LOCAL INTERSECTION COHOMOLOGY OF BAILY-BOREL COMPACTIFICATIONS

M. GORESKY[1], G. HARDER, R. MACPHERSON[2], AND A. NAIR[3]

1. Introduction

1.1. Suppose D is a Hermitian symmetric domain, Γ is a neat arithmetic group of automorphisms of D, and $X = \Gamma\backslash D$ is the corresponding locally symmetric space. The Baily-Borel Satake compactification $\widehat{X}$ of X is a projective algebraic variety. The Zucker conjecture (proven by Looijenga [L], Saper and Stern [SS]) states that the complex of sheaves of L^2 differential forms on $\widehat{X}$ is canonically isomorphic to the complex of sheaves $\mathbf{IC}^\bullet(\widehat{X})$ of intersection chains on $\widehat{X}$. Both proofs proceed by showing that, in some sense, these two complexes of sheaves have the same stalk cohomology at any point $x \in \widehat{X}$. These stalk cohomology groups $IH^i_x(\widehat{X})$ are usually viewed as being extremely complicated objects. In this paper (Theorem 6.3) we give an explicit interpretation for the stalk cohomology, together with its weight filtration which arises from mixed Hodge theory. In Theorem 7.8, we evaluate the formula of Theorem 6.3 for the case $X = \Gamma(p)\backslash \mathbf{Sp_4}(\mathbb{R})/\mathbf{U_2}$.

The identity mapping $X \to X$ has a unique continuous extension $\Phi : \overline{X} \to \widehat{X}$ to the "reductive Borel- Serre compactification" $\overline{X}$, which may be thought of as a (nonalgebraic) partial resolution of singularities of $\widehat{X}$. The projection Φ is stratified by the natural stratifications of $\overline{X}$ and of $\widehat{X}$, but the singularities of $\overline{X}$ are particularly explicit and easy to understand. The fiber $\Phi^{-1}(x)$ over a boundary point $x \in \widehat{X}$ is itself the reductive Borel-Serre compactification $\overline{X}_\ell$ of a certain "linear" locally symmetric space (6.1.2),

$$X_\ell = \Gamma_\ell\backslash Q_\ell/A_Q K_\ell. \tag{1.1.1}$$

Our formula expresses the stalk cohomology of the intersection cohomology of $\widehat{X}$ as a direct sum of *weighted cohomology* groups of this auxiliary space $\Phi^{-1}(x)$.

1.2. Weighted cohomology groups for locally symmetric spaces were introduced in [GHM] and the present paper relies heavily on the results and notations of [GHM] and[GM2]. Like intersection cohomology, the weighted cohomology $W^pH^*(\overline{X})$ is the hypercohomology of a complex of sheaves $\mathbf{W}^p\mathbf{C}^\bullet(\overline{X})$ which is obtained by "truncation" of the direct image sheaf

1. partially supported by NSF grant # DMS 9626616 and DMS 9900324.
2. partially supported by NSF grant # DMS 0106522.
3. partially supported by NSF grant # DMS 9304580.

$Ri_*(\mathbb{C})$ (where $i : X \to \overline{X}$ denotes the inclusion). However the weighted cohomology complex is obtained by truncating with respect to weights of a certain torus action, rather than with respect to dimension.

A central result in [GHM] states that the pushforward $\Phi_* \mathbf{W}^\nu \mathbf{C}^\bullet(\overline{X})$ of the weighted cohomology complex (with "middle" weight ν) on $\overline{X}$ is canonically isomorphic to the ("middle") intersection complex $\mathbf{IC}^\bullet(\widehat{X})$ of $\widehat{X}$. It follows that the stalk intersection cohomology at a point $x \in \widehat{X}$ is isomorphic to the hypercohomology of the restriction

$$\mathbf{W}^\nu \mathbf{C}^\bullet | \Phi^{-1}(x) \tag{1.2.1}$$

of the weighted cohomology complex to the fiber $\Phi^{-1}(x)$:

