrightarrow \mathbb{C}$$

by

$$\mathcal{I}_v^\natural(\phi_{1,v}, \phi_{2,v}; f_{1,v}, f_{2,v})$$

$$= \int_{Z_{H,v} \backslash H_v} \mathcal{B}_{\sigma_v}(\sigma_v(h_v)\phi_{1,v}, \phi_{2,v}) \mathcal{B}_{\theta(\tau_v)}(\theta(\tau_v)(h_v)f_{1,v}, f_{2,v}) dh_v$$

for $\phi_{1,v}, \phi_{2,v} \in \sigma_v$ and $f_{1,v}, f_{2,v} \in \theta(\tau_v)$. By Proposition 3.2, we have

$$\mathcal{I} = 2^c \cdot \zeta_{E \otimes K}(2) \cdot \frac{L(\frac{1}{2}, \sigma \times \theta(\tau))}{L(1, \sigma, \mathrm{Ad})L(1, \tau_K, \mathrm{Ad})} \cdot \prod_v \mathcal{I}_v.$$

Here

$$c = \begin{cases} -4 & \text{if } E = K = F \times F, \\ -1 & \text{if } E = F \times F \text{ and } K \text{ is a quadratic extension of } F, \\ -3 & \text{if } E \text{ is a quadratic extension of } F \text{ and } K = F \times F, \\ -2 & \text{if } E \text{ and } K \text{ are quadratic extensions of } F \text{ and } E = K, \\ -1 & \text{if } E \text{ and } K \text{ are quadratic extensions of } F \text{ and } E \neq K, \end{cases}$$

and

$$\mathcal{I}_v = \frac{1}{\zeta_{E_v \otimes K_v}(2)} \cdot \frac{L_v(1, \sigma_v, \mathrm{Ad})L_v(1, \tau_{K,v}, \mathrm{Ad})}{L_v(\frac{1}{2}, \sigma_v \times \theta(\tau_v))} \cdot \mathcal{I}_v^\sharp.$$

By Lemma 10.2, we have

$$\mathcal{P} \circ \mathcal{T} = 2^c \cdot \zeta_{E \otimes K}(2) \cdot \frac{L(\frac{1}{2}, \sigma \times \theta(\tau))}{L(1, \sigma, \mathrm{Ad})L(1, \tau_K, \mathrm{Ad})} \cdot \prod_v \mathcal{I}_v \circ \mathcal{T}'_v.$$

Let $\mathcal{B}_{\theta(\tau_v)}^\sharp : \theta(\tau_v) \otimes \overline{\theta(\tau_v)} \rightarrow \mathbb{C}$ be the pairing defined by (9.2) and

$$\mathcal{I}_v^\sharp : (\sigma_v \boxtimes \bar{\sigma}_v) \otimes (\theta(\tau_v) \boxtimes \overline{\theta(\tau_v)}) \longrightarrow \mathbb{C}$$

the $H_v \times H_v$ -invariant functional defined by (9.4). By Proposition 6.10, we have

$$\mathcal{B}_{\theta(\tau)} = 2^{\beta''} \cdot \frac{L(1, \tau, \mathrm{Ad} \otimes \omega_{E \otimes K/E})}{\zeta_{E \otimes K}(2)} \cdot \prod_v \mathcal{B}_{\theta(\tau_v)}^\sharp.$$

Here

$$\beta'' = \begin{cases} 0 & \text{if } E = K = F \times F, \\ -2 & \text{if } E = F \times F \text{ and } K \text{ is a quadratic extension of } F, \\ 0 & \text{if } E \text{ is a quadratic extension of } F \text{ and } K = F \times F, \\ 0 & \text{if } E \text{ and } K \text{ are quadratic extensions of } F \text{ and } E = K, \\ -1 & \text{if } E \text{ and } K \text{ are quadratic extensions of } F \text{ and } E \neq K. \end{cases}$$

Hence we have

$$\mathcal{P} \circ \mathcal{T} = 2^{\beta''+c} \cdot \frac{L(\frac{1}{2}, \sigma \times \theta(\tau))}{L(1, \sigma, \mathrm{Ad})L(1, \tau, \mathrm{Ad})} \cdot \prod_v \mathcal{I}_v^\sharp \circ \mathcal{T}'_v.$$

Let $\mathcal{B}_{\pi_v}^\sharp : \pi_v \otimes \bar{\pi}_v \rightarrow \mathbb{C}$ be the pairing defined by (9.1) and

$$\mathcal{P}_v^\sharp : (\pi_v \boxtimes \bar{\pi}_v) \otimes (\pi'_v \boxtimes \bar{\pi}'_v) \longrightarrow \mathbb{C}$$

the $G'_v \times G'_v$ -invariant functional defined by (9.3). By Lemma 9.2, we have

$$\mathcal{P} \circ \mathcal{T} = 2^{\beta''+c} \cdot \frac{L(\frac{1}{2}, \sigma \times \theta(\tau))}{L(1, \sigma, \mathrm{Ad})L(1, \tau, \mathrm{Ad})} \cdot \prod_v \mathcal{P}_v^\sharp \circ \mathcal{T}'_v.$$

By Proposition 7.13 and Lemma 10.1, we have

$$\mathcal{P} \circ \mathcal{T} = \frac{2^{\beta''+c}}{2^{\beta'} |\mathfrak{X}_\tau|} \cdot \frac{L(\frac{1}{2}, \sigma \times \theta(\tau))}{L(1, \sigma, \mathrm{Ad})L(1, \tau, \mathrm{Ad})} \cdot \frac{\zeta(2)\zeta(4)}{L(1, \sigma, \mathrm{std})} \cdot \prod_v \mathcal{P}_v \circ \mathcal{T}'_v.$$

This shows the desired identity of invariant functionals on the image of \mathcal{T} , which is the subspace $(V_{\theta(\sigma)} \boxtimes \bar{V}_{\theta(\sigma)}) \otimes (V_{\pi'} \boxtimes \bar{V}_{\pi'})$ of $(V_{\pi} \boxtimes \bar{V}_{\pi}) \otimes (V_{\pi'} \boxtimes \bar{V}_{\pi'})$. Since $G(\mathbb{A}) = G'(\mathbb{A})G(\mathbb{A})^+$ and V_{π} is generated by $V_{\theta(\sigma)}$, we have

$$\mathcal{P} = \frac{\zeta(2)\zeta(4)}{2^{\beta' - \beta'' - c} |\mathfrak{X}_{\tau}|} \cdot \frac{L(\frac{1}{2}, \sigma \times \theta(\tau))}{L(1, \sigma, \text{std})L(1, \sigma, \text{Ad})L(1, \tau, \text{Ad})} \cdot \prod_v \mathcal{P}_v.$$

This completes the proof of Theorem 1.1.

Appendix A. Explicit local theta correspondence for $\text{GO}(V) \times \text{GSp}_4$

In this appendix, let F be a non-archimedean local field of characteristic zero and residual characteristic p . We consider an arbitrary four dimensional quadratic space V and a four dimensional symplectic space W over F . The discriminant algebra of V is an étale quadratic F -algebra K . Let ν_V and ν_W denote the similitude characters of the corresponding similitude groups $\text{GO}(V)$ and $\text{GSp}(W)$, respectively. The image of ν_V is the subgroup $\text{N}_{K/F}(K^\times) \subset F^\times$ and we set

$$\text{GSp}(W)^+ = \{g \in \text{GSp}(W) \mid \nu_W(g) \in \text{N}_{K/F}(K^\times)\}.$$

For a non-trivial additive character ψ of F , one has an induced Weil representation Ω_ψ of the similitude dual pair $\text{GO}(V) \times \text{GSp}(W)^+$. If K is split, then Ω_ψ is independent of ψ , and in general, Ω_ψ depends only on the orbit of ψ under the natural action of $\text{N}_{K/F}(K^\times)$. Thus, when K is a field, there are two such induced Weil representations. Henceforth, we shall fix an orbit of ψ and write Ω for Ω_ψ , suppressing ψ from the notation. However, because we shall be dealing with various different dual pairs, we shall sometimes write $\Omega_{V,W}$ to indicate the particular dual pair we are considering.

If σ is an irreducible representation of $\text{GO}(V)$, then the maximal σ -isotypic quotient of Ω is of the form $\sigma \boxtimes \Theta(\sigma)$ for some smooth representation $\Theta(\sigma)$ (the big theta lift of σ) of $\text{GSp}(W)^+$. One knows that $\Theta(\sigma)$ is a representation of finite length. We let $\theta(\sigma)$ (the small theta lift of σ) denote the maximal semisimple quotient of $\Theta(\sigma)$. Moreover, we set

$$\tilde{\Theta}(\sigma) = \text{ind}_{\text{GSp}(W)^+}^{\text{GSp}(W)}(\Theta(\sigma)) \quad \text{and} \quad \tilde{\theta}(\sigma) = \text{ind}_{\text{GSp}(W)^+}^{\text{GSp}(W)}(\theta(\sigma)).$$

Again, we shall sometimes write $\Theta_{V,W}(\sigma)$, $\theta_{V,W}(\sigma)$ and so on if there is a need to be specific about the dual pair we are considering.

We should remark that the definition of the induced Weil representation Ω used in this appendix is as given in [9], which is slightly different from that given in [47] (and used in the main body of this paper). Moreover, our definition of $\Theta(\sigma)$ is slightly different from that given in the main body of this paper. The upshot is that these two changes cancel each other, so that the local theta correspondence defined in this appendix agrees with that defined in the main body of this paper. In particular, the local theta correspondence preserves central characters.

The main result of this appendix is:

Theorem A.1. *Let σ be an irreducible representation of $\text{GO}(V)$.*

- (i) $\Theta(\sigma)$ is multiplicity free (possibly zero) and has a unique irreducible quotient $\theta(\sigma)$.
- (ii) $\theta(\sigma)$ can be precisely determined in terms of σ (in terms of the local Langlands correspondence for GSp_4 established in [9]).

For the purpose of this paper, we really only need part (i) of the theorem, but we find it useful to include part (ii) as well. Part (ii) of the theorem will be stated in full details in the respective cases later on. In order to do that, we first need to introduce some notations for representations of $\text{GO}(V)$ and $\text{GSp}(W)^+$.

A.1. Principal series representations of GSp_4 . We have a Witt decomposition $W = Y^* \oplus Y$ with a two dimensional isotropic space Y . We can write

$$Y^* = F \cdot e_1 \oplus F \cdot e_2 \quad \text{and} \quad Y = F \cdot f_1 \oplus F \cdot f_2$$

with $\langle e_i, f_j \rangle = \delta_{ij}$ and consider the decomposition $W = F \cdot e_1 \oplus W' \oplus F \cdot f_1$, where $W' = F \cdot e_2 \oplus F \cdot f_2$. Let $Q(Z) = L(Z) \cdot U(Z)$ be the parabolic subgroup stabilizing $Z = F \cdot f_1$, so that

$$L(Z) \cong \mathrm{GL}(Z) \times \mathrm{GSp}(W')$$

and $U(Z)$ is a Heisenberg group:

$$1 \longrightarrow \mathrm{Sym}^2 Z \longrightarrow U(Z) \longrightarrow \mathrm{Hom}(W', Z) \longrightarrow 1$$

This is typically called the Klingen or Heisenberg parabolic subgroup. An irreducible representation of $L(Z)$ is thus of the form $\chi \boxtimes \tau$ with a character χ of F^\times and an irreducible representation τ of $\mathrm{GSp}(W') \cong \mathrm{GL}_2$. We denote the corresponding normalized induced representation by $I_{Q(Z)}(\chi, \tau)$. If $I_{Q(Z)}(\chi, \tau)$ is a standard module, then it has a unique irreducible quotient (the Langlands quotient), which we shall denote by $J_{Q(Z)}(\chi, \tau)$. The same notation applies to other principal series representations to be introduced later.

The module structure of $I_{Q(Z)}(\chi, \tau)$ is known by Sally and Tadić [51] and a convenient reference is [50]. In particular, we note the following:

Lemma A.2. (a) *Suppose that τ is a supercuspidal representation of GL_2 . Then $I_{Q(Z)}(\chi, \tau)$ is reducible if and only if one of the following holds:*

(i) $\chi = 1$;

(ii) $\chi = \chi_0 | \cdot |^{\pm 1}$ with a non-trivial quadratic character χ_0 such that $\tau \cdot \chi_0 = \tau$.

In case (i), $I_{Q(Z)}(1, \tau)$ is the direct sum of two irreducible quasi-tempered representations, exactly one of which is generic. In case (ii), assuming without loss of generality that $\chi = \chi_0 | \cdot |$, one has a non-split short exact sequence:

$$0 \longrightarrow \mathrm{St}(\chi_0, \tau_0) \longrightarrow I_{Q(Z)}(\chi_0 | \cdot |, \tau_0 | \cdot |^{-1/2}) \longrightarrow \mathrm{Sp}(\chi_0, \tau_0) \longrightarrow 0$$

Here $\mathrm{St}(\chi_0, \tau_0)$ is a generic discrete series representation and the Langlands quotient $\mathrm{Sp}(\chi_0, \tau_0)$ is non-generic.

(b) *Suppose that τ is a twisted Steinberg representation of GL_2 . Then $I_{Q(Z)}(\chi, \tau)$ is reducible if and only if one of the following holds:*

(i) $\chi = 1$;

(ii) $\chi = | \cdot |^{\pm 2}$.

In case (i), $I_{Q(Z)}(1, \mathrm{st}_\mu)$ is the direct sum of two irreducible quasi-tempered representations, exactly one of which is generic. In case (ii), $I_{Q(Z)}(| \cdot |^2, \mathrm{st}_\mu | \cdot |^{-1})$ has the twisted Steinberg representation $\mathrm{St}_{\mathrm{PGSp}_4} \cdot \mu$ as its unique irreducible submodule.

(c) *There is a standard intertwining operator*

$$I_{Q(Z)}(\chi^{-1}, \tau \cdot \chi) \longrightarrow I_{Q(Z)}(\chi, \tau),$$

which is an isomorphism if $I_{Q(Z)}(\chi, \tau)$ is irreducible. If $I_{Q(Z)}(\chi^{-1}, \tau \cdot \chi)$ is a standard module, then the image of this operator is the unique irreducible submodule of $I_{Q(Z)}(\chi, \tau)$.

Let $P(Y) = M(Y) \cdot N(Y)$ be the Siegel parabolic subgroup stabilizing Y , so that

$$M(Y) \cong \mathrm{GL}(Y) \times \mathbb{G}_m \quad \text{and} \quad N(Y) \cong \mathrm{Sym}^2 Y.$$

An irreducible representation of $M(Y)$ is thus of the form $\tau \boxtimes \mu$ with an irreducible representation τ of $\mathrm{GL}(Y) \cong \mathrm{GL}_2$ and a character μ of F^\times . We denote the corresponding normalized induced representation by $I_{P(Y)}(\tau, \mu)$. As before, the module structure of $I_{P(Y)}(\tau, \mu)$ is completely known by [51] and a convenient reference is [50]. In particular, we note the following:

Lemma A.3. (a) *Suppose that τ is a supercuspidal representation of GL_2 . Then $I_{P(Y)}(\tau, \mu)$ is reducible if and only if $\tau = \tau_0| \cdot |^{\pm 1/2}$ with τ_0 having trivial central character. In this case, one has a non-split short exact sequence:*

$$0 \longrightarrow \mathrm{St}(\tau_0, \mu_0) \longrightarrow I_{P(Y)}(\tau_0| \cdot |^{1/2}, \mu_0| \cdot |^{-1/2}) \longrightarrow \mathrm{Sp}(\tau_0, \mu_0) \longrightarrow 0$$

Here $\mathrm{St}(\tau_0, \mu_0)$ is a generic discrete series representation and the Langlands quotient $\mathrm{Sp}(\tau_0, \mu_0)$ is non-generic.

(b) *Suppose that τ is a twisted Steinberg representation of GL_2 . Then $I_{P(Y)}(\tau, \mu)$ is reducible if and only if one of the following holds:*

- (i) $\tau = \mathrm{st}| \cdot |^{\pm 1/2}$; in this case, $I_{P(Y)}(\mathrm{st}| \cdot |^{1/2}, \mu| \cdot |^{-1/2})$ has a unique irreducible Langlands quotient and a unique irreducible quasi-tempered submodule, which is the unique generic constituent of $I_{Q(Z)}(1, \mathrm{st}_\mu)$.
- (ii) $\tau = \mathrm{st}_\chi| \cdot |^{\pm 1/2}$ with a non-trivial quadratic character χ ; in this case, $I_{P(Y)}(\mathrm{st}_\chi| \cdot |^{1/2}, \mu_0| \cdot |^{-1/2})$ has a unique irreducible Langlands quotient and a unique irreducible submodule, which is a generic discrete series representation $\mathrm{St}(\mathrm{st}_\chi, \mu_0)$. Moreover, $\mathrm{St}(\mathrm{st}_\chi, \chi\mu_0) = \mathrm{St}(\mathrm{st}_\chi, \mu_0)$.
- (iii) $\tau = \mathrm{st}| \cdot |^{\pm 3/2}$; in this case, $I_{P(Y)}(\mathrm{st}| \cdot |^{3/2}, \mu| \cdot |^{-3/2})$ has the twisted Steinberg representation $\mathrm{St}_{\mathrm{PGSp}_4} \cdot \mu$ as its unique irreducible submodule.

(c) *There is a standard intertwining operator*

$$I_{P(Y)}(\tau, \mu) \longrightarrow I_{P(Y)}(\tau^\vee, \mu\omega_\tau),$$

which is an isomorphism if $I_{P(Y)}(\tau, \mu)$ is irreducible. If $I_{P(Y)}(\tau, \mu)$ is a standard module, then the image of this operator is the unique irreducible submodule of $I_{P(Y)}(\tau^\vee, \mu\omega_\tau)$.

Finally, let $B = P(Y) \cap Q(Z) = T \cdot U$ be a Borel subgroup of $\mathrm{GSp}(W)$, so that

$$T \cong (\mathrm{GL}(F \cdot f_1) \times \mathrm{GL}(F \cdot f_2)) \times \mathbb{G}_m.$$

In particular, for characters χ_1, χ_2 , and χ of F^\times , we let $I_B(\chi_1, \chi_2; \chi)$ denote the normalized induced representation. Again, we refer the reader to [50] for the reducibility points and module structure of $I_B(\chi_1, \chi_2; \chi)$. We simply note here that $I_B(\chi_1, \chi_2; \chi)$ is multiplicity free, and if χ_1 and χ_2 are unitary, then $I_B(\chi_1, \chi_2; \chi)$ is irreducible.

A.2. Representations of $\mathrm{GO}(V)$. Now we come to representations of $\mathrm{GO}(V)$. We consider the various cases separately:

K is split.

In this case, we have

$$V = (D, -N_D)$$

where D is a quaternion F -algebra (possibly split) with reduced norm N_D . We have the identification

$$\mathrm{GSO}(V) \cong (D^\times \times D^\times) / \{(z, z^{-1}) \mid z \in F^\times\}$$

via

$$(g_1, g_2) : x \longmapsto g_1 \cdot x \cdot \bar{g}_2.$$

Moreover, the main involution $x \mapsto \bar{x}$ on D gives an order two element \mathfrak{t} of $O(V)$ with determinant -1 , so that $\mathrm{GO}(V) = \mathrm{GSO}(V) \rtimes \langle \mathfrak{t} \rangle$. The conjugation of \mathfrak{t} on $\mathrm{GSO}(V)$ is given by

$$(g_1, g_2) \longmapsto (g_2, g_1).$$

Thus, an irreducible representation of $\mathrm{GSO}(V)$ is of the form $\tau_1 \boxtimes \tau_2$ with an irreducible representation τ_i of D^\times such that $\omega_{\tau_1} = \omega_{\tau_2}$. Moreover, the action of \mathfrak{t} sends $\tau_1 \boxtimes \tau_2$ to $\tau_2 \boxtimes \tau_1$.

In particular, if $\tau_1 = \tau_2 = \tau$, then there are two extensions of $\tau \boxtimes \tau$ to $\mathrm{GO}(V)$, which we denote by $(\tau \boxtimes \tau)^\pm$. To distinguish these two extensions, we note that exactly one of them participates in the theta correspondence with $\mathrm{GSp}(W') \cong \mathrm{GL}_2$, and we denote this distinguished extension by $(\tau \boxtimes \tau)^+$.

On the other hand, if $\tau_1 \neq \tau_2$, then

$$\mathrm{ind}_{\mathrm{GSO}(V)}^{\mathrm{GO}(V)}(\tau_1 \boxtimes \tau_2) = \mathrm{ind}_{\mathrm{GSO}(V)}^{\mathrm{GO}(V)}(\tau_2 \boxtimes \tau_1)$$

is irreducible, in which case we denote this irreducible representation by $(\tau_1 \boxtimes \tau_2)^+ = (\tau_1 \boxtimes \tau_2)^-$.

When D is split, the quadratic space V is split and we have a Witt decomposition $V = X \oplus X^*$ with a two dimensional isotropic space X . Let $P(X) = M(X) \cdot N(X)$ be the parabolic subgroup stabilizing X , so that

$$M(X) \cong \mathrm{GL}(X) \times \mathbb{G}_m \quad \text{and} \quad N(X) \cong \wedge^2 X.$$

For an irreducible representation $\tau \boxtimes \chi$ of $\mathrm{GL}(X) \times \mathbb{G}_m \cong \mathrm{GL}_2 \times F^\times$, we let $I_{P(X)}(\tau, \chi)$ denote the normalized induced representation. The following lemma is easy to check.

Lemma A.4. *Under the identification $\mathrm{GSO}(V) \cong (\mathrm{GL}_2 \times \mathrm{GL}_2)/F^\times$, we have*

$$\pi(\chi_1, \chi_2) \boxtimes \tau = I_{P(X)}(\tau^\vee \cdot \chi_1, \chi_2) = I_{P(X)}(\tau \cdot \chi_2^{-1}, \chi_2).$$

K is a field.

In this case, we have two quadratic spaces

$$V^+ = \mathbb{H} \oplus V_K^+ \quad \text{and} \quad V^- = \mathbb{H} \oplus V_K^-$$

where \mathbb{H} is the hyperbolic plane and

$$V_K^+ = (K, N_{K/F}) \quad \text{and} \quad V_K^- = (K, \delta \cdot N_{K/F})$$

with $\delta \in F^\times \setminus N_{K/F}(K^\times)$. One can realize these quadratic spaces on the space V of 2×2 Hermitian matrices with entries in K . The determinant map defines a quadratic form on V and we have

$$V^+ = (V, -\det) \quad \text{and} \quad V^- = (V, -\delta \cdot \det).$$

The similitude groups of $V = V^\pm$ are isomorphic:

$$\mathrm{GO}(V) = \mathrm{GSO}(V) \rtimes \langle \mathfrak{t} \rangle$$

with

$$\mathrm{GSO}(V) \cong (\mathrm{GL}_2(K) \times F^\times) / \{(z, N_{K/F}(z)^{-1}) \mid z \in K^\times\}$$

acting via

$$(g, \lambda) : x \longmapsto \lambda \cdot g \cdot x \cdot {}^t g^c,$$

and $\mathfrak{t} \in O(V)$ with determinant -1 acting via

$$\mathfrak{t} : x \longmapsto x^c$$

where c is the non-trivial element of $\mathrm{Gal}(K/F)$. Without loss of generality, we shall henceforth fix

$$V = V^+.$$

The conjugation of \mathfrak{t} on $\mathrm{GSO}(V)$ is given by

$$(g, \lambda) \longmapsto (g^c, \lambda).$$

Moreover, we let

$$\text{sgn} : \text{GO}(V) \longrightarrow \{\pm 1\}$$

be the unique non-trivial quadratic character of $\text{GO}(V)$ trivial on $\text{GSO}(V)$.

Thus, an irreducible representation of $\text{GSO}(V)$ is of the form $\tau \boxtimes \chi$ with an irreducible representation τ of $\text{GL}_2(K)$ and a character χ of F^\times such that $\omega_\tau = \chi \circ \mathbf{N}_{K/F}$. Such a representation is invariant under the action of \mathfrak{t} if and only if τ is obtained by base change from $\text{GL}_2(F)$, in which case there are two extensions of $\tau \boxtimes \chi$ to $\text{GO}(V)$, which we denote by $(\tau \boxtimes \chi)^\pm$. How can one distinguish between these two extensions of $\tau \boxtimes \chi$? As we now explain, one can do this using the Whittaker model when τ is generic.

More precisely, if U_0 is the unipotent radical of a \mathfrak{t} -stable Borel subgroup of $\text{GSO}(V)$, let ψ_0 be a generic character of U_0 which is fixed by the action of the outer automorphism \mathfrak{t} . Then, if τ is invariant and generic, \mathfrak{t} acts on the one dimensional space $\text{Hom}_{U_0}((\tau \boxtimes \chi)^\pm, \mathbb{C}_{\psi_0})$ with $\mathfrak{t}^2 = 1$. Then $(\tau \boxtimes \chi)^+$ is the extension of $\tau \boxtimes \chi$ such that \mathfrak{t} acts by $+1$ on $\text{Hom}_{U_0}((\tau \boxtimes \chi)^+, \mathbb{C}_{\psi_0})$. Note that this characterization is independent of the choice of the generic character ψ_0 which is fixed by \mathfrak{t} .

There is another way of specifying the two extensions of $\tau \boxtimes \chi$ in the invariant case, using their behaviour under the theta correspondence. Following Roberts [49], we distinguish two mutually exclusive scenarios in the invariant case.

- **Invariant and distinguished representations:** these are the representations $\tau \boxtimes \chi$, where τ is the base change of an irreducible representation τ_F of $\text{GL}_2(F)$ with central character $\chi \omega_{K/F}$. In this case, by [49, Theorem 3.4], one of the extensions $(\tau \boxtimes \chi)^+$ of $\tau \boxtimes \chi$ to $\text{GO}(V)$ participates in the theta correspondence with $\text{GSp}(W')^+ \cong \text{GL}_2^+$ (and hence with GSp_4^+), whereas the other extension $(\tau \boxtimes \chi)^-$ does not participate in the theta correspondence with GSp_4^+ . When τ is generic, it follows from Corollary A.17 below that the extension $(\tau \boxtimes \chi)^+$ is the same as the one defined above using the Whittaker model.
- **Invariant but not distinguished representations:** these are the remaining invariant representations $\tau \boxtimes \chi$. In this case, [49, Theorem 3.4] says that neither of the two extensions $(\tau \boxtimes \chi)^\pm$ participates in the theta correspondence with $\text{GSp}(W')^+ \cong \text{GL}_2^+$, but both participate in the theta correspondence with GSp_4^+ . Thus, the theta correspondence does not allow one to distinguish between the two extensions. When τ is generic, it follows from Corollary A.17 below that the theta lift of exactly one of the extensions, namely the extension $(\tau \boxtimes \chi)^+$ defined above, is a generic representation of GSp_4^+ .

We should remark that Roberts' definition of distinguished representations uses the existence of $\text{SO}(2, 1)$ -invariant functionals. In [48], he showed that his definition agrees with the one above when the residual characteristic of F is $p \neq 2$. It is not difficult to prove the same assertion for all p by computing the theta correspondence for $\text{GL}_2^+ \times \text{GSO}(V)$ and the Whittaker modules of the induced Weil representations.

On the other hand, if $\tau \boxtimes \chi$ is not invariant, then

$$\text{ind}_{\text{GSO}(V)}^{\text{GO}(V)}(\tau \boxtimes \chi) = \text{ind}_{\text{GSO}(V)}^{\text{GO}(V)}(\tau^c \boxtimes \chi)$$

is irreducible, in which case we denote this irreducible representation by $(\tau \boxtimes \chi)^+ = (\tau \boxtimes \chi)^-$.

Now we describe principal series representations of $\text{GO}(V)$. We have a Witt decomposition $V = J \oplus V_K \oplus J^*$ with an isotropic line J . Let $Q(J) = L(J) \cdot U(J)$ be the parabolic subgroup stabilizing J , so that

$$L(J) \cong \text{GL}(J) \times \text{GO}(V_K) \quad \text{and} \quad U(J) \cong \text{Hom}(V_K, J).$$

We set $Q(J)^+ = Q(J) \cap \text{GSO}(V)$.

Let $\chi \boxtimes \mu$ be an irreducible representation of $\text{GL}(J) \times \text{GSO}(V_K) \cong F^\times \times K^\times$. If μ is invariant under the Galois action, then μ has two extensions μ^\pm to $\text{GO}(V_K)$, whereas if μ is not invariant under the Galois

action, then we set

$$\mu^+ = \mu^- = \mathrm{ind}_{\mathrm{GSO}(V_K)}^{\mathrm{GO}(V_K)}(\mu).$$

We consider the normalized induced representations $I_{Q(J)}(\chi, \mu^\pm)$ and $I_{Q(J)^+}(\chi, \mu)$ of $\mathrm{GO}(V)$ and $\mathrm{GSO}(V)$, respectively. If we take J to be the isotropic line spanned by the matrix $\mathrm{diag}(1, 0) \in V$, then the following lemma is easy to check.

Lemma A.5. *Under the identification $\mathrm{GSO}(V) \cong (\mathrm{GL}_2(K) \times F^\times)/K^\times$, we have*

$$I_{Q(J)}(\chi, \mu^\pm) = (\pi((\chi \circ \mathrm{N}_{K/F}) \cdot \mu, \mu^c) \boxtimes (\chi \cdot \mu|_{F^\times}))^\pm.$$

From this lemma, it is not difficult to deduce the following:

Lemma A.6. (i) $I_{Q(J)}(\chi, \mu^\pm)$ is reducible if and only if one of the following holds:

- $\chi \circ \mathrm{N}_{K/F} = \mu^c / \mu \cdot | \cdot |_K^{\pm 1}$; in this case, $I_{Q(J)^+}(\chi, \mu)$ is reducible.
- $\mu^c \neq \mu$ and $\chi = 1$ or $\omega_{K/F}$; in this case, $I_{Q(J)^+}(\chi, \mu)$ is irreducible, but

$$I_{Q(J)}(\chi, \mu^\pm) = \sigma \oplus \sigma \cdot \mathrm{sgn}$$

for some irreducible representation σ of $\mathrm{GO}(V)$.

(ii) $I_{Q(J)^+}(\chi, \mu)$ is invariant if and only if one of the following mutually exclusive conditions holds:

- $\mu^c = \mu$;
- $\mu^c \neq \mu$ and $\chi = 1$ or $\omega_{K/F}$.

In this case, it is distinguished unless $\mu^c \neq \mu$ and $\chi = \omega_{K/F}$.

A.3. Theta lifts from $\mathrm{GO}(V_K)$. Before coming to our main results, let us recall the theta lifts from $\mathrm{GO}(V_K)$ to $\mathrm{GSp}(W') \cong \mathrm{GL}_2$ and $\mathrm{GSp}(W) \cong \mathrm{GSp}_4$. The following proposition is well-known:

Proposition A.7. *Let μ be an irreducible representation of $\mathrm{GSO}(V_K) \cong K^\times$.*

(i) *If μ is not Galois invariant (so that $\mu^+ = \mu^-$), then*

$$\Theta(\mu^\pm) = \theta(\mu^\pm)$$

is a non-zero irreducible supercuspidal representation of GL_2^+ such that

$$\pi(\mu) := \mathrm{ind}_{\mathrm{GL}_2^+}^{\mathrm{GL}_2}(\Theta(\mu^\pm))$$

is irreducible supercuspidal. These are precisely the supercuspidal representations of GL_2 which are dihedral with respect to K .

(ii) *If μ is Galois invariant so that $\mu = \mu_F \circ \mathrm{N}_{K/F}$ for some μ_F , then*

$$\Theta(\mu^+) = \theta(\mu^+)$$

is a non-zero irreducible representation of GL_2^+ such that

$$\pi(\mu) := \mathrm{ind}_{\mathrm{GL}_2^+}^{\mathrm{GL}_2}(\Theta(\mu^+)) = \pi(\mu_F, \mu_F \omega_{K/F}).$$

Moreover,

$$\Theta(\mu^-) = 0.$$

Proposition A.8. *Let μ be an irreducible representation of $\mathrm{GSO}(V_K) \cong K^\times$.*

(i) If μ is not Galois invariant (so that $\mu^+ = \mu^-$), then

$$\Theta(\mu^\pm) = \theta(\mu^\pm)$$

is a non-zero irreducible representation of GSp_4^+ such that

$$\tilde{\Theta}(\mu^\pm) := \mathrm{ind}_{\mathrm{GSp}_4^+}^{\mathrm{GSp}_4}(\Theta(\mu^\pm)) = J_{Q(Z)}(\omega_{K/F}|\cdot|, \pi(\mu)|\cdot|^{-1/2}).$$

(ii) If μ is Galois invariant so that $\mu = \mu_F \circ N_{K/F}$ for some μ_F , then

$$\Theta(\mu^+) = \theta(\mu^+)$$

is a non-zero irreducible representation of GSp_4^+ such that

$$\tilde{\Theta}(\mu^+) := \mathrm{ind}_{\mathrm{GSp}_4^+}^{\mathrm{GSp}_4}(\Theta(\mu^+))$$

is the unique irreducible quotient of

$$I_{Q(Z)}(\omega_{K/F}|\cdot|, \pi(\mu)|\cdot|^{-1/2}) = I_B(\omega_{K/F}|\cdot|, \omega_{K/F}; \mu_F|\cdot|^{-1/2}).$$

On the other hand,

$$\Theta(\mu^-) = \theta(\mu^-)$$

is a non-zero irreducible representation of GSp_4^+ such that $\tilde{\Theta}(\mu^-)$ is the irreducible non-generic supercuspidal representation of GSp_4 with L -parameter $(\mu_F \boxtimes S_2) \oplus (\mu_F \omega_{K/F} \boxtimes S_2)$ and similitude character μ_F^2 . Note that the L -parameter is a representation of the Weil-Deligne group $W_F \times \mathrm{SL}_2(\mathbb{C})$ and S_2 is the irreducible two dimensional representation of $\mathrm{SL}_2(\mathbb{C})$.