1.3. The key technical achievement in this paper is the identification of the restriction (1.2.1) as a sum of shifted weighted cohomology complexes of the reductive Borel-Serre compactification $\overline{X}_\ell$. In Theorem 4.3 and Corollary 4.6 we show, more generally, that for any weight profile p, the restriction $\mathbf{WC}^\bullet(\overline{X})|\overline{X}_Q$ of the weighted cohomology complex to the closure $\overline{X}_Q$ of any boundary stratum $X_Q \subset \overline{X}$ breaks into a direct sum of weighted cohomology complexes of $\overline{X}_Q$ with shifts. (This result even holds when we drop the assumption that D is Hermitian.)

Combining this with (1.2.1) gives a formula (6.5.1) (notation explained in §6) for the local intersection cohomology at a point x in the Baily-Borel compactification,

$$IH^i_x(\widehat{X}, \mathbf{E}) \cong \bigoplus_{\beta \geq -\rho_Q} \bigoplus_i W^{\nu \boxplus \beta} H^{k-i}(\overline{X}_\ell, \mathbf{H}^i(\mathfrak{N}_Q, \mathbf{E})_\beta). \tag{1.3.1}$$

Theorem 4.3 is one of several properties which make weighted cohomology a somewhat simpler object to study than intersection cohomology. The analogous statements which may be formulated for the intersection cohomology of $\widehat{X}$ or of $\overline{X}$ are false. (The restriction of the intersection complex $\mathbf{IC}^\bullet(\widehat{X})$ to the closure $\widehat{Y} \subset \widehat{X}$ of a boundary stratum $Y \subset \widehat{X}$ may fail to be isomorphic to a sum of intersection cohomology complexes of $\widehat{Y}$.)

The results of [LR] show that the weight filtration from mixed Hodge theory on the stalk cohomology $IH^i_x(\widehat{X})$ is given by the torus weights which define the truncation for weighted cohomology. This gives a precise formula (6.3.1) for the associated graded of the weight filtration of the stalk cohomology.

In [F], J. Franke introduced an important family of invariants, the weighted L^2 cohomology groups of X, which are the Lie algebra cohomology groups of a certain space of functions on X. In [N] it was shown that Franke's weighted L^2 cohomology groups coincide with the weighted cohomology groups. Consequently, the above formula (1.3.1) for the stalk cohomology $IH^*_x(\widehat{X})$ may be translated into Lie algebra cohomology.

Finally, in §7 we evaluate the local cohomology and intersection cohomology Euler characteristic for the Siegel modular threefolds given by principal congruence subgroups of level ≥ 3.

1.4. The authors would like to thank J. Franke, R. Pink, G. Prasad, and M. Stern for useful conversations. We are grateful to an anonymous referee for his many useful comments and we wish to thank the Editor for patiently waiting while this paper was revised. The first and fourth authors would like to thank the Institute for Advanced Study in Princeton N.J. for its hospitality and support.

2. Some linear algebra

2.1. Suppose X is a $\mathbb{Q}$-vector space. Denote by $X^* = \mathrm{Hom}(X, \mathbb{Q})$ the dual vector space. Let $\Delta \subset X^*$ be a finite collection of linearly independent functionals. Define

$$C(\Delta) = \left\{ \sum_{\alpha\in\Delta} m_\alpha \alpha \,\middle|\, m_\alpha \geq 0 \right\} \text{ and } \ker(\Delta) = \bigcap_{\alpha\in\Delta} \ker(\alpha).$$

The set $C(\Delta)$ is called the positive cone spanned by the elements of Δ. If $\nu \in X^*$ define

$$(X^*)_{\geq\nu(\Delta)} = \{\gamma \in X^* \,|\gamma - \nu \in C(\Delta)\}\,. \tag{2.1.1}$$

Every element $\gamma \in (X^*)_{\geq\nu(\Delta)}$ satisfies $\gamma|\ker(\Delta) = \nu|\ker(\Delta)$.