Proof. We shall only give a sketch of the proof. Applying the normalized Jacquet module functor $R_{Q(Z)}$ to the induced Weil representation $\Omega_{V_K, W}$ of $\mathrm{GO}(V_K) \times \mathrm{GSp}(W)^+$, one sees that there is a $\mathrm{GL}(Z) \times (\mathrm{GO}(V_K) \times \mathrm{GSp}(W')^+)$ -equivariant surjective map

$$R_{Q(Z)}(\Omega_{V_K, W}) \longrightarrow \omega_{K/F}|\cdot|^{-1} \boxtimes (\Omega_{V_K, W'} \cdot |\det_{W'}|^{1/2}).$$

By the previous proposition, one has a $\mathrm{GO}(V_K) \times \mathrm{GSp}(W')^+$ -equivariant surjective map

$$\Omega_{V_K, W'} \cdot |\det_{W'}|^{1/2} \longrightarrow \mu^+ \boxtimes (\Theta_{V_K, W'}(\mu^+)|\cdot|^{-1/2}).$$

Frobenius reciprocity shows that there is a non-zero equivariant map

$$\Omega_{V_K, W} \longrightarrow \mu^+ \boxtimes I_{Q(Z)^+}(\omega_{K/F}|\cdot|^{-1}, \Theta_{V_K, W'}(\mu^+)|\cdot|^{-1/2}).$$

Since $\Theta_{V_K, W}(\mu^+)$ is an irreducible representation (as $\mathrm{O}(V_K)$ is anisotropic), we have

$$\Theta_{V_K, W}(\mu^+) \hookrightarrow I_{Q(Z)^+}(\omega_{K/F}|\cdot|^{-1}, \Theta_{V_K, W'}(\mu^+)|\cdot|^{-1/2}),$$

so that

$$\tilde{\Theta}_{V_K, W}(\mu^+) \hookrightarrow I_{Q(Z)}(\omega_{K/F}|\cdot|^{-1}, \pi(\mu)|\cdot|^{-1/2}).$$

The latter representation has $J_{Q(Z)}(\omega_{K/F}|\cdot|, \pi(\mu)|\cdot|^{-1/2})$ as its unique irreducible submodule and this proves the proposition for μ^+ .

To complete the proof of the proposition, we need to show the claim in (ii) that $\tilde{\Theta}(\mu^-)$ is the irreducible non-generic supercuspidal representation with the desired L -parameter. Proposition A.7 shows that $\tilde{\Theta}(\mu^-)$ is supercuspidal and it is non-zero since we are in the stable range. Moreover, it is not difficult to show that the Whittaker module of $\Omega_{V_K, W}$ is zero, so that $\tilde{\Theta}(\mu^-)$ is non-generic. Now by [9], $\tilde{\Theta}(\mu^-)$ has a non-zero theta lift to the anisotropic group $\mathrm{GSO}(D, -N_D)$, where D is the quaternion division F -algebra. We need to show that its theta lift to $\mathrm{GSO}(D, -N_D)$ is $\mu_F \boxtimes \mu_F \omega_{K/F}$.

For this, we resort to a global argument:

- Choose number fields $\mathbb{F} \subset \mathbb{K}$ such that for some place v of \mathbb{F} , one has $\mathbb{K}_v/\mathbb{F}_v = K/F$.

- Choose a quaternion \mathbb{F} -algebra \mathbb{D} such that $\mathbb{D}_v = D$ and $\mathbb{K} \subset \mathbb{D}$.
- Let Ξ be a Hecke character of $\mathbb{A}_{\mathbb{F}}^{\times}$ such that $\Xi_v = \mu_F$.
- One has the automorphic representation $\Xi \boxtimes \Xi\omega_{\mathbb{K}/\mathbb{F}}$ of $\text{GSO}(\mathbb{D}, -N_{\mathbb{D}})(\mathbb{A}_{\mathbb{F}})$ and one may consider its theta lift $\Theta(\Xi \boxtimes \Xi\omega_{\mathbb{K}/\mathbb{F}})$ to GSp_4 .

We claim that this global theta lift is non-zero. To see this, one computes a Fourier coefficient of this theta lift along the Siegel parabolic subgroup $P(Y)$. More precisely, the generic $M(Y)$ -orbits of Fourier coefficients are naturally parametrized by étale quadratic \mathbb{F} -algebras. If one takes a character Ψ of $N(Y)$ corresponding to \mathbb{K} , then the identity component of the stabilizer of Ψ in $M(Y)$ is isomorphic to $\text{GSO}(V_{\mathbb{K}}) \cong \mathbb{K}^{\times}$. One can then compute the Bessel period of the theta lift defined by the character Ψ of $N(Y)(\mathbb{A}_{\mathbb{F}})$ and the character $\Xi \circ N_{\mathbb{K}/\mathbb{F}}$ of $\mathbb{A}_{\mathbb{K}}^{\times}$. By a standard computation, one sees that this Bessel period is non-zero precisely when both the representations Ξ and $\Xi\omega_{\mathbb{K}/\mathbb{F}}$ have non-zero period integrals over the torus $\mathbb{A}_{\mathbb{K}}^{\times}$ against the character $\Xi^{-1} \circ N_{\mathbb{K}/\mathbb{F}}$. Since this last condition evidently holds, we conclude that $\Theta(\Xi \boxtimes \Xi\omega_{\mathbb{K}/\mathbb{F}})$ is non-zero.

In addition, we know that the theta lift $\Theta(\Xi \boxtimes \Xi\omega_{\mathbb{K}/\mathbb{F}})$ is irreducible, and its local component at v is non-generic supercuspidal with L -parameter

$$(\mu_F \boxtimes S_2) \oplus (\mu_F\omega_{K/F} \boxtimes S_2).$$

Moreover, $\Theta(\Xi \boxtimes \Xi\omega_{\mathbb{K}/\mathbb{F}})$ is nearly equivalent to the global theta lift $\Theta(\Xi \circ N_{\mathbb{K}/\mathbb{F}})$ of $\Xi \circ N_{\mathbb{K}/\mathbb{F}}$ from $\text{GSO}(V_{\mathbb{K}})$ to GSp_4^+ . In particular, it is CAP with respect to the Borel subgroup of GSp_4 .

According to a result of Soudry [53], all such CAP representations are obtained by theta lifts from $\text{GO}(V_{\mathbb{K}})$ and so one concludes that $\Theta(\Xi \boxtimes \Xi\omega_{\mathbb{K}/\mathbb{F}})$ is an irreducible constituent of $\Theta(\Xi \circ N_{\mathbb{K}/\mathbb{F}})$. By extracting the local component at v , one concludes that

$$\Theta_{D,W}(\mu_F \boxtimes \mu_F\omega_{K/F}) = \tilde{\Theta}_{V_K,W}(\mu^+) \quad \text{or} \quad \tilde{\Theta}_{V_K,W}(\mu^-).$$

Since we have already seen that $\tilde{\Theta}_{V_K,W}(\mu^+)$ is non-supercuspidal, we must have

$$\Theta_{D,W}(\mu_F \boxtimes \mu_F\omega_{K/F}) = \tilde{\Theta}_{V_K,W}(\mu^-).$$

This completes the proof of the proposition. \square

A.4. Explicit determination of local theta lifts. Now we can state the main results of this appendix. These are contained in the following three theorems. Together, they imply Theorem A.1.

Theorem A.9. *Let V be the anisotropic quadratic space and let $\tau_1 \boxtimes \tau_2$ be an irreducible representation of $\text{GSO}(V) = \text{GSO}(4)$.*

(i) $\Theta((\tau_1 \boxtimes \tau_2)^{\pm})$ is either zero or an irreducible representation of GSp_4 .

(ii) If $\tau_1 = \tau_2 = \tau$, then

$$\Theta((\tau \boxtimes \tau)^+) = \pi_{ng}(\text{JL}(\tau)),$$

which is the unique non-generic constituent of $I_{Q(Z)}(1, \text{JL}(\tau))$, whereas

$$\Theta((\tau \boxtimes \tau)^-) = 0.$$

(iii) If $\tau_1 \neq \tau_2$, then

$$\Theta((\tau_1 \boxtimes \tau_2)^{\pm}) = \Theta((\tau_2 \boxtimes \tau_1)^{\pm})$$

is the irreducible non-generic supercuspidal representation of GSp_4 with L -parameter $\phi_{\tau_1} \oplus \phi_{\tau_2}$ and similitude character $\omega_{\tau_1} = \omega_{\tau_2}$.

Theorem A.10. *Let V be the split quadratic space and let $\tau_1 \boxtimes \tau_2$ be an irreducible representation of $\text{GSO}(V) = \text{GSO}(2, 2)$.*

(i) If $\tau_1 = \tau_2 = \tau$ is a discrete series representation, then

$$\Theta((\tau \boxtimes \tau)^+) = \theta((\tau \boxtimes \tau)^+) = \pi_{gen}(\tau),$$

which is the unique generic constituent of $I_{Q(Z)}(1, \tau)$, whereas

$$\Theta((\tau \boxtimes \tau)^-) = 0.$$

(ii) If $\tau_1 \neq \tau_2$ are both supercuspidal, then $\Theta((\tau_1 \boxtimes \tau_2)^\pm) = \theta((\tau_1 \boxtimes \tau_2)^\pm)$ is the irreducible generic supercuspidal representation of GSp_4 with L -parameter $\phi_{\tau_1} \oplus \phi_{\tau_2}$ and similitude character $\omega_{\tau_1} = \omega_{\tau_2}$.

(iii) If τ_1 is supercuspidal and $\tau_2 = st_\chi$, then

$$\Theta((\tau_1 \boxtimes \tau_2)^+) = \theta((\tau_1 \boxtimes \tau_2)^+) = St(\tau_1 \cdot \chi^{-1}, \chi).$$

(iv) Suppose that $\tau_1 = st_{\chi_1}$ and $\tau_2 = st_{\chi_2}$ with $\chi_1 \neq \chi_2$ but $\chi_1^2 = \chi_2^2$. Then

$$\Theta((\tau_1 \boxtimes \tau_2)^+) = \theta((\tau_1 \boxtimes \tau_2)^+) = St(st_{\chi_1/\chi_2}, \chi_2) = St(st_{\chi_2/\chi_1}, \chi_1).$$

(v) Suppose that τ_1 is a discrete series representation and $\tau_2 \hookrightarrow \pi(\chi, \chi')$ with $|\chi/\chi'| = |-|^{-s_0}$ and $s_0 \geq 0$, so that τ_2 is a non-discrete series representation. Then

$$I_{P(Y)}(\tau_1 \cdot \chi^{-1}, \chi) \twoheadrightarrow \Theta((\tau_1 \boxtimes \tau_2)^+),$$

so that the latter representation is multiplicity free and

$$\theta((\tau_1 \boxtimes \tau_2)^+) = J_{P(Y)}(\tau_1 \cdot \chi^{-1}, \chi).$$

(vi) Suppose that

$$\tau_1 \hookrightarrow \pi(\chi_1, \chi'_1) \quad \text{and} \quad \tau_2 \hookrightarrow \pi(\chi_2, \chi'_2)$$

with

$$|\chi_i/\chi'_i| = |-|^{-s_i} \quad \text{and} \quad s_1 \geq s_2 \geq 0.$$

Then

$$I_{P(Y)}(\pi(\chi'_2, \chi_2) \cdot \chi_1^{-1}, \chi_1) = I_B(\chi'_2/\chi_1, \chi_2/\chi_1; \chi_1) \twoheadrightarrow \Theta((\tau_1 \boxtimes \tau_2)^+),$$

so that the latter representation is multiplicity free and

$$\theta((\tau_1 \boxtimes \tau_2)^+) = J_B(\chi'_2/\chi_1, \chi_2/\chi_1; \chi_1).$$

If $\tau_1 = \tau_2 = \tau$, then

$$\Theta((\tau \boxtimes \tau)^-) = 0.$$

Theorem A.11. Suppose that the discriminant algebra K of V is a field. Let $\tau \boxtimes \chi$ be an irreducible representation of

$$\mathrm{GSO}(V) = \mathrm{GSO}(3, 1) \cong (\mathrm{GL}_2(K) \times F^\times)/K^\times,$$

so that $\omega_\tau = \chi \circ \mathbb{N}_{K/F}$.

(i) If σ is an irreducible representation of $\mathrm{GO}(V)$, then $\Theta(\sigma) = 0$ if and only if $\sigma = (\tau \boxtimes \chi)^-$ for an invariant and distinguished $\tau \boxtimes \chi$; we shall say that such a σ is of **forbidden type**. If σ is not of forbidden type, then $\theta(\sigma)$ is an irreducible representation of GSp_4^+ such that $\tilde{\theta}(\sigma)$ is irreducible.

(ii) Suppose that τ is supercuspidal. Then we have the following situations:

- (Non-invariant case) If $\tau^c \neq \tau$, then $\Theta((\tau \boxtimes \chi)^+) = \theta((\tau \boxtimes \chi)^+)$ is generic supercuspidal.
- (Invariant and distinguished case) Suppose that $\tau^c = \tau$ and τ is obtained by base change of some supercuspidal representation τ_F of $\mathrm{GL}_2(F)$ and $\chi = \omega_{\tau_F} \omega_{K/F}$. Then

$$\tilde{\Theta}((\tau \boxtimes \chi)^+) = \tilde{\theta}((\tau \boxtimes \chi)^+) = I_{Q(Z)}(\omega_{E/F}, \tau_F)$$

with L -parameter $\phi_{\tau_F} \oplus \phi_{\tau_F} \cdot \omega_{K/F}$ and similitude character $\omega_{\tau_F} \omega_{K/F}$.

- (Invariant but not distinguished case) Suppose that $\tau^c = \tau$ and τ is obtained by base change of some supercuspidal representation τ_F of $\mathrm{GL}_2(F)$ but $\chi = \omega_{\tau_F}$. Then

$$\tilde{\Theta}((\tau \boxtimes \chi)^+) \quad \text{and} \quad \tilde{\Theta}((\tau \boxtimes \chi)^-)$$

are both irreducible supercuspidal with L -parameter $\phi_{\tau_F} \oplus \phi_{\tau_F} \cdot \omega_{K/F}$ and similitude character ω_{τ_F} . Exactly one of them, namely $\tilde{\Theta}((\tau \boxtimes \chi)^+)$, is generic.

- (iii) Suppose that $\tau = St_\mu$ is a twisted Steinberg representation so that $\mu^2 = \chi \circ \mathrm{N}_{K/F}$. Then there is a quadratic character η (possibly trivial) of F^\times such that

$$\mu^c/\mu = \eta \circ \mathrm{N}_{K/F} \quad \text{and} \quad \chi = \eta \cdot \mu|_{F^\times},$$

so that

$$St_\mu \boxtimes \chi \hookrightarrow I_{Q(J)^+}(\eta|\cdot|, \mu^c|\cdot|_K^{-1/2}).$$

Then we have the following situations:

- (Non-invariant case) If $\mu^c \neq \mu$, then $\eta \neq 1$ or $\omega_{K/F}$, and

$$\tilde{\Theta}((\tau \boxtimes \chi)^+) = \tilde{\theta}((\tau \boxtimes \chi)^+) = St(\eta\omega_{K/F}, \pi(\mu)).$$

- (Invariant and distinguished case) In this case, we have $\eta = \omega_{K/F}$,

$$\mu = \mu_F \circ \mathrm{N}_{K/F} \quad \text{and} \quad \chi = \mu_F^2 \omega_{K/F}.$$

Then

$$St_\mu \boxtimes \chi \hookrightarrow I_{Q(J)^+}(\omega_{K/F}|\cdot|, \mu|\cdot|_K^{-1/2})$$

and

$$\tilde{\Theta}((\tau \boxtimes \chi)^+) = \tilde{\theta}((\tau \boxtimes \chi)^+) = I_{Q(Z)}(\omega_{K/F}, st_{\mu_F}).$$

- (Invariant but not distinguished case) In this case, we have $\eta = 1$,

$$\mu = \mu_F \circ \mathrm{N}_{K/F} \quad \text{and} \quad \chi = \mu_F^2.$$

Then

$$\tilde{\Theta}((\tau \boxtimes \chi)^+) \quad \text{and} \quad \tilde{\Theta}((\tau \boxtimes \chi)^-)$$

are both irreducible discrete series representations of GSp_4 with L -parameter

$$(\mu_F \boxtimes S_2) \oplus (\mu_F \omega_{K/F} \boxtimes S_2)$$

and similitude character μ_F^2 . In particular,

$$\tilde{\Theta}((\tau \boxtimes \chi)^+) = St(st_{\omega_{K/F}}, \mu_F)$$

is generic, whereas $\tilde{\Theta}((\tau \boxtimes \chi)^-)$ is non-generic supercuspidal.