Let $W = \ker(\Delta)$ and let $\{t_\alpha \,|\alpha \in \Delta\} \subset X/W$ be the basis of X/W which is dual to the basis determined by Δ. If $\gamma \in X^*$ satisfies $\gamma|W = \nu|W$ then $\gamma - \nu$ passes to a linear functional on X/W. Hence

$$(X^*)_{\geq\nu(\Delta)} = \{\gamma \in X^*|\ \gamma|W = \nu|W \text{ and } \langle\gamma - \nu, t_\alpha\rangle \geq 0, \forall\alpha \in \Delta\} \tag{2.1.2}$$

where $\langle\cdot,\cdot\rangle : (X/W)^* \times (X/W) \to \mathbb{Q}$ is the canonical pairing.

Fix a subset $J \subset \Delta$ and set $Y = \ker(J)$ and $Z = \ker(\Delta - J)$. The sequence

$$0 \to W \to Y \oplus Z \to X \to 0$$

is exact, where the maps are given by $w \mapsto (w, -w)$ and $y \oplus z \mapsto y + z$. It follows by duality that if $\nu \in X^*$ and if $\beta \in Y^*$ and if $\nu|W = \beta|W$ then there is a unique element

$$q = \nu \boxplus \beta \in X^*$$

so that $q(y + z) = \beta(y) + \nu(z)$ for all $y \in Y$ and $z \in Z$. Moreover, $q|W = \nu|W = \beta|W$.

If $\alpha \in \Delta - J$ then $t_\alpha \in Y/W$. The elements of $\Delta - J$ restrict to a basis of $(Y/W)^*$ whose dual basis is $\{t_\alpha \,|\alpha \in \Delta - J\}$. The composition $Z/W \subset X/W \to X/Y$ is an isomorphism. If $\alpha \in J$ then $t_\alpha \in Z/W$ projects to a nonzero element $\bar{t}_\alpha \in X/Y$. The elements of J determine a basis of $(X/Y)^*$ whose dual basis is $\{\bar{t}_\alpha \,|\alpha \in J\}$.

2.2. **Proposition.** *Let X be a rational vector space, let $\Delta \subset X^*$ be a finite set of linearly independent elements, and set $W = \ker(\Delta)$. Let $J \subset \Delta$ and let $Y = \ker(J)$. Fix $\nu, \gamma \in X^*$ with $\nu|W = \gamma|W$. Let $\beta = \gamma|Y \in Y^*$. Then*

$$\gamma \in (X^*)_{\geq\nu(\Delta)} \iff \beta \in (Y^*)_{\geq\nu(\Delta-J)} \text{ and } \gamma \in (X^*)_{\geq\nu\boxplus\beta(J)}$$

(where we have also written ν for its restriction to Y).

2.3. **Proof.** If $\alpha \in \Delta - J$ then $t_\alpha \in Y/W$ and the first condition says that $\langle \beta - \nu, t_\alpha \rangle \geq 0$. But $\beta = \gamma | Y$ so $\langle \gamma - \nu, t_\alpha \rangle \geq 0$ for all $\alpha \in \Delta - J$.

If $\alpha \in J$ then $t_\alpha \in Z/W$. Let $\hat{t}_\alpha \in Z \subset X$ be any lift of t_α; then $\nu \boxplus \beta(\hat{t}_\alpha) = \nu(\hat{t}_\alpha)$. Therefore the second condition says

$$0 \leq \langle \gamma - \nu \boxplus \beta, \bar{t}_\alpha \rangle = \langle \gamma - \nu \boxplus \beta, \hat{t}_\alpha \rangle = \langle \gamma - \nu, \hat{t}_\alpha \rangle = \langle \gamma - \nu, t_\alpha \rangle$$