- (iv) Suppose that σ is a non-discrete series representation of $\mathrm{GO}(V)$ which is not of forbidden type, so that

$$\sigma \hookrightarrow I_{Q(J)}(\chi, \mu^+)$$

with

$$|\chi| = |\cdot|^{-s_0} \quad \text{and} \quad s_0 \geq 0.$$

Then we have the following situations:

- (Non-invariant or invariant and distinguished case) In this case, we have

$$I_{Q(Z)}(\chi^{-1}\omega_{K/F}, \pi(\mu) \cdot \chi) \twoheadrightarrow \tilde{\Theta}(\sigma)$$

where $\pi(\mu)$ is as given in Proposition A.7. In particular, $\tilde{\Theta}(\sigma)$ is multiplicity free and has a unique irreducible quotient.

- (Invariant but not distinguished case) In this case,

$$\tilde{\Theta}(\sigma) \quad \text{and} \quad \tilde{\Theta}(\sigma \cdot \text{sgn})$$

are the two irreducible constituents of $I_{Q(Z)}(1, \pi(\mu))$.

A.5. Jacquet and Whittaker modules. Because Theorem A.11 is the most subtle part of the three theorems, we shall give its proof in detail here and then give a sketch of the proof of Theorem A.10 later. (Theorem A.9 is the easiest part and its proof will be omitted.) Hence we shall assume that K is a field until §A.7. A key step in the proof of Theorem A.11 is the computation of normalized Jacquet modules of the induced Weil representation Ω with respect to $Q(J)$ and $Q(Z)$. This is a by-now-standard computation, following the lines of [29], and we shall simply state the results below.

Proposition A.12. *Let $R_{Q(J)}(\Omega)$ denote the normalized Jacquet module of Ω along $Q(J)$. Then we have a short exact sequence of $L(J) \times \text{GSp}(W)^+$ -modules:*

$$0 \longrightarrow A \longrightarrow R_{Q(J)}(\Omega) \longrightarrow B \longrightarrow 0$$

Here, as $\text{GL}(J) \times (\text{GO}(V_K) \times \text{GSp}(W)^+)$ -modules,

$$B \cong |\det_J| \boxtimes (\Omega_{V_K, W} \otimes |\nu_{V_K}|^{-1/2})$$

where $\Omega_{V_K, W}$ is the induced Weil representation of $\text{GO}(V_K) \times \text{GSp}(W)^+$, and

$$A \cong I_{Q(Z)^+}(S(F^\times) \otimes \Omega_{V_K, W'} \otimes |\det_J| \cdot |\det_Z|^{-1} \cdot (\omega_{K/F} \circ \det_Z) \cdot |\nu_{V_K}|^{-1})$$

where the action of $(\text{GL}(J) \times \text{GO}(V_K)) \times (\text{GL}(Z) \times \text{GSp}(W')^+)$ on $S(F^\times)$ is given by

$$((a, h), (b, g)) \cdot f(x) = f(b^{-1} \cdot x \cdot a \cdot \nu_{W'}(g))$$

and $\Omega_{V_K, W'}$ is the induced Weil representation of $\text{GO}(V_K) \times \text{GSp}(W')^+$.

Corollary A.13. *Let $\chi \boxtimes \mu$ be an irreducible representation of $\text{GL}(J) \times \text{GSO}(V_K)$.*

- (i) *We have*

$$\text{Hom}_{\text{GL}(J) \times \text{GO}(V_K)}(B, \chi \boxtimes \mu^\pm) \neq 0$$

if and only if $\chi = | \cdot |$, in which case

$$\text{Hom}_{\text{GL}(J) \times \text{GO}(V_K)}(B, \chi \boxtimes \mu^\epsilon) = (\Theta_{V_K, W}(\mu^\epsilon) \cdot |\nu_W|^{1/2})^*.$$

In particular,

$$\text{Hom}_{\text{GL}(J) \times \text{GO}(V_K)}(B, \chi \boxtimes \mu^+) = J_{Q(Z)^+}(\omega_{K/F} | \cdot |, \Theta_{V_K, W'}(\mu^+))^*.$$

- (ii) *We have*

$$\text{Hom}_{\text{GL}(J) \times \text{GO}(V_K)}(A, \chi \boxtimes \mu^\epsilon) = 0$$

if and only if μ is invariant and $\epsilon = -$. Outside of this case, we have

$$\text{Hom}_{\text{GL}(J) \times \text{GO}(V_K)}(A, \chi \boxtimes \mu^+) = I_{Q(Z)^+}(\chi^{-1} \omega_{K/F}, \Theta_{V_K, W'}(\mu^+) \cdot \chi)^*.$$

- (iii) *In particular, if $\chi \neq | \cdot |$, then*

$$\text{Hom}_{\text{GO}(V)}(\Omega, I_{Q(J)}(\chi, \mu^+)) = I_{Q(Z)^+}(\chi^{-1} \omega_{K/F}, \Theta_{V_K, W'}(\mu^+) \cdot \chi)^*.$$

Proposition A.14. *Let $R_{Q(Z)^+}(\Omega)$ denote the normalized Jacquet module of Ω along $Q(Z)^+$. Then we have a short exact sequence of $\text{GO}(V) \times L(Z)^+$ -modules:*

$$0 \longrightarrow A' \longrightarrow R_{Q(Z)^+}(\Omega) \longrightarrow B' \longrightarrow 0$$

Here, as $\text{GL}(Z) \times (\text{GO}(V) \times \text{GSp}(W')^+)$ -modules,

$$B' \cong (\omega_{K/F} \circ \det_Z) \boxtimes \Omega_{V, W'}$$

where $\Omega_{V,W'}$ is the induced Weil representation of $\mathrm{GO}(V) \times \mathrm{GSp}(W')^+$, and

$$A' \cong I_{Q(J)}(S(F^\times) \otimes \Omega_{V_K, W'} \otimes (\omega_{K/F} \circ \det_Z) \cdot |\nu_{V_K}|^{-1} \cdot |\nu_{W'}|^{-1})$$

where the action of $(\mathrm{GL}(J) \times \mathrm{GO}(V_K)) \times (\mathrm{GL}(Z) \times \mathrm{GSp}(W')^+)$ on $S(F^\times)$ is given by

$$(((a, h), (b, g)) \cdot f)(x) = f(a^{-1} \cdot \nu_{W'}(g)^{-1} \cdot x \cdot b)$$

and $\Omega_{V_K, W'}$ is the induced Weil representation of $\mathrm{GO}(V_K) \times \mathrm{GSp}(W')^+$.

Corollary A.15. (i) Suppose that $\tau \boxtimes \chi$ is an irreducible representation of $\mathrm{GSO}(V) \cong (\mathrm{GL}_2(K) \times F^\times)/K^\times$ which is invariant and distinguished, so that $(\tau \boxtimes \chi)^+$ participates in the theta correspondence with $\mathrm{GSp}(W')^+ \cong \mathrm{GL}_2^+$. If the small theta lift of $(\tau \boxtimes \chi)^+$ to GL_2^+ is denoted by τ_F^+ , then

$$\mathrm{Hom}_{\mathrm{GO}(V) \times \mathrm{GSp}(W)^+}(\Omega, (\tau \boxtimes \chi)^+ \boxtimes I_{Q(Z)^+}(\omega_{K/F}, \tau_F^+)) \neq 0.$$

(ii) We have

$$\mathrm{Hom}_{\mathrm{GL}(Z) \times \mathrm{GSp}(W')^+}(A', \chi \boxtimes \theta_{V_K, W'}(\mu^+)) = I_{Q(J)}(\chi^{-1} \omega_{K/F}, \mu^+ \cdot (\chi \circ \nu_{V_K}))^*.$$

We also need the computation of the Whittaker module of the induced Weil representation Ω . This is given by:

Proposition A.16. Let U be the unipotent radical of the Borel subgroup $B = P(Y) \cap Q(Z)$ of GSp_4 . Let ψ and ψ' be representatives of the two orbits of generic characters of U under the action of B^+ . Similarly, let U_0 be the unipotent radical of a \mathfrak{t} -stable Borel subgroup of $\mathrm{GO}(V)$ and ψ_0 a generic character of U_0 which is fixed by \mathfrak{t} . Then (perhaps after relabelling ψ and ψ')

$$(\Omega_{V,W})_{U,\psi} \cong \mathrm{c}\text{-ind}_{U_0 \rtimes \langle \mathfrak{t} \rangle}^{\mathrm{GO}(V)}(\psi_0 \boxtimes \mathbf{1}),$$

whereas

$$(\Omega_{V,W})_{U,\psi'} = 0.$$

Corollary A.17. Let $\tau \boxtimes \chi$ be an irreducible representation of $\mathrm{GSO}(V)$ for a generic τ . Then $\Theta_{V,W}((\tau \boxtimes \chi)^+)$ is ψ -generic, whereas in the invariant case, $\Theta_{V,W}((\tau \boxtimes \chi)^-)$ is non-generic with respect to any generic character.

A.6. Proof of Theorem A.11. We are now ready to give the proof of Theorem A.11. Suppose that σ is an irreducible representation of $\mathrm{GO}(V)$. We first note the following:

- If σ is infinite dimensional, then [49, Theorem 3.4] says that $\Theta(\sigma) = 0$ if and only if $\sigma = (\tau \boxtimes \chi)^-$ for an invariant and distinguished $\tau \boxtimes \chi$. The case where σ is finite dimensional can be established in the course of the proof below, but since it is not relevant to the application to this paper, we shall omit the details.
- By a result of Muić [42, Theorem 6.2], $\Theta(\sigma)$ is irreducible or zero if σ is a discrete series representation, at least when the residual characteristic of F is $p \neq 2$. The reason for this restriction on p in [42] is that the Howe duality conjecture on the irreducibility of $\theta(\sigma)$ is known to hold in general for $p \neq 2$ but not for $p = 2$. However, our proof below actually verifies the Howe duality conjecture for all p , so that [42, Theorem 6.2] holds without restriction on residual characteristic, at least for the dual pair considered here. Note however that this information is not necessary for Theorem A.1.

In view of the above, we may assume henceforth that σ is not of forbidden type. We now consider the various cases in Theorem A.11 in turn.

Non-discrete series representations.

Let σ be a non-discrete series representation of $\mathrm{GO}(V)$ which is not of forbidden type. Then, as in Theorem A.11 (iv), we have

$$\sigma \hookrightarrow I_{Q(J)}(\chi, \mu^+)$$

with

$$|\chi| = |\cdot|^{-s_0} \quad \text{and} \quad s_0 \geq 0.$$

By Frobenius reciprocity, one has

$$\Theta(\sigma)^* = \mathrm{Hom}_{\mathrm{GO}(V)}(\Omega, \sigma) \hookrightarrow \mathrm{Hom}_{\mathrm{GO}(V)}(\Omega, I_{Q(J)}(\chi, \mu^+)) = \mathrm{Hom}_{\mathrm{GL}(J) \times \mathrm{GO}(V_K)}(R_{Q(J)}(\Omega), \chi \boxtimes \mu^+).$$

By Corollary A.13, we see that

$$\mathrm{Hom}_{\mathrm{GL}(J) \times \mathrm{GO}(V_K)}(R_{Q(J)}(\Omega), \chi \boxtimes \mu^+) = I_{Q(Z)+}(\chi^{-1}\omega_{K/F}, \theta_{V_K, W'}(\mu^+) \cdot \chi)^*$$

and hence

$$I_{Q(Z)+}(\chi^{-1}\omega_{K/F}, \theta_{V_K, W'}(\mu^+) \cdot \chi) \twoheadrightarrow \Theta(\sigma),$$

so that

$$I_{Q(Z)}(\chi^{-1}\omega_{K/F}, \pi(\mu) \cdot \chi) \twoheadrightarrow \tilde{\Theta}(\sigma).$$

We now examine the various cases in Theorem A.11 (iv):

- (Non-invariant case) In this case, $\mu^c \neq \mu$ and $\chi \neq 1$ or $\omega_{K/F}$ and $\theta_{V_K, W'}(\mu^+)$ is supercuspidal. By Lemma A.2, $I_{Q(Z)}(\chi^{-1}\omega_{K/F}, \pi(\mu) \cdot \chi)$ is either irreducible or of length two with a unique irreducible quotient. This shows that $\tilde{\Theta}(\sigma)$ is multiplicity free and $\tilde{\theta}(\sigma)$ is irreducible, as desired.
- (Invariant and distinguished case) In this case, either $\mu^c = \mu$ or $\mu^c \neq \mu$ and $\chi = 1$. When $\mu^c = \mu$ so that $\mu = \mu_F \circ \mathbb{N}_{K/F}$, we have $\pi(\mu) = \pi(\mu_F, \mu_F\omega_{K/F})$ and

$$I_{Q(Z)}(\chi^{-1}\omega_{K/F}, \pi(\mu_F, \mu_F\omega_{K/F}) \cdot \chi) \twoheadrightarrow \tilde{\Theta}(\sigma).$$

Moreover,

$$I_{Q(Z)}(\chi^{-1}\omega_{K/F}, \pi(\mu_F, \mu_F\omega_{K/F}) \cdot \chi) = I_B(\chi^{-1}\omega_{K/F}, \omega_{K/F}; \chi\mu_F).$$

This is irreducible unless $\chi = |\cdot|^{-1}$ or $\omega_{K/F}|\cdot|^{-1}$. In any case, it is multiplicity free and has a unique irreducible quotient (cf. [50, p. 270, Table A.1, Type V]).

When $\mu^c \neq \mu$ and $\chi = 1$, we have

$$I_{Q(J)}(1, \mu^+) = \sigma \oplus \sigma \cdot \mathrm{sgn}.$$

Hence we have

$$\Theta(\sigma) \oplus \Theta(\sigma \cdot \mathrm{sgn}) = I_{Q(Z)+}(\omega_{K/F}, \theta_{V_K, W'}(\mu^+)),$$

so that

$$\tilde{\Theta}(\sigma) \oplus \tilde{\Theta}(\sigma \cdot \mathrm{sgn}) = I_{Q(Z)}(\omega_{K/F}, \pi(\mu))$$

where $\pi(\mu)$ is supercuspidal. By Lemma A.2, $I_{Q(Z)}(\omega_{K/F}, \pi(\mu))$ is irreducible. Moreover, $\sigma \cdot \mathrm{sgn}$ is of forbidden type, so that $\Theta(\sigma \cdot \mathrm{sgn}) = 0$. Hence we conclude that

$$\tilde{\Theta}(\sigma) = I_{Q(Z)}(\omega_{K/F}, \pi(\mu)).$$

- (Invariant but not distinguished case) In this case, $\mu^c \neq \mu$ but $\chi = \omega_{K/F}$. Then

$$I_{Q(J)}(\omega_{K/F}, \mu^+) = \sigma \oplus \sigma \cdot \mathrm{sgn}.$$

Hence we have

$$\Theta(\sigma) \oplus \Theta(\sigma \cdot \mathrm{sgn}) = I_{Q(Z)+}(1, \theta_{V_K, W'}(\mu^+)),$$

so that

$$\tilde{\Theta}(\sigma) \oplus \tilde{\Theta}(\sigma \cdot \mathrm{sgn}) = I_{Q(Z)}(1, \pi(\mu))$$

where $\pi(\mu)$ is supercuspidal. By Lemma A.2, $I_{Q(Z)}(1, \pi(\mu))$ is the direct sum of a generic representation and a non-generic one, which constitute an L -packet of size two.

Twisted Steinberg representations.