for all $\alpha \in J$. The reverse implication is similar. $\square$

3. Weighted cohomology

3.1. **Locally symmetric spaces.** Algebraic groups will be designated by bold face type ($\mathbf{G}, \mathbf{P}$, etc.). If an algebraic group is defined over the real numbers $\mathbb{R}$ then its group of real points will be in Roman ($G = \mathbf{G}(\mathbb{R})$). Throughout this paper we fix a connected reductive group $\mathbf{G}$ which is defined over $\mathbb{Q}$. Let $\mathbf{S_G}$ be the maximal $\mathbb{Q}$-split torus in the center of $\mathbb{G}$ and let $A_G = \mathbf{S_G}(\mathbb{R})^0$ be the identity component of its group of real points. Let K be a maximal compact subgroup of G, set $K' = KA_G$ and $D = G/K'$. This is a homogeneous space on which G acts transitively. The group K' corresponds to a choice of basepoint $x_0 \in D$. We also fix a neat arithmetic subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$ and set $X = \Gamma \backslash D$. By abuse of terminology we will refer to X as a locally symmetric space.

In this paper, $\widetilde{X}$ denotes the Borel-Serre compactification of X ([BS]) and $\overline{X}$ denotes the reductive Borel-Serre compactification ([Z3] §4.2, [GHM] §8). If X is Hermitian then $\widehat{X}$ will denote the Baily-Borel Satake compactification.

3.2. **Parabolic subgroups.** Let $\mathbf{P}$ be a rationally defined parabolic subgroup of $\mathbf{G}$. Then we have the following groups:

- $\boldsymbol{\mathcal{U}}_\mathbf{P}$ = unipotent radical of $\mathbf{P}$
- $\mathfrak{N}_\mathbf{P} = \mathrm{Lie}(\boldsymbol{\mathcal{U}}_\mathbf{P})$
- $\mathbf{R_d P}$ = the $\mathbb{Q}$-split radical of $\mathbf{P}$ ([BS] §0.3)
- $\mathbf{L_P}$ = the Levi quotient, $\nu_P : \mathbf{P} \to \mathbf{L_P}$ the projection
- $\mathbf{S_P} = \mathbf{R_d P}/\boldsymbol{\mathcal{U}}_\mathbf{P}$
- $\chi^*_\mathbb{Q}(\mathbf{S_P}) = \chi^*(\mathbf{S_P}) \otimes_\mathbb{Z} \mathbb{Q}$
- $\chi_*^\mathbb{Q}(\mathbf{S_P}) = \mathrm{Hom}(\chi^*_\mathbb{Q}(\mathbf{S_P}), \mathbb{Q})$
- $A_P = \mathbf{S_P}(\mathbb{R})^0$
- $\Gamma_P = \Gamma \cap P$ and $\Gamma_L = \nu_P(\Gamma_P)$
- $K_P = K_P(x_0) = K \cap P$ and $K'_P = K' \cap P$.

The torus $\mathbf{S_P}$ may also be identified as the greatest $\mathbb{Q}$-split torus in the center of $\mathbf{L_P}$. It contains $\mathbf{S_G}$ and we denote the quotient by $\mathbf{S'_P} = \mathbf{S_P}/\mathbf{S_G}$.

The choice of basepoint $x_0 \in D$ determines a Cartan involution $\theta : G \to G$ with fixed point set $K(x_0)$. There is a unique lift ([BS] §1.6, §1.9) $i_{x_0} : L_P \to P$ of the Levi quotient such that the image $L_P(x_0) = i_{x_0}(L_P)$ is θ-stable. This determines a lift of $\mathbf{S_P}(\mathbb{R})$. Note that $K_P \subset L_P(x_0)$.

Fix once and for all a minimal rational parabolic subgroup $\mathbf{P_0} \subset \mathbf{G}$. The (rational "relative") root system $\Phi(\mathbf{S_{P_0}}, \mathbf{G})$ admits a linear order such that the positive roots are those in $\mathfrak{N}_0 = \mathrm{Lie}(\boldsymbol{\mathcal{U}}_{\mathbf{P_0}})$. Let Δ denote the set of simple (rational) roots. The rational parabolic subgroups containing $\mathbf{P_0}$ are called *standard*. They form a unique set of representatives of the $\mathbf{G}(\mathbb{Q})$ conjugacy classes of rational parabolic subgroups of $\mathbb{G}$. They are in one to one correspondence with subsets $J \subset \Delta$, with the parabolic subgroup $\mathbf{P}(J)$ corresponding to J determined by the condition that

$$\mathbf{S}_{\mathbf{P}(J)} \subset \ker(\alpha) \text{ for all } \alpha \in J.$$

The (restrictions of the) roots $\alpha \in \Delta - J$ form a basis of the rational (quasi-) character module $\chi^*_{\mathbb{Q}}(\mathbf{S'}_{\mathbf{P}(J)})$. Denote by $\Delta_{\mathbf{P}(J)}$ the collection of restrictions $\left\{\alpha|\mathbf{S}_{\mathbf{P}(J)}\right\}$ (for $\alpha \in \Delta - J$), which we refer to as the set of simple roots of $\mathbf{P}(J)$ occurring in $\mathfrak{N}_J = \mathrm{Lie}(\boldsymbol{\mathcal{U}}_{\mathbf{P}(J)})$. These notions extend to arbitrary rational parabolic subgroups $\mathbf{P} \subset \mathbf{G}$ by conjugation.

3.3. **Two parabolic subgroups.** If $\mathbf{P} \subset \mathbf{Q} \subseteq \mathbf{G}$ are rational parabolic subgroups of $\mathbf{G}$, then $\boldsymbol{\mathcal{U}}_{\mathbf{Q}} \subset \boldsymbol{\mathcal{U}}_{\mathbf{P}}$ and $\mathbf{R_d Q} \subset \mathbf{R_d P}$. This gives an embedding $\mathbf{S_Q} \hookrightarrow \mathbf{S_P}$. Let $\nu_Q : \mathbf{Q} \to \mathbf{L_Q}$ denote the projection to the Levi quotient and set $\overline{\mathbf{P}} = \nu_Q(\mathbf{P}) = \mathbf{P}/\boldsymbol{\mathcal{U}}_{\mathbf{Q}}$. Then $\boldsymbol{\mathcal{U}}_{\overline{\mathbf{P}}} = \boldsymbol{\mathcal{U}}_{\mathbf{P}}/\boldsymbol{\mathcal{U}}_{\mathbf{Q}}$ and the resulting isomorphism

$$\mathbf{L}_{\overline{\mathbf{P}}} = \overline{\mathbf{P}}/\boldsymbol{\mathcal{U}}_{\overline{\mathbf{P}}} = (\mathbf{P}/\boldsymbol{\mathcal{U}}_{\mathbf{Q}})/(\boldsymbol{\mathcal{U}}_{\mathbf{P}}/\boldsymbol{\mathcal{U}}_{\mathbf{Q}}) \cong \mathbf{L_P}$$

induces an isomorphism $\mathbf{S_P} \cong \mathbf{S}_{\overline{\mathbf{P}}}$. However, $\mathbf{P}$ is regarded as a parabolic subgroup of $\mathbf{G}$, so $\mathbf{S'_P} = \mathbf{S_P}/\mathbf{S_G}$, while $\overline{\mathbf{P}}$ is regarded as a parabolic subgroup of $\mathbf{L_Q}$ so $\mathbf{S'}_{\overline{\mathbf{P}}} = \mathbf{S_P}/\mathbf{S_Q}$.