Let $\tau = St_\mu$ be a twisted Steinberg representation so that $\mu^2 = \chi \circ \mathbf{N}_{K/F}$ and $\sigma = (St_\mu \boxtimes \chi)^\pm$. It is easy to see that there is a quadratic character η (possibly trivial) of F^\times such that

$$\mu^c/\mu = \eta \circ \mathbf{N}_{K/F} \quad \text{and} \quad \chi = \eta \cdot \mu|_{F^\times}.$$

The representation $St_\mu \boxtimes \chi$ is invariant but not distinguished if and only if η is trivial. We now examine the various cases in Theorem A.11 (iii):

- (Non-invariant or invariant and distinguished case) In this case, η is non-trivial and we have

$$\sigma \hookrightarrow I_{Q(J)}(\eta|-|, \mu^+|\nu_{V_K}|^{-1/2}).$$

By Corollary A.13, we have

$$I_{Q(Z)^+}(\eta\omega_{K/F}|-|^{-1}, \theta_{V_K, W'}(\mu^+)|-|^{1/2}) \twoheadrightarrow \Theta(\sigma),$$

so that

$$I_{Q(Z)}(\eta\omega_{K/F}|-|^{-1}, \pi(\mu)|-|^{1/2}) \twoheadrightarrow \tilde{\Theta}(\sigma).$$

By Lemma A.2 and [50, p. 270, Table A.1, Type III], $I_{Q(Z)}(\eta\omega_{K/F}|-|^{-1}, \pi(\mu)|-|^{1/2})$ is multiplicity free with a unique irreducible quotient

$$\begin{cases} St(\eta\omega_{K/F}, \pi(\mu)) & \text{if } \eta \neq \omega_{K/F} \text{ (non-invariant);} \\ I_{Q(Z)}(\omega_{K/F}, st_{\mu_F}) & \text{if } \eta = \omega_{K/F} \text{ (invariant and distinguished),} \end{cases}$$

where μ_F is a character of F^\times such that $\mu = \mu_F \circ \mathbf{N}_{K/F}$ in the invariant and distinguished case. This verifies Theorem A.1 in this case. It does not quite show that $\Theta(\sigma)$ is irreducible, but as we explained above, this follows from a result of Muić [42, Theorem 6.2].

- (Invariant but not distinguished case) To a certain extent, this is the most non-trivial case of Theorem A.11. If $\sigma = (St_\mu \boxtimes \mu_F^2)^-$, then

$$\sigma \hookrightarrow I_{Q(J)}(|-|, \mu^-|\nu_{V_K}|^{-1/2}).$$

By Corollary A.13 (i), (ii) and Proposition A.8 (ii), we deduce that

$$\tilde{\Theta}(\sigma) = \tilde{\Theta}_{V_K, W'}(\mu^-)$$

is the non-generic supercuspidal representation with the desired L -parameter. On the other hand, if $\sigma = (St_\mu \boxtimes \mu_F^2)^+$, then

$$\sigma \hookrightarrow I_{Q(J)}(|-|, \mu^+|\nu_{V_K}|^{-1/2}).$$

In this case, Corollary A.13 implies that one has an exact sequence:

$$\begin{aligned} 0 &\longrightarrow J_{Q(Z)^+}(\omega_{K/F}|-|, \theta_{V_K, W'}(\mu^+)|-|^{-1/2})^* \\ &\longrightarrow \mathrm{Hom}_{\mathrm{GO}(V)}(\Omega, I_{Q(J)}(|-|, \mu^+|\nu_{V_K}|^{-1/2})) \\ &\xrightarrow{\delta} I_{Q(Z)^+}(\omega_{K/F}|-|^{-1}, \theta_{V_K, W'}(\mu^+)|-|^{1/2})^* \end{aligned}$$

Since

$$i : \Theta(\sigma)^* \hookrightarrow \mathrm{Hom}_{\mathrm{GO}(V)}(\Omega, I_{Q(J)}(|-|, \mu^+|\nu_{V_K}|^{-1/2})),$$

we obtain by composition with δ a map

$$\delta \circ i : \Theta(\sigma)^* \longrightarrow I_{Q(Z)^+}(\omega_{K/F}|-|^{-1}, \theta_{V_K, W'}(\mu^+)|-|^{1/2})^*.$$

We claim that this map is still injective; this will be sufficient to establish the desired result in this case. Indeed, it will give

$$I_{Q(Z)}(\omega_{K/F}|\cdot|^{-1}, \pi(\mu)|\cdot|^{1/2}) \twoheadrightarrow \tilde{\Theta}(\sigma),$$

and one knows by [51] (cf. also [50, p. 270, Table A.1, Type V]) that

$$I_{Q(Z)}(\omega_{K/F}|\cdot|^{-1}, \pi(\mu)|\cdot|^{1/2}) = I_B(\omega_{K/F}|\cdot|^{-1}, \omega_{K/F}; \mu_F|\cdot|^{1/2})$$

is multiplicity free with a unique irreducible quotient $St(st_{\omega_{K/F}}, \mu_F)$. This verifies Theorem A.1 in this case, and together with [42, Theorem 6.2], one has

$$\tilde{\Theta}(\sigma) = St(st_{\omega_{K/F}}, \mu_F).$$

It remains to show that $\delta \circ i$ is injective. Suppose on the contrary that it is not. Then we would have a non-zero equivariant map

$$\Omega \longrightarrow \sigma \boxtimes J_{Q(Z)^+}(\omega_{K/F}|\cdot|, \theta_{V_K, W'}(\mu^+)|\cdot|^{-1/2}),$$

so that

$$\begin{aligned} \sigma^* &\hookrightarrow \text{Hom}_{\text{GSp}_4^+}(\Omega, J_{Q(Z)^+}(\omega_{K/F}|\cdot|, \theta_{V_K, W'}(\mu^+)|\cdot|^{-1/2})) \\ &\hookrightarrow \text{Hom}_{\text{GSp}_4^+}(\Omega, I_{Q(Z)^+}(\omega_{K/F}|\cdot|^{-1}, \theta_{V_K, W'}(\mu^+)|\cdot|^{1/2})). \end{aligned}$$

Now we compute the latter Hom space using Corollary A.15 (ii). We conclude that

$$\text{Hom}_{\text{GSp}_4^+}(\Omega, I_{Q(Z)^+}(\omega_{K/F}|\cdot|^{-1}, \theta_{V_K, W'}(\mu^+)|\cdot|^{1/2})) = I_{Q(J)}(|\cdot|, \mu^+|\nu_{V_K}|^{-1/2})^*,$$

so that

$$I_{Q(J)}(|\cdot|, \mu^+|\nu_{V_K}|^{-1/2}) \twoheadrightarrow \sigma.$$

This is a contradiction, since $I_{Q(J)}(|\cdot|, \mu^+|\nu_{V_K}|^{-1/2})$ has σ as a submodule but not a quotient.

Supercuspidal representations.

Let $\tau \boxtimes \chi$ be a supercuspidal representation of $\text{GSO}(V)$. Finally, we examine the various cases in Theorem A.11 (ii):

- (Non-invariant case) In this case, one knows by [49, Theorem 3.4] that $\Theta((\tau \boxtimes \chi)^+)$ is non-zero and irreducible supercuspidal. Moreover, the L -parameter of $\tilde{\Theta}((\tau \boxtimes \chi)^+)$ is identified in [9, §11].
- (Invariant and distinguished case) In this case, τ is the base change of some supercuspidal representation τ_F of $\text{GL}_2(F)$ and $\chi = \omega_{\tau_F} \omega_{K/F}$. Moreover, one knows that the extension $(\tau \boxtimes \chi)^+$ participates in the theta correspondence with $\text{GSp}(W')^+ \cong \text{GL}_2^+$ and its theta lift to GL_2^+ is a constituent τ_F^+ of $\tau_F|_{\text{GL}_2^+}$. By Corollary A.15, we deduce that

$$\Theta((\tau \boxtimes \chi)^+) = I_{Q(Z)^+}(\omega_{K/F}, \tau_F^+),$$

and hence

$$\tilde{\Theta}((\tau \boxtimes \chi)^+) = I_{Q(Z)}(\omega_{K/F}, \tau_F),$$

which is irreducible.

- (Invariant but not distinguished case) In this case, τ is the base change of some supercuspidal representation τ_F of $\text{GL}_2(F)$ but $\chi = \omega_{\tau_F}$. One knows by [49, Theorem 3.4] that both extensions $(\tau \boxtimes \chi)^\pm$ have non-zero big theta lifts to GSp_4^+ and $\tilde{\Theta}((\tau \boxtimes \chi)^\pm)$ is irreducible supercuspidal. It remains to show that these two supercuspidal representations of GSp_4 make up an L -packet with L -parameter $\phi_{\tau_F} \oplus \phi_{\tau_F} \cdot \omega_{K/F}$ and similitude character ω_{τ_F} . Note that τ_F is necessarily non-dihedral with respect to K , so that $\tau_F \cdot \omega_{K/F} \neq \tau_F$.

For this, we consider the representations

$$\begin{cases} \tau_F \boxtimes \tau_F \cdot \omega_{K/F} & \text{of GSO}(2, 2); \\ \mathrm{JL}(\tau_F) \boxtimes \mathrm{JL}(\tau_F) \cdot \omega_{K/F} & \text{of GSO}(4), \end{cases}$$

and their theta lifts to GSp_4 . Then we are required to show that

$$\begin{aligned} \tilde{\Theta}((\tau \boxtimes \chi)^+) &= \Theta(\tau_F \boxtimes \tau_F \cdot \omega_{K/F}), \\ \tilde{\Theta}((\tau \boxtimes \chi)^-) &= \Theta(\mathrm{JL}(\tau_F) \boxtimes \mathrm{JL}(\tau_F) \cdot \omega_{K/F}). \end{aligned}$$

We achieve this by using a global argument:

- Choose a totally real number field \mathbb{F} such that for two places v and v' of \mathbb{F} , one has $\mathbb{F}_v = \mathbb{F}_{v'} = F$.
- Choose a totally real quadratic extension \mathbb{K} of \mathbb{F} such that $\mathbb{K}_v = \mathbb{K}_{v'} = K$.
- Let Σ be a cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{F}})$ such that $\Sigma_v = \Sigma_{v'} = \tau_F$ and the archimedean component Σ_{∞} of Σ is a discrete series representation. This can be achieved by using a simple trace formula. By [4], such a Σ is tempered. Then $\Sigma \boxtimes \Sigma \cdot \omega_{\mathbb{K}/\mathbb{F}}$ is a tempered cuspidal representation of $\mathrm{GSO}(2, 2)(\mathbb{A}_{\mathbb{F}})$.
- Consider the global theta lift

$$\Pi := \Theta(\Sigma \boxtimes \Sigma \cdot \omega_{\mathbb{K}/\mathbb{F}})$$

of $\Sigma \boxtimes \Sigma \cdot \omega_{\mathbb{K}/\mathbb{F}}$ to GSp_4 . It is an irreducible globally generic cuspidal representation.

On the other hand, we may consider the base change $\mathrm{BC}(\Sigma)$ of Σ to $\mathrm{GL}_2(\mathbb{A}_{\mathbb{K}})$, so that $\mathrm{BC}(\Sigma) \boxtimes \omega_{\Sigma}$ is a globally generic tempered cuspidal representation of $\mathrm{GSO}(\mathbb{V})(\mathbb{A}_{\mathbb{F}}) \cong (\mathrm{GL}_2(\mathbb{A}_{\mathbb{K}}) \times \mathbb{A}_{\mathbb{F}}^{\times})/\mathbb{A}_{\mathbb{K}}^{\times}$, where \mathbb{V} is the quadratic space $\mathbb{H} \oplus (\mathbb{K}, \mathrm{N}_{\mathbb{K}/\mathbb{F}})$. Observe that $\mathrm{BC}(\Sigma) \boxtimes \omega_{\Sigma}$ is a globally invariant representation and almost all of its local components are distinguished, but its local components at v and v' are isomorphic to $\tau \boxtimes \chi$ which is not distinguished.

Because $\mathrm{BC}(\Sigma) \boxtimes \omega_{\Sigma}$ is globally invariant, it can be abstractly extended to an irreducible representation of $\mathrm{GO}(\mathbb{V})(\mathbb{A}_{\mathbb{F}})$ in infinitely many ways; more precisely, at each place of \mathbb{F} , one has two possible extensions. One knows that at least half of these extensions occur in the space of cusp forms on $\mathrm{GO}(\mathbb{V})(\mathbb{A}_{\mathbb{F}})$. This is because at least one of these extensions is automorphic, and one can twist an automorphic extension by an automorphic sign character of $\mathrm{GO}(\mathbb{V})(\mathbb{A}_{\mathbb{F}})$. In particular, one can find an automorphic extension of $\mathrm{BC}(\Sigma) \boxtimes \omega_{\Sigma}$ whose local component at v is any one of the two extensions $(\tau \boxtimes \chi)^{\pm}$. We denote one such automorphic extension by $(\mathrm{BC}(\Sigma) \boxtimes \omega_{\Sigma})^{\pm}$. By using the place v' , one can further ensure that at any place w such that $\mathrm{BC}(\Sigma_w) \boxtimes \omega_{\Sigma_w}$ is distinguished, the local component of $(\mathrm{BC}(\Sigma) \boxtimes \omega_{\Sigma})^{\pm}$ is the $+$ -extension

$$(\mathrm{BC}(\Sigma_w) \boxtimes \omega_{\Sigma_w})^+.$$

Thus, we may ensure that all the local components of $(\mathrm{BC}(\Sigma) \boxtimes \omega_{\Sigma})^{\pm}$ have non-zero local theta lifts to GSp_4^+ .

Now by [49, Theorem 8.3], the global theta lift of $(\mathrm{BC}(\Sigma) \boxtimes \omega_{\Sigma})^{\pm}$ to GSp_4 is non-zero and irreducible cuspidal. Thus, we obtain an irreducible cuspidal representation

$$\Pi^{\pm} := \tilde{\Theta}((\mathrm{BC}(\Sigma) \boxtimes \omega_{\Sigma})^{\pm})$$

of $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{F}})$. By the local unramified theta correspondence, one sees that Π^{\pm} is nearly equivalent to Π , so that the partial standard L -function $L^S(s, \Pi^{\pm}, \mathrm{std})$ of degree five has a pole at $s = 1$. By a result of Kudla and Rallis [34], this implies that Π^{\pm} has a non-zero global theta lift to an inner form of $\mathrm{GSO}(2, 2)$. Such an inner form is associated to a quaternion \mathbb{F} -algebra \mathbb{D}_{\pm} (possibly split) and is isomorphic to $(\mathbb{D}_{\pm}^{\times} \times \mathbb{D}_{\pm}^{\times})/\mathbb{F}^{\times}$. If we denote the theta lift of Π^{\pm} to such an inner form by $\Theta_{\mathbb{D}_{\pm}}(\Pi^{\pm})$, then

$\Theta_{\mathbb{D}_{\pm}}(\Pi^{\pm})$ is a cuspidal representation which is nearly equivalent to $\Sigma \boxtimes \Sigma \cdot \omega_{\mathbb{K}/\mathbb{F}}$. Thus, $\Sigma \boxtimes \Sigma \cdot \omega_{\mathbb{K}/\mathbb{F}}$ must be the Jacquet-Langlands transfer of $\Theta_{\mathbb{D}_{\pm}}(\Pi^{\pm})$. Note that at the place v , we necessarily have $(\mathbb{D}_+)v \neq (\mathbb{D}_-)v$. By extracting the local component at v , we conclude that

$$\{\tilde{\Theta}((\tau \boxtimes \chi)^+), \tilde{\Theta}((\tau \boxtimes \chi)^-)\} = \{\Theta(\tau_F \boxtimes \tau_F \cdot \omega_{K/F}), \Theta(\text{JL}(\tau_F) \boxtimes \text{JL}(\tau_F) \cdot \omega_{K/F})\}.$$

Since we know that $\tilde{\Theta}((\tau \boxtimes \chi)^+)$ and $\Theta(\tau_F \boxtimes \tau_F \cdot \omega_{K/F})$ are generic and the other two representations are not, we obtain the desired result.

This completes the proof of Theorem A.11.

A.7. Proof of Theorem A.10. For the sake of completeness, we shall give a sketch of the proof of Theorem A.10. As before, a key step is the computation of normalized Jacquet modules of the induced Weil representation $\Omega_{V,W}$, where V is now the split four dimensional quadratic space. Before coming to this computation, we first introduce some more notations.

Recall that $V = X \oplus X^*$, where X is a two dimensional isotropic space. We can write

$$X = F \cdot u_1 \oplus F \cdot u_2 \quad \text{and} \quad X^* = F \cdot v_1 \oplus F \cdot v_2$$

with $\langle u_i, v_j \rangle = \delta_{ij}$. Let $P(X)$ be the parabolic subgroup of $\text{GSO}(V)$ stabilizing X with Levi factor

$$M(X) \cong \text{GL}(X) \times \mathbb{G}_m.$$

Let $J = F \cdot u_1$ be the isotropic line spanned by u_1 in X and let $B(J)$ be the stabilizer of J in $M(X)$; it is also the stabilizer of the isotropic line spanned by v_2 in X^* . With respect to the basis $\{u_1, u_2\}$ of X , $B(J)$ is the group of upper triangular matrices in $M(X) \cong \text{GL}(X) \times \mathbb{G}_m$. We write

$$(t(a, b), \lambda) = \left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \lambda \right) \in B(J) \subset M(X).$$

Similarly, recall that $W = Y^* \oplus Y$,

$$Y^* = F \cdot e_1 \oplus F \cdot e_2 \quad \text{and} \quad Y = F \cdot f_1 \oplus F \cdot f_2$$

with $\langle e_i, f_j \rangle = \delta_{ij}$. The stabilizer of Y in $\text{GSp}(W)$ is the Siegel parabolic subgroup $P(Y)$ with Levi factor

$$M(Y) \cong \text{GL}(Y) \times \mathbb{G}_m$$

and the stabilizer of $Z = F \cdot f_1$ in $\text{GSp}(W)$ is the Klingen parabolic subgroup $Q(Z)$ with Levi factor

$$L(Z) \cong \text{GL}(Z) \times \text{GSp}(W')$$

where $W' = F \cdot e_2 \oplus F \cdot f_2$.

With the above notations, we have:

Proposition A.18. *The normalized Jacquet module $R_{P(X)}(\Omega_{V,W})$ of $\Omega_{V,W}$ along $P(X)$ has a natural three step filtration as an $M(X) \times \text{GSp}(W)$ -module whose successive quotients are described as follows:*

(i) *The top quotient is*

$$C \cong S(F^{\times}).$$

Here the action of $(m, \lambda) \in M(X) \cong \text{GL}(X) \times \mathbb{G}_m$ on $S(F^{\times})$ is given by

$$((m, \lambda) \cdot f)(t) = |\det_X(m)|^{3/2} \cdot |\lambda|^{-3/2} \cdot f(\lambda \cdot t).$$

(ii) *The middle subquotient is*

$$B \cong I_{B(J) \times Q(Z)}(S(F^{\times}) \otimes S(F^{\times} \cdot v_2 \otimes f_1)).$$

Here the action of $(t(a, b), \lambda) \in B(J)$ on $S(F^{\times}) \otimes S(F^{\times} \cdot v_2 \otimes f_1)$ is given by

$$((t(a, b), \lambda) \cdot f)(t, x) = |a| \cdot |b|^2 \cdot |\lambda|^{-3/2} \cdot f(\lambda \cdot t, b \cdot x),$$

whereas the action of $(\alpha, g) \in L(Z) \cong \mathrm{GL}(Z) \times \mathrm{GSp}(W')$ is given by

$$((\alpha, g) \cdot f)(t, x) = |\alpha|^{-2} \cdot |\nu_{W'}(g)| \cdot f(\nu_{W'}(g) \cdot t, \alpha^{-1} \cdot \nu_{W'}(g) \cdot x).$$

(iii) Finally, the submodule is

$$A \cong I_{P(Y)}(S(F^\times) \otimes S(\mathrm{Isom}(X, Y)))$$

where $\mathrm{Isom}(X, Y)$ is the set of isomorphisms from X to Y as vector spaces (which is a torsor for $\mathrm{GL}(X)$ as well as for $\mathrm{GL}(Y)$). Here the action of $(m, \lambda) \in M(X) \cong \mathrm{GL}(X) \times \mathbb{G}_m$ on $S(F^\times) \otimes S(\mathrm{Isom}(X, Y))$ is given by

$$((m, \lambda) \cdot f)(t, h) = |\det_X(m)|^{3/2} \cdot |\lambda|^{-3/2} \cdot f(\lambda \cdot t, h \circ m),$$

whereas the action of $(m', \lambda') \in M(Y) \cong \mathrm{GL}(Y) \times \mathbb{G}_m$ is given by

$$((m', \lambda') \cdot f)(t, h) = |\lambda'|^{3/2} \cdot |\det_Y(m')|^{-3/2} \cdot f(\lambda' \cdot t, \lambda' \cdot m'^{-1} \circ h).$$

Corollary A.19. Let $\sigma = \pi(\chi_1, \chi_2) \boxtimes \tau$ be a representation of $\mathrm{GSO}(V) \cong (\mathrm{GL}_2 \times \mathrm{GL}_2)/F^\times$ such that τ is irreducible but $\pi(\chi_1, \chi_2)$ may be reducible, so that $\omega_\tau = \chi_1 \chi_2$ and

$$\sigma = I_{P(X)}(\tau^\vee \cdot \chi_1, \chi_2).$$

Then

$$\mathrm{Hom}_{\mathrm{GSO}(V)}(\Omega, \sigma) = \mathrm{Hom}_{M(X)}(R_{P(X)}(\Omega), \tau^\vee \cdot \chi_1 \boxtimes \chi_2).$$

(i) If $\chi_1/\chi_2 \neq |\cdot|^3$, then

$$\mathrm{Hom}_{M(X)}(C, \tau^\vee \cdot \chi_1 \boxtimes \chi_2) = 0.$$

(ii) If $R_B(\tau)$ does not have $\chi_1 |\cdot|^{-1} \boxtimes \eta$ as a subquotient for any character η , then

$$\mathrm{Hom}_{M(X)}(B, \tau^\vee \cdot \chi_1 \boxtimes \chi_2) = 0.$$

(iii) If the conditions in (i) and (ii) hold, then

$$\mathrm{Hom}_{M(X)}(R_{P(X)}(\Omega), \tau^\vee \cdot \chi_1 \boxtimes \chi_2) = \mathrm{Hom}_{M(X)}(A, \tau^\vee \cdot \chi_1 \boxtimes \chi_2) = I_{P(Y)}(\tau \cdot \chi_1^{-1}, \chi_1)^*.$$

Proposition A.20. Let U be the unipotent radical of the Borel subgroup $P(Y) \cap Q(Z)$ of GSp_4 and ψ a generic character of U . Similarly, let U_0 be the unipotent radical of a Borel subgroup of $\mathrm{GSO}(V)$ and ψ_0 a generic character of U_0 . Then

$$(\Omega_{V,W})_{U,\psi} \cong \mathrm{c}\text{-ind}_{U_0}^{\mathrm{GSO}(V)}(\psi_0).$$

In particular, if σ is an irreducible generic representation of $\mathrm{GSO}(V)$, then its big theta lift $\Theta(\sigma)$ to GSp_4 is generic and hence non-zero.

We are now ready to give the proof of Theorem A.10. Let $\tau_1 \boxtimes \tau_2$ be an irreducible representation of $\mathrm{GSO}(V) \cong (\mathrm{GL}_2 \times \mathrm{GL}_2)/F^\times$. Then one knows by results of Roberts that

$$\Theta(\tau_1 \boxtimes \tau_2) = \Theta(\tau_2 \boxtimes \tau_1) \neq 0.$$

We now consider the various cases in Theorem A.10 in turn.

Supercuspidal representations.

Suppose that $\tau_1 \boxtimes \tau_2$ is supercuspidal. Then one knows that $\Theta(\tau_1 \boxtimes \tau_2) = \theta(\tau_1 \boxtimes \tau_2)$ is non-zero and irreducible. Moreover, if $\tau_1 \neq \tau_2$, then the theta lift of $\tau_1 \boxtimes \tau_2$ to $\mathrm{GSp}(W') \cong \mathrm{GL}_2$ is zero and hence $\theta(\tau_1 \boxtimes \tau_2)$ is supercuspidal. By definition, the L -parameter of $\theta(\tau_1 \boxtimes \tau_2)$ is $\phi_{\tau_1} \oplus \phi_{\tau_2}$ with similitude character $\omega_{\tau_1} = \omega_{\tau_2}$.

On the other hand, if $\tau_1 = \tau_2 = \tau$, then $\tau \boxtimes \tau$ participates in the theta correspondence with $\mathrm{GSp}(W') \cong \mathrm{GL}_2$ and its big theta lift to GL_2 is τ . By an analogue of Proposition A.14 for the split V , there is a $\mathrm{GSO}(V) \times L(Z)$ -equivariant surjective map

$$R_{Q(Z)}(\Omega_{V,W}) \longrightarrow \Omega_{V,W'}.$$

By Frobenius reciprocity, one has a non-zero $\mathrm{GSO}(V) \times \mathrm{GSp}(W)$ -equivariant map

$$\Omega_{V,W} \longrightarrow (\tau \boxtimes \tau) \boxtimes I_{Q(Z)}(1, \tau).$$

Thus, we see that

$$\Theta(\tau \boxtimes \tau) \hookrightarrow I_{Q(Z)}(1, \tau).$$

We know that $I_{Q(Z)}(1, \tau)$ is the direct sum of two irreducible constituents with a unique generic constituent $\pi_{gen}(\tau)$. It follows from Proposition A.20 that

$$\Theta(\tau \boxtimes \tau) = \pi_{gen}(\tau).$$

Discrete series representations.

Suppose that $\sigma = st_\chi \boxtimes \tau$ where st_χ is a twisted Steinberg representation and τ is a discrete series representation so that $\omega_\tau = \chi^2$. Note that τ is either supercuspidal or equal to st_μ . Then

$$\sigma \hookrightarrow \pi(\chi|\cdot|^{-1/2}, \chi|\cdot|^{-1/2}) \boxtimes \tau = I_{P(X)}(\tau^\vee \cdot \chi|\cdot|^{-1/2}, \chi|\cdot|^{-1/2}).$$

We would like to apply Corollary A.19 (iii) and so we need to verify that the conditions in Corollary A.19 (i) and (ii) hold. The condition in Corollary A.19 (i) obviously holds, and that in Corollary A.19 (ii) holds when τ is supercuspidal. If $\tau = st_\mu$ is a twisted Steinberg representation (so that $\chi^2 = \mu^2$), then

$$R_B(\tau) = \mu|\cdot|^{-1/2} \boxtimes \mu|\cdot|^{-1/2} \neq \chi|\cdot|^{-1/2} \boxtimes \eta$$

for any character η . Hence the condition in Corollary A.19 (ii) also holds when τ is a twisted Steinberg representation. In particular, we conclude by Corollary A.19 (iii) that

$$I_{P(Y)}(\tau \cdot \chi^{-1}|\cdot|^{-1/2}, \chi|\cdot|^{-1/2}) \twoheadrightarrow \Theta(\sigma).$$

By Lemma A.3, the above induced representation is multiplicity free and of length two with a unique irreducible quotient, so that $\Theta(\sigma)$ is multiplicity free and $\theta(\sigma)$ is irreducible. Moreover,

$$\theta(\sigma) = \begin{cases} St(\tau \cdot \chi^{-1}, \chi) & \text{if } \tau \neq st_\chi; \\ \pi_{gen}(\tau) & \text{if } \tau = st_\chi. \end{cases}$$

There remains the issue of whether $\Theta(\sigma) = \theta(\sigma)$. This follows from a result of Muić [42], but we can also give a brief sketch of the proof. Suppose on the contrary that $\Theta(\sigma) = I_{P(Y)}(\tau \cdot \chi^{-1}|\cdot|^{-1/2}, \chi|\cdot|^{-1/2})$. Then we would have

$$\sigma^* \hookrightarrow \mathrm{Hom}_{\mathrm{GSp}(W)}(\Omega_{V,W}, I_{P(Y)}(\tau \cdot \chi^{-1}|\cdot|^{-1/2}, \chi|\cdot|^{-1/2})).$$

Now one compute the latter Hom space, which amounts to the computation of the normalized Jacquet module $R_{P(Y)}(\Omega_{V,W})$. A short computation shows that

$$\mathrm{Hom}_{\mathrm{GSp}(W)}(\Omega_{V,W}, I_{P(Y)}(\tau \cdot \chi^{-1}|\cdot|^{-1/2}, \chi|\cdot|^{-1/2})) = I_{P(X)}(\tau^\vee \cdot \chi|\cdot|^{-1/2}, \chi|\cdot|^{-1/2})^*,$$

so that

$$I_{P(X)}(\tau^\vee \cdot \chi|\cdot|^{-1/2}, \chi|\cdot|^{-1/2}) \twoheadrightarrow \sigma.$$

This is a contradiction, since $I_{P(X)}(\tau^\vee \cdot \chi|\cdot|^{-1/2}, \chi|\cdot|^{-1/2})$ has σ as a submodule but not a quotient. Thus, we conclude that $\Theta(\sigma) = \theta(\sigma)$ is irreducible.

Non-discrete series representations I.

Suppose that

$$\sigma \longmapsto \pi(\chi_1, \chi_2) \boxtimes \tau = I_{P(X)}(\tau^\vee \cdot \chi_1, \chi_2)$$

where τ is a discrete series representation with $\omega_\tau = \chi_1\chi_2$ and

$$|\chi_1/\chi_2| = |\cdot|^{-s_0} \quad \text{and} \quad s_0 \geq 0.$$

Again, we would like to apply Corollary A.19 (iii) and so we need to verify the conditions there. As before, the only issue is the condition in Corollary A.19 (ii) when $\tau = st_\chi$ is a twisted Steinberg representation, in which case

$$R_B(\tau) = \chi|\cdot|^{1/2} \boxtimes \chi|\cdot|^{-1/2}$$

and we need to show that this is different from $\chi_1|\cdot|^{-1} \boxtimes \eta$ for any character η . In other words, we need to show that

$$\chi/\chi_1 \neq |\cdot|^{-3/2}.$$

But observe that

$$|\chi|^2 = |\chi_1\chi_2| = |\chi_1|^2 \cdot |\chi_2/\chi_1| = |\chi_1|^2 \cdot |\cdot|^{s_0},$$

so that

$$|\chi/\chi_1| = |\cdot|^{s_0/2} \neq |\cdot|^{-3/2}.$$

This verifies that the conditions in Corollary A.19 (i) and (ii) hold, so that we conclude that

$$I_{P(Y)}(\tau \cdot \chi_1^{-1}, \chi_1) \longrightarrow \Theta(\sigma).$$

Since the above induced representation is multiplicity free with a unique irreducible quotient, we conclude that $\Theta(\sigma)$ is multiplicity free and $\theta(\sigma) = J_{P(Y)}(\tau \cdot \chi_1^{-1}, \chi_1)$ is irreducible.

Non-discrete series representations II.

Finally, we consider the case where

$$\sigma \longmapsto \pi(\chi_1, \chi'_1) \boxtimes \pi(\chi_2, \chi'_2)$$

with $\chi_1\chi'_1 = \chi_2\chi'_2$ and

$$|\chi_i/\chi'_i| = |\cdot|^{-s_i} \quad \text{and} \quad s_1 \geq s_2 \geq 0.$$

We consider two subcases:

(a) $\chi_2/\chi'_2 \neq |\cdot|^{-1}$; in this case $\pi(\chi_2, \chi'_2) = \pi(\chi'_2, \chi_2)$ is irreducible and

$$\sigma \longmapsto I_{P(X)}(\pi(\chi_2, \chi'_2)^\vee \cdot \chi_1, \chi'_1).$$

Again, to apply Corollary A.19 (iii), we need to verify the conditions there, and in particular the condition in Corollary A.19 (ii). We have

$$R_B(\pi(\chi_2, \chi'_2)) = (\chi_2 \boxtimes \chi'_2) \oplus (\chi'_2 \boxtimes \chi_2)$$

up to semisimplification and so we need to verify that

$$\chi_2 \neq \chi_1|\cdot|^{-1} \quad \text{and} \quad \chi'_2 \neq \chi_1|\cdot|^{-1}.$$

To see these, we argue by contradiction. If $\chi_2 = \chi_1|\cdot|^{-1}$, then $\chi'_2 = \chi'_1|\cdot|^{-1}$, so that

$$|\cdot|^{-s_2} = |\chi_2/\chi'_2| = |\chi_1/\chi'_1| \cdot |\cdot|^{-2} = |\cdot|^{-s_1-2}.$$

This would give $s_2 = s_1 + 2$ which contradicts $s_1 \geq s_2$. On the other hand, if $\chi'_2 = \chi_1|\cdot|^{-1}$, then $\chi_2 = \chi'_1|\cdot|^{-1}$, so that

$$|\cdot|^{s_2} = |\chi'_2/\chi_2| = |\chi_1/\chi'_1| \cdot |\cdot|^{-2} = |\cdot|^{-s_1-2}.$$

This would give $s_2 = -s_1 - 2 < 0$, which is a contradiction. Thus, we may apply Corollary A.19 (iii) to conclude that

$$I_{P(Y)}(\pi(\chi'_2, \chi_2) \cdot \chi_1^{-1}, \chi_1) = I_B(\chi'_2/\chi_1, \chi_2/\chi_1; \chi_1) \longrightarrow \Theta(\sigma).$$

This shows that $\Theta(\sigma)$ is multiplicity free with a unique irreducible quotient

$$\theta(\sigma) = J_B(\chi'_2/\chi_1, \chi_2/\chi_1; \chi_1).$$

- (b) $\chi_2/\chi'_2 = |\cdot|^{-1}$; in this case, $\pi(\chi_2, \chi'_2)$ is reducible and has the one dimensional representation $\chi_2|\cdot|^{1/2}$ as its unique irreducible submodule. Then

$$\sigma \longmapsto \pi(\chi_1, \chi'_1) \boxtimes \chi_2|\cdot|^{1/2} = I_{P(X)}(\chi_1\chi_2^{-1}|\cdot|^{-1/2}, \chi'_1).$$

Applying Corollary A.19 (iii) (we leave the verification of the conditions there to the reader), we conclude that

$$I_{P(Y)}(\chi_1^{-1}\chi_2|\cdot|^{1/2}, \chi_1) \twoheadrightarrow \Theta(\sigma).$$

Observe that

$$I_B(\chi'_2/\chi_1, \chi_2/\chi_1; \chi_1) \twoheadrightarrow I_{P(Y)}(\chi_1^{-1}\chi_2|\cdot|^{1/2}, \chi_1)$$

and the former induced representation is a standard module. This shows that $\Theta(\sigma)$ is multiplicity free with a unique irreducible quotient

$$\theta(\sigma) = J_B(\chi'_2/\chi_1, \chi_2/\chi_1; \chi_1).$$

This completes the proof of Theorem A.10.