3.4. **Boundary strata.** Let $\widetilde{X}$ denote the Borel-Serre compactification of X ([BS]) and let $\overline{X}$ denote the reductive Borel-Serre compactification of X ([Z3] §4.2, [GHM] §8). These spaces are Whitney stratified by their boundary strata $Y_P \subset \widetilde{X}$ and $X_P \subset \overline{X}$ which are in one to one correspondence with Γ-conjugacy classes of proper rational parabolic subgroups of $\mathbf{G}$. A choice of representative $\mathbf{P}$ in this Γ-conjugacy class determines an identification

$$Y_P = \Gamma_P \backslash P / K_P A_P(x_0) \tag{3.4.1}$$

and an identification

$$X_P = \Gamma_P \backslash P / \mathcal{U}_P K_P A_P(x_0) \tag{3.4.2}$$

The identity map $X \to X$ has a unique continuous extension to a mapping $\pi : \widetilde{X} \to \overline{X}$. Then π is surjective, takes strata to strata, and its restriction to each boundary stratum Y_P is the smooth proper fiber bundle $\pi_P : Y_P \to X_P$ which is given by collapsing the "orbits" of $\boldsymbol{\mathcal{U}}_P$ ([GHM] §7.4).

3.5. **Weighted cohomology.** Fix a minimal rational parabolic subgroup $\mathbf{P_0} \subset \mathbf{G}$. A *weight profile* is an element of $\chi^*_{\mathbb{Q}}(\mathbf{S_{P_0}})$. (This is slightly more general than the definition in [GHM], but it agrees with the definition in [GKM] and [GM2].) For any standard rational parabolic subgroup $\mathbf{P} \supset \mathbf{P_0}$ the weight profile together with the set $\Delta_P \subset \chi^*_{\mathbb{Q}}(\mathbf{S_P})$ determines a "high" subset of weights as in (2.1.1) and (2.1.2),

$$\chi^*_{\mathbb{Q}}(\mathbf{S_P})_{\geq p(\Delta_P)} = \left\{ \gamma \in \chi^*_{\mathbb{Q}}(\mathbf{S_P}) \middle| \ \gamma|\mathbf{S_G} = p|\mathbf{S_G} \text{ and } \langle \gamma - p, t_\alpha \rangle \geq 0, \ \forall \alpha \in \Delta_P \right\} \tag{3.5.1}$$

where $\{t_\alpha \mid \alpha \in \Delta_P\}$ is the basis of the rational co-character module $\chi_*^{\mathbb{Q}}(\mathbf{S'_P})$ which is dual to the basis of $\chi^*_{\mathbb{Q}}(\mathbf{S'_P})$ determined by Δ_P. When there is no possibility of confusion we will abbreviate the notation to $\chi^*_{\mathbb{Q}}(\mathbf{S_P})_{\geq p}$ (which agrees with the notation of [GKM] and [GM2], and which was denoted $\chi^*_{\mathbb{Q}}(\mathbf{S_P})_+$ in [GHM]). If a rational vectorspace V is a module over $\mathbf{S_P}$ let V_α be the subspace of weight $\alpha \in \chi^*_{\mathbb{Q}}(\mathbf{S_P})$ and set

$$V_{\geq p} = V_{\geq p(\Delta_P)} = \bigoplus_{\alpha \in \chi^*_{\mathbb{Q}}(\mathbf{S_P})_{\geq p}} V_\alpha. \tag{3.5.2}$$

Let $\psi : \mathbf{G} \to \mathbf{GL}(E)$ be an irreducible representation of $\mathbf{G}$ on some finite dimensional complex vectorspace E and let $\mathbf{E} = E \times_\Gamma D$ be the resulting local coefficient system on X. Since E is irreducible, the torus $\mathbf{S_G}$ acts by a single character $\lambda_E \in \chi^*_{\mathbb{Q}}(\mathbf{S_G})$. Let p be a weight profile such that $p|\mathbf{S_G} = \lambda_E$. The construction of [GHM] defines a weighted complex of sheaves $\mathbf{W^pC^\bullet}(\overline{X}, \mathbf{E})$ on the reductive Borel-Serre compactification $\overline{X}$ of X. This is a (cohomologically) constructible complex of sheaves on $\overline{X}$ which is obtained by truncating the sheaf of smooth differential forms $i_*\Omega^\bullet(X, \mathbf{E})$ along the boundary strata X_P so as to keep only the differential forms with "weights" in $\chi^*_{\mathbb{Q}}(\mathbf{S_P})_{\geq p}$. (Here, $i : X \to \overline{X}$ denotes the inclusion of X into its reductive Borel-Serre compactification.)