Appendix B. Spherical Eisenstein series on GO_{2n}

Let F be a number field. Let $\xi(s) = \mathfrak{D}^{s/2}\zeta(s)$, where \mathfrak{D} is the absolute value of the discriminant of F and $\zeta(s)$ is the zeta function of F including archimedean factors. Then the functional equation $\xi(1-s) = \xi(s)$ holds. We write

$$\xi(s) = \frac{\rho}{s-1} + \gamma + O(s-1).$$

For each $s \in \mathbb{C}$, let

$$[s](a) = |a|^s, \quad [s]'(a) = |a|^s \log |a|,$$

for $a \in \mathbb{A}^\times$. For an automorphic form ϕ on $\mathrm{GO}_{2n}(\mathbb{A})$, let

$$\phi[s](h) = \phi(h) \cdot [s](\nu(h)), \quad \phi[s]'(h) = \phi(h) \cdot [s]'(\nu(h)),$$

for $h \in \mathrm{GO}_{2n}(\mathbb{A})$. Let $\mathbf{1}$ denote the constant function on $\mathrm{GO}_{2n}(\mathbb{A})$.

For each $r \in \mathbb{N}$ with $r \leq n$, let $P_{n,r}$ be the parabolic subgroup of GO_{2n} and $E^{(n,r)}(s)$ the spherical Eisenstein series given in §7.3. Note that $E^{(n,0)}(s) = \mathbf{1}$. For each $s_0 \in \mathbb{C}$, let

$$E^{(n,r)}(s) = \sum_{d \gg -\infty} (s - s_0)^d E_d^{(n,r)}(s_0)$$

be the Laurent expansion of $E^{(n,r)}(s)$ at $s = s_0$.

Let $Q = P_{n,1}$. For an automorphic form ϕ on $\mathrm{GO}_{2n}(\mathbb{A})$, let ϕ_Q denote the constant term of ϕ along Q . We regard ϕ_Q as an automorphic form on $\mathbb{A}^\times \times \mathrm{GO}_{2n-2}(\mathbb{A})$ via the embedding

$$(a, h') \longmapsto \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a' & 0 & b' \\ 0 & 0 & \nu(h')a^{-1} & 0 \\ 0 & c' & 0 & d' \end{pmatrix},$$

where

$$h' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}.$$

Lemma B.1. *Let ϕ be a \mathbf{K} -invariant automorphic form on $\mathrm{GO}_{2n}(\mathbb{A})$. Assume that ϕ is concentrated on the Borel subgroup. If $\phi_Q = 0$, then*

$$\phi = 0.$$

Proof. The assertion follows from the Langlands lemma (cf. [24, Corollary 3.1]). \square

We have a double coset decomposition

$$\mathrm{GO}_{2n} = \begin{cases} P_{n,r}Q \cup P_{n,r}w_1^{(n,r)}Q \cup P_{n,r}w_2^{(n,r)}Q & \text{if } 1 \leq r < n, \\ P_{n,r}Q \cup P_{n,r}w_2^{(n,r)}Q & \text{if } r = n, \end{cases}$$

where

$$w_1^{(n,r)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1}_{r-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_{n-r-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1}_{r-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1}_{n-r-1} \end{pmatrix}$$

and

$$w_2^{(n,r)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & \mathbf{1}_{n-1} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_{n-1} \end{pmatrix}.$$

As in [24, Proposition 2.6], a routine calculation shows the following proposition.

Proposition B.2. *If $1 \leq r < n$, then $E^{(n,r)}(s)_Q$ is equal to*

$$\begin{aligned} & [s + \frac{2n-r-1}{2}] \otimes E^{(n-1,r-1)}(s + \frac{1}{2})[-\frac{s}{2} - \frac{2n-r-1}{4}] \\ & + \frac{\xi(s + \frac{2n-3r-1}{2})}{\xi(s + \frac{2n-r-1}{2})} \cdot [r] \otimes E^{(n-1,r)}(s)[-r] \\ & + \frac{\xi(s + \frac{r-1}{2})\xi(s - \frac{2n-3r-1}{2})\xi(2s)}{\xi(s + \frac{r+1}{2})\xi(s + \frac{2n-r-1}{2})\xi(2s+r-1)} \\ & \times [-s + \frac{2n-r-1}{2}] \otimes E^{(n-1,r-1)}(s - \frac{1}{2})[\frac{s}{2} - \frac{2n-r-1}{4}]. \end{aligned}$$

If $r = n$, then $E^{(n,n)}(s)_Q$ is equal to

$$\begin{aligned} & [s + \frac{n-1}{2}] \otimes E^{(n-1,n-1)}(s + \frac{1}{2})[-\frac{s}{2} - \frac{n-1}{4}] \\ & + \frac{\xi(2s)}{\xi(2s+n-1)} \cdot [-s + \frac{n-1}{2}] \otimes E^{(n-1,n-1)}(s - \frac{1}{2})[\frac{s}{2} - \frac{n-1}{4}]. \end{aligned}$$

The case $n = 1$.

Obviously, $E^{(1,1)}(s)$ is entire. We have

$$E_0^{(1,1)}(0) = 2 \cdot \mathbf{1}, \quad E_1^{(1,1)}(0) = 0.$$

The case $n = 2$.

Let $r = 2$. By Proposition B.2, we have

$$\begin{aligned} E^{(2,2)}(s)_Q &= [s + \frac{1}{2}] \otimes E^{(1,1)}(s + \frac{1}{2})[-\frac{s}{2} - \frac{1}{4}] \\ &\quad + \frac{\xi(2s)}{\xi(2s+1)} \cdot [-s + \frac{1}{2}] \otimes E^{(1,1)}(s - \frac{1}{2})[\frac{s}{2} - \frac{1}{4}]. \end{aligned}$$

Hence $E^{(2,2)}(s)$ has a simple pole at $s = \frac{1}{2}$ and is holomorphic at $s = \frac{3}{2}$. Also, the functional equation

$$\xi(2s)E^{(2,2)}(-s) = \xi(2s+1)E^{(2,2)}(s)$$

holds. We have

$$E_{-1}^{(2,2)}(\frac{1}{2}) = \frac{\rho}{\xi(2)} \cdot \mathbf{1}.$$

Let $r = 1$. By Proposition B.2, we have

$$\begin{aligned} E^{(2,1)}(s)_Q &= [s + 1] \otimes \mathbf{1}[-\frac{s}{2} - \frac{1}{2}] \\ &\quad + \frac{\xi(s)}{\xi(s+1)} \cdot [1] \otimes E^{(1,1)}(s)[- \frac{1}{2}] \\ &\quad + \frac{\xi(s)^2}{\xi(s+1)^2} \cdot [-s + 1] \otimes \mathbf{1}[\frac{s}{2} - \frac{1}{2}]. \end{aligned}$$

Hence $E^{(2,1)}(s)$ has a double pole at $s = 1$ and is holomorphic at $s = 2$. We have

$$E_{-2}^{(2,1)}(1) = \frac{\rho^2}{\xi(2)^2} \cdot \mathbf{1}.$$

Lemma B.3.

$$E_{-1}^{(2,1)}(1) = \frac{\rho}{\xi(2)} \cdot E_0^{(2,2)}(\frac{1}{2}).$$

Proof. We have

$$\begin{aligned} &E_0^{(2,2)}(\frac{1}{2})_Q \\ &= [1] \otimes E_0^{(1,1)}(1)[- \frac{1}{2}] \\ &\quad + \frac{\rho}{2\xi(2)} \cdot (-[0]' \otimes E_0^{(1,1)}(0)[0] + \frac{1}{2}[0] \otimes E_0^{(1,1)}(0)[0]' + [0] \otimes E_1^{(1,1)}(0)[0]) \\ &\quad + \frac{\rho}{\xi(2)} \left(\frac{\gamma}{\rho} - \frac{\xi'(2)}{\xi(2)} \right) \cdot [0] \otimes E_0^{(1,1)}(0)[0] \\ &= [1] \otimes E_0^{(1,1)}(1)[- \frac{1}{2}] \\ &\quad - \frac{\rho}{\xi(2)} \cdot ([0]' \otimes \mathbf{1}[0] - \frac{1}{2}[0] \otimes \mathbf{1}[0]') + \frac{2\rho}{\xi(2)} \left(\frac{\gamma}{\rho} - \frac{\xi'(2)}{\xi(2)} \right) \cdot [0] \otimes \mathbf{1}[0]. \end{aligned}$$

On the other hand,

$$\begin{aligned} &E_{-1}^{(2,1)}(1)_Q \\ &= \frac{\rho}{\xi(2)} \cdot [1] \otimes E_0^{(1,1)}(1)[- \frac{1}{2}] \\ &\quad + \frac{\rho^2}{\xi(2)^2} \cdot (-[0]' \otimes \mathbf{1}[0] + \frac{1}{2}[0] \otimes \mathbf{1}[0]') + \frac{2\rho^2}{\xi(2)^2} \left(\frac{\gamma}{\rho} - \frac{\xi'(2)}{\xi(2)} \right) \cdot [0] \otimes \mathbf{1}[0]. \end{aligned}$$

By Lemma B.1, this yields the lemma. □

The case $n = 3$.

Let $r = 3$. By Proposition B.2, we have

$$\begin{aligned} E^{(3,3)}(s)_Q &= [s+1] \otimes E^{(2,2)}(s+\frac{1}{2})[-\frac{s}{2}-\frac{1}{2}] \\ &\quad + \frac{\xi(2s)}{\xi(2s+2)} \cdot [-s+1] \otimes E^{(2,2)}(s-\frac{1}{2})[\frac{s}{2}-\frac{1}{2}] \\ &= [s+1] \otimes E^{(2,2)}(s+\frac{1}{2})[-\frac{s}{2}-\frac{1}{2}] \\ &\quad + \frac{\xi(2s-1)}{\xi(2s+2)} \cdot [-s+1] \otimes E^{(2,2)}(-s+\frac{1}{2})[\frac{s}{2}-\frac{1}{2}]. \end{aligned}$$

Hence $E^{(3,3)}(s)$ has a simple pole at $s = 1$ and is holomorphic at $s = 0$. Also, the functional equation

$$\xi(2s-1)E^{(3,3)}(-s) = \xi(2s+2)E^{(3,3)}(s)$$

holds. We have

$$E_1^{(3,3)}(0) = -\frac{2\xi'(2)}{\xi(2)} \cdot E_0^{(3,3)}(0), \quad E_{-1}^{(3,3)}(1) = \frac{\rho}{\xi(4)} \cdot \mathbf{1}.$$

Let $r = 2$. By Proposition B.2, we have

$$\begin{aligned} E^{(3,2)}(s)_Q &= [s+\frac{3}{2}] \otimes E^{(2,1)}(s+\frac{1}{2})[-\frac{s}{2}-\frac{3}{4}] \\ &\quad + \frac{\xi(s-\frac{1}{2})}{\xi(s+\frac{3}{2})} \cdot [2] \otimes E^{(2,2)}(s)[1] \\ &\quad + \frac{\xi(s+\frac{1}{2})^2\xi(2s)}{\xi(s+\frac{3}{2})^2\xi(2s+1)} \cdot [-s+\frac{3}{2}] \otimes E^{(2,1)}(s-\frac{1}{2})[\frac{s}{2}-\frac{3}{4}]. \end{aligned}$$

Hence $E^{(3,2)}(s)$ has a double pole at $s = \frac{3}{2}$. We have

$$E_{-2}^{(3,2)}(\frac{3}{2}) = \frac{\rho^2}{\xi(3)\xi(4)} \cdot \mathbf{1}.$$

Let $r = 1$. By Proposition B.2, we have

$$\begin{aligned} E^{(3,1)}(s)_Q &= [s+2] \otimes \mathbf{1}[-\frac{s}{2}-1] \\ &\quad + \frac{\xi(s+1)}{\xi(s+2)} \cdot [1] \otimes E^{(2,1)}(s)[-1] \\ &\quad + \frac{\xi(s)\xi(s-1)}{\xi(s+1)\xi(s+2)} \cdot [-s+2] \otimes \mathbf{1}[\frac{s}{2}-1]. \end{aligned}$$

Hence $E^{(3,1)}(s)$ has a simple pole at $s = 2$ and has a simple pole at $s = 1$. We have

$$E_{-1}^{(3,1)}(2) = \frac{\rho\xi(2)}{\xi(3)\xi(4)} \cdot \mathbf{1}.$$

Lemma B.4.

$$E_{-1}^{(3,2)}(\frac{3}{2}) = \frac{\rho}{\xi(3)} \cdot E_0^{(3,3)}(1).$$

Proof. We have

$$\begin{aligned} &E_0^{(3,3)}(1)_Q \\ &= [2] \otimes E_0^{(2,2)}(\frac{3}{2})[1] \\ &\quad + \frac{\xi(2)}{\xi(4)} \cdot (-[0]' \otimes E_{-1}^{(2,2)}(\frac{1}{2})[0] + \frac{1}{2}[0] \otimes E_{-1}^{(2,2)}(\frac{1}{2})[0]' + [0] \otimes E_0^{(2,2)}(\frac{1}{2})[0]) \end{aligned}$$

$$\begin{aligned}
& + \frac{2\xi(2)}{\xi(4)} \left(\frac{\xi'(2)}{\xi(2)} - \frac{\xi'(4)}{\xi(4)} \right) \cdot [0] \otimes E_{-1}^{(2,2)}(\tfrac{1}{2})[0] \\
= & [2] \otimes E_0^{(2,2)}(\tfrac{3}{2})[1] - \frac{\rho}{\xi(4)} \cdot ([0]' \otimes \mathbf{1}[0] - \tfrac{1}{2}[0] \otimes \mathbf{1}[0]') \\
& + \frac{\xi(2)}{\xi(4)} \cdot [0] \otimes E_0^{(2,2)}(\tfrac{1}{2})[0] + \frac{2\rho}{\xi(4)} \left(\frac{\xi'(2)}{\xi(2)} - \frac{\xi'(4)}{\xi(4)} \right) \cdot [0] \otimes \mathbf{1}[0].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& E_{-1}^{(3,2)}(\tfrac{3}{2})_Q \\
= & \frac{\rho}{\xi(3)} \cdot [2] \otimes E_0^{(2,2)}(\tfrac{3}{2})[1] \\
& + \frac{\xi(2)^2}{\xi(3)\xi(4)} \cdot (-[0]' \otimes E_{-2}^{(2,1)}(1)[0] + \tfrac{1}{2}[0] \otimes E_{-2}^{(2,1)}(1)[0]' + [0] \otimes E_{-1}^{(2,1)}(1)[0]) \\
& + \frac{2\xi(2)^2}{\xi(3)\xi(4)} \left(\frac{\xi'(2)}{\xi(2)} - \frac{\xi'(4)}{\xi(4)} \right) \cdot [0] \otimes E_{-2}^{(2,1)}(1)[0] \\
= & \frac{\rho}{\xi(3)} \cdot [2] \otimes E_0^{(2,2)}(\tfrac{3}{2})[1] - \frac{\rho^2}{\xi(3)\xi(4)} \cdot ([0]' \otimes \mathbf{1}[0] - \tfrac{1}{2}[0] \otimes \mathbf{1}[0]') \\
& + \frac{\xi(2)^2}{\xi(3)\xi(4)} \cdot [0] \otimes E_{-1}^{(2,1)}(1)[0] + \frac{2\rho^2}{\xi(3)\xi(4)} \left(\frac{\xi'(2)}{\xi(2)} - \frac{\xi'(4)}{\xi(4)} \right) \cdot [0] \otimes \mathbf{1}[0].
\end{aligned}$$

Hence the assertion follows from Lemmas B.1 and B.3. \square

Lemma B.5.

$$E_{-1}^{(3,1)}(1) = \frac{\rho}{2\xi(3)} \cdot E_0^{(3,3)}(0).$$

Proof. We have

$$\begin{aligned}
& E_0^{(3,3)}(0)_Q \\
= & [1]' \otimes E_{-1}^{(2,2)}(\tfrac{1}{2})[-\tfrac{1}{2}] - \tfrac{1}{2}[1] \otimes E_{-1}^{(2,2)}(\tfrac{1}{2})[-\tfrac{1}{2}]' + [1] \otimes E_0^{(2,2)}(\tfrac{1}{2})[-\tfrac{1}{2}] \\
& + [1]' \otimes E_{-1}^{(2,2)}(\tfrac{1}{2})[-\tfrac{1}{2}] - \tfrac{1}{2}[1] \otimes E_{-1}^{(2,2)}(\tfrac{1}{2})[-\tfrac{1}{2}]' + [1] \otimes E_0^{(2,2)}(\tfrac{1}{2})[-\tfrac{1}{2}] \\
& + \frac{4\xi'(2)}{\xi(2)} \cdot [1] \otimes E_{-1}^{(2,2)}(\tfrac{1}{2})[-\tfrac{1}{2}] \\
= & \frac{2\rho}{\xi(2)} \cdot ([1]' \otimes \mathbf{1}[-\tfrac{1}{2}] - \tfrac{1}{2}[1] \otimes \mathbf{1}[-\tfrac{1}{2}]') + 2 \cdot [1] \otimes E_0^{(2,2)}(\tfrac{1}{2})[-\tfrac{1}{2}] \\
& + \frac{4\rho\xi'(2)}{\xi(2)^2} \cdot [1] \otimes \mathbf{1}[-\tfrac{1}{2}].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& E_{-1}^{(3,1)}(1)_Q \\
= & \frac{\xi(2)}{\xi(3)} \cdot [1] \otimes E_{-1}^{(2,1)}(1)[- \tfrac{1}{2}] + \frac{\xi(2)}{\xi(3)} \left(\frac{\xi'(2)}{\xi(2)} - \frac{\xi'(3)}{\xi(3)} \right) \cdot [1] \otimes E_{-2}^{(2,1)}(1)[- \tfrac{1}{2}] \\
& - \frac{\rho^2}{\xi(2)\xi(3)} \cdot (-[1]' \otimes \mathbf{1}[-\tfrac{1}{2}] + \tfrac{1}{2}[1] \otimes \mathbf{1}[-\tfrac{1}{2}]') \\
& + \frac{\rho^2}{\xi(2)\xi(3)} \left(\frac{\xi'(2)}{\xi(2)} + \frac{\xi'(3)}{\xi(3)} \right) \cdot [1] \otimes \mathbf{1}[-\tfrac{1}{2}]
\end{aligned}$$

$$\begin{aligned}
&= \frac{\xi(2)}{\xi(3)} \cdot [1] \otimes E_{-1}^{(2,1)}(1)[- \frac{1}{2}] + \frac{2\rho^2\xi'(2)}{\xi(2)^2\xi(3)} \cdot [1] \otimes \mathbf{1}[- \frac{1}{2}] \\
&\quad + \frac{\rho^2}{\xi(2)\xi(3)} \cdot ([1]' \otimes \mathbf{1}[- \frac{1}{2}] - \frac{1}{2}[1] \otimes \mathbf{1}[- \frac{1}{2}]').
\end{aligned}$$

Hence the assertion follows from Lemmas B.1 and B.3. \square

The case $n = 4$.

Let $r = 4$. By Proposition B.2, we have

$$\begin{aligned}
E^{(4,4)}(s)_Q &= [s + \frac{3}{2}] \otimes E^{(3,3)}(s + \frac{1}{2})[- \frac{s}{2} - \frac{3}{4}] \\
&\quad + \frac{\xi(2s)}{\xi(2s+3)} \cdot [-s + \frac{3}{2}] \otimes E^{(3,3)}(s - \frac{1}{2})[\frac{s}{2} - \frac{3}{4}].
\end{aligned}$$

Hence $E^{(4,4)}(s)$ has a simple pole at $s = \frac{1}{2}$.

Let $r = 2$. By Proposition B.2, we have

$$\begin{aligned}
E^{(4,2)}(s)_Q &= [s + \frac{5}{2}] \otimes E^{(3,1)}(s + \frac{1}{2})[- \frac{s}{2} - \frac{5}{4}] \\
&\quad + \frac{\xi(s + \frac{1}{2})}{\xi(s + \frac{5}{2})} \cdot [2] \otimes E^{(3,2)}(s)[-1] \\
&\quad + \frac{\xi(s + \frac{1}{2})\xi(s - \frac{1}{2})\xi(2s)}{\xi(s + \frac{5}{2})\xi(s + \frac{3}{2})\xi(2s+1)} \cdot [-s + \frac{5}{2}] \otimes E^{(3,1)}(s - \frac{1}{2})[\frac{s}{2} - \frac{5}{4}].
\end{aligned}$$

Hence $E^{(4,2)}(s)$ has a double pole at $s = \frac{3}{2}$.

Let $r = 1$. By Proposition B.2, we have

$$\begin{aligned}
E^{(4,1)}(s)_Q &= [s + 3] \otimes \mathbf{1}[- \frac{s}{2} - \frac{3}{2}] \\
&\quad + \frac{\xi(s+2)}{\xi(s+3)} \cdot [1] \otimes E^{(3,1)}(s)[- \frac{1}{2}] \\
&\quad + \frac{\xi(s)\xi(s-2)}{\xi(s+1)\xi(s+3)} \cdot [-s + 3] \otimes \mathbf{1}[\frac{s}{2} - \frac{3}{2}].
\end{aligned}$$

Hence $E^{(4,1)}(s)$ has a simple pole at $s = 1$.

Lemma B.6.

$$E_{-2}^{(4,2)}(\frac{3}{2}) = \frac{\rho\xi(2)}{\xi(3)\xi(4)} \cdot E_{-1}^{(4,4)}(\frac{1}{2}).$$

Proof. We have

$$\begin{aligned}
E_{-1}^{(4,4)}(\frac{1}{2})_Q &= [2] \otimes E_{-1}^{(3,3)}(1)[-1] + \frac{\rho}{2\xi(4)} \cdot [1] \otimes E_0^{(3,3)}(0)[- \frac{1}{2}] \\
&= \frac{\rho}{\xi(4)} \cdot [2] \otimes \mathbf{1}[-1] + \frac{\rho}{2\xi(4)} \cdot [1] \otimes E_0^{(3,3)}(0)[- \frac{1}{2}].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
E_{-2}^{(4,2)}(\frac{3}{2})_Q &= \frac{\xi(2)}{\xi(4)} \cdot [2] \otimes E_{-2}^{(3,2)}(\frac{3}{2})[-1] + \frac{\rho\xi(2)}{\xi(4)^2} \cdot [1] \otimes E_{-1}^{(3,1)}(1)[- \frac{1}{2}] \\
&= \frac{\rho^2\xi(2)}{\xi(3)\xi(4)^2} \cdot [2] \otimes \mathbf{1}[-1] + \frac{\rho\xi(2)}{\xi(4)^2} \cdot [1] \otimes E_{-1}^{(3,1)}(1)[- \frac{1}{2}].
\end{aligned}$$

Hence the assertion follows from Lemmas B.1 and B.5. \square

Lemma B.7.

$$E_{-1}^{(4,1)}(1) = E_{-1}^{(4,4)}(\frac{1}{2}).$$

Proof. We have

$$E_{-1}^{(4,1)}(1)_Q = \frac{\xi(3)}{\xi(4)} \cdot [1] \otimes E_{-1}^{(3,1)}(1)[- \frac{1}{2}] + \frac{\rho}{\xi(4)} \cdot [2] \otimes \mathbf{1}[-1].$$

Hence the assertion follows from Lemmas B.1 and B.5. \square

Proposition B.8.

$$\left(\frac{\rho\xi(2)}{\xi(3)\xi(4)} \right)^{-1} \cdot E_{-1}^{(4,2)}(\frac{3}{2}) = E_0^{(4,4)}(\frac{1}{2}) + E_0^{(4,1)}(1) + \left(-\frac{\gamma}{\rho} + \frac{3\xi'(2)}{\xi(2)} \right) \cdot E_{-1}^{(4,4)}(\frac{1}{2}).$$

Proof. We have

$$\begin{aligned} & E_0^{(4,4)}(\frac{1}{2})_Q \\ &= [2]' \otimes E_{-1}^{(3,3)}(1)[-1] - \frac{1}{2}[2] \otimes E_{-1}^{(3,3)}(1)[-1]' + [2] \otimes E_0^{(3,3)}(1)[-1] \\ &\quad + \frac{\rho}{2\xi(4)} \cdot (-[1]' \otimes E_0^{(3,3)}(0)[- \frac{1}{2}] + \frac{1}{2}[1] \otimes E_0^{(3,3)}(0)[- \frac{1}{2}]' + [1] \otimes E_1^{(3,3)}(0)[- \frac{1}{2}]) \\ &\quad + \frac{\rho}{\xi(4)} \left(\frac{\gamma}{\rho} - \frac{\xi'(4)}{\xi(4)} \right) \cdot [1] \otimes E_0^{(3,3)}(0)[- \frac{1}{2}] \\ &= \frac{\rho}{\xi(4)} \cdot ([2]' \otimes \mathbf{1}[-1] - \frac{1}{2}[2] \otimes \mathbf{1}[-1]') + [2] \otimes E_0^{(3,3)}(1)[-1] \\ &\quad - \frac{\rho}{2\xi(4)} \cdot ([1]' \otimes E_0^{(3,3)}(0)[- \frac{1}{2}] - \frac{1}{2}[1] \otimes E_0^{(3,3)}(0)[- \frac{1}{2}]') \\ &\quad + \frac{\rho}{\xi(4)} \left(\frac{\gamma}{\rho} - \frac{\xi'(2)}{\xi(2)} - \frac{\xi'(4)}{\xi(4)} \right) \cdot [1] \otimes E_0^{(3,3)}(0)[- \frac{1}{2}]. \end{aligned}$$

By Lemmas B.4 and B.5, we have

$$\begin{aligned} & E_{-1}^{(4,2)}(\frac{3}{2})_Q \\ &= [4] \otimes E_{-1}^{(3,1)}(2)[-2] \\ &\quad + \frac{\xi(2)}{\xi(4)} \cdot [2] \otimes E_{-1}^{(3,2)}(\frac{3}{2})[-1] + \frac{\xi(2)}{\xi(4)} \left(\frac{\xi'(2)}{\xi(2)} - \frac{\xi'(4)}{\xi(4)} \right) \cdot [2] \otimes E_{-2}^{(3,2)}(\frac{3}{2})[-1] \\ &\quad + \frac{\rho\xi(2)}{\xi(4)^2} \cdot (-[1]' \otimes E_{-1}^{(3,1)}(1)[- \frac{1}{2}] + \frac{1}{2}[1] \otimes E_{-1}^{(3,1)}(1)[- \frac{1}{2}]' + [1] \otimes E_0^{(3,1)}(1)[- \frac{1}{2}]) \\ &\quad + \frac{\rho\xi(2)}{\xi(4)^2} \left(\frac{\gamma}{\rho} + \frac{\xi'(2)}{\xi(2)} + \frac{\xi'(3)}{\xi(3)} - \frac{3\xi'(4)}{\xi(4)} \right) \cdot [1] \otimes E_{-1}^{(3,1)}(1)[- \frac{1}{2}] \\ &= \frac{\rho\xi(2)}{\xi(3)\xi(4)} \cdot [4] \otimes \mathbf{1}[-2] \\ &\quad + \frac{\rho\xi(2)}{\xi(3)\xi(4)} \cdot [2] \otimes E_0^{(3,3)}(1)[-1] + \frac{\rho^2\xi(2)}{\xi(3)\xi(4)^2} \left(\frac{\xi'(2)}{\xi(2)} - \frac{\xi'(4)}{\xi(4)} \right) \cdot [2] \otimes \mathbf{1}[-1] \\ &\quad - \frac{\rho^2\xi(2)}{2\xi(3)\xi(4)^2} \cdot ([1]' \otimes E_0^{(3,3)}(0)[- \frac{1}{2}] - \frac{1}{2}[1] \otimes E_0^{(3,3)}(0)[- \frac{1}{2}]') \\ &\quad + \frac{\rho\xi(2)}{\xi(4)^2} \cdot [1] \otimes E_0^{(3,1)}(1)[- \frac{1}{2}] \\ &\quad + \frac{\rho^2\xi(2)}{2\xi(3)\xi(4)^2} \left(\frac{\gamma}{\rho} + \frac{\xi'(2)}{\xi(2)} + \frac{\xi'(3)}{\xi(3)} - \frac{3\xi'(4)}{\xi(4)} \right) \cdot [1] \otimes E_0^{(3,3)}(0)[- \frac{1}{2}] \end{aligned}$$

and

$$\begin{aligned}
& E_0^{(4,1)}(1)_Q \\
&= [4] \otimes \mathbf{1}[-2] \\
&\quad + \frac{\xi(3)}{\xi(4)} \cdot [1] \otimes E_0^{(3,1)}(1)[- \frac{1}{2}] + \frac{\xi(3)}{\xi(4)} \left(\frac{\xi'(3)}{\xi(3)} - \frac{\xi'(4)}{\xi(4)} \right) \cdot [1] \otimes E_{-1}^{(3,1)}(1)[- \frac{1}{2}] \\
&\quad + \frac{\rho}{\xi(4)} \cdot (-[2]' \otimes \mathbf{1}[-1] + \frac{1}{2}[2] \otimes \mathbf{1}[-1]') \\
&\quad + \frac{\rho}{\xi(4)} \left(\frac{\gamma}{\rho} - \frac{2\xi'(2)}{\xi(2)} - \frac{\xi'(4)}{\xi(4)} \right) \cdot [2] \otimes \mathbf{1}[-1] \\
&= [4] \otimes \mathbf{1}[-2] \\
&\quad + \frac{\xi(3)}{\xi(4)} \cdot [1] \otimes E_0^{(3,1)}(1)[- \frac{1}{2}] + \frac{\rho}{2\xi(4)} \left(\frac{\xi'(3)}{\xi(3)} - \frac{\xi'(4)}{\xi(4)} \right) \cdot [1] \otimes E_0^{(3,3)}(0)[- \frac{1}{2}] \\
&\quad - \frac{\rho}{\xi(4)} \cdot ([2]' \otimes \mathbf{1}[-1] - \frac{1}{2}[2] \otimes \mathbf{1}[-1]') \\
&\quad + \frac{\rho}{\xi(4)} \left(\frac{\gamma}{\rho} - \frac{2\xi'(2)}{\xi(2)} - \frac{\xi'(4)}{\xi(4)} \right) \cdot [2] \otimes \mathbf{1}[-1].
\end{aligned}$$

Hence we have

$$\begin{aligned}
& \left(\frac{\rho\xi(2)}{\xi(3)\xi(4)} \right)^{-1} \cdot E_{-1}^{(4,2)}(\frac{3}{2})_Q - E_0^{(4,4)}(\frac{1}{2})_Q - E_0^{(4,1)}(1)_Q \\
&= \frac{\rho}{\xi(4)} \left(-\frac{\gamma}{\rho} + \frac{3\xi'(2)}{\xi(2)} \right) \cdot [2] \otimes \mathbf{1}[-1] \\
&\quad + \frac{\rho}{2\xi(4)} \left(-\frac{\gamma}{\rho} + \frac{3\xi'(2)}{\xi(2)} \right) \cdot [1] \otimes E_0^{(3,3)}(0)[- \frac{1}{2}] \\
&= \left(-\frac{\gamma}{\rho} + \frac{3\xi'(2)}{\xi(2)} \right) \cdot E_{-1}^{(4,4)}(\frac{1}{2})_Q.
\end{aligned}$$

By Lemma B.1, this yields the proposition. \square

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SAN DIEGO, LA JOLLA, CA 92093, USA

E-mail address: wgan@math.ucsd.edu

SCHOOL OF MATHEMATICS, INSTITUTE FOR ADVANCED STUDY, PRINCETON, NJ 08540, USA

E-mail address: ichino@math.ias.edu