Let $\mathfrak{N}_P = \mathrm{Lie}(\mathcal{U}_P)$ be the Lie algebra of the unipotent radical of P. The Lie algebra cohomology $H^i(\mathfrak{N}_P, E)$ is a module over $\mathbf{L_P}$ and hence also over $\mathbf{S_P}$. The torus $\mathbf{S_G}$ acts on $H^*(\mathfrak{N}_P, E)$ via the weight λ_E. Then ([GHM] §17) the stalk cohomology at a (singular) point $x \in X_P$ of the weighted cohomology complex $\mathbf{W^pC^\bullet}(\overline{X}, \mathbf{E})$ is the (finite) sum,

$$W^pH^i_x \cong H^i(\mathfrak{N}_P, E)_{\geq p} = \bigoplus_{\beta \in \chi^*_{\mathbb{Q}}(\mathbf{S_P})_{\geq p}} H^i(\mathfrak{N}_P, E)_\beta. \tag{3.5.3}$$

3.6. For some purposes it is necessary to consider weight truncations of the form

$$\chi^*_{\mathbb{Q}}(\mathbf{S_P})_{> p} = \left\{ \gamma \in \chi^*_{\mathbb{Q}}(\mathbf{S_P}) \middle| \ \gamma|\mathbf{S_G} = p|\mathbf{S_G} \text{ and } \langle \gamma - p, t_\alpha \rangle > 0, \forall \alpha \in \Delta_P \right\}.$$

Since only finitely many weights occur in $H^*(\mathfrak{N}_P, E)$, for any $\epsilon > 0$ sufficiently small we have

$$\bigoplus_{\beta \in \chi^*_{\mathbb{Q}}(\mathbf{S_P})_{\geq p+\epsilon}} H^i(\mathfrak{N}_P, E)_\beta = \bigoplus_{\beta \in \chi^*_{\mathbb{Q}}(\mathbf{S_P})_{> p}} H^i(\mathfrak{N}_P, E)_\beta.$$

The weighted cohomology sheaf constructed with respect to this weight truncation will be denoted $\mathbf{W^{p+\epsilon}C^\bullet}(\overline{X}, \mathbf{E})$.

Choose a Cartan subgroup $\mathbf{H}$ and a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ so that

$$\mathbf{S_{P_0}} \subset \mathbf{H} \subset \mathbf{B} \subset \mathbf{P_0} \subset \mathbf{G}.$$

Let $\Phi^+ = \Phi^+(\mathbf{H}(\mathbb{C}), \mathbf{G}(\mathbb{C}))$ be the resulting set of positive roots and let $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+}$. The "lower middle" weight profile is $\nu = -\rho|\mathbf{S_{P_0}}$, the "upper middle" weight profile is $\nu = -\rho|\mathbf{S_{P_0}} + \epsilon$. (The modification by ϵ corresponds exactly to the $\pm \log$ modification which occurs in the weighted L^2 theory.) The "dualizing" weight profile is $d = -2\rho|\mathbf{S_{P_0}} + \epsilon$. The weighted cohomology sheaf $\mathbf{W^dC^\bullet}(\overline{X}, \mathbb{C})$ is canonically (quasi-) isomorphic ([GHM] §19.4) to the dualizing complex $\mathbb{D}_{\overline{X}}$ on $\overline{X}$.

3.7. **Duality.** A morphism $E_1 \otimes E_2 \to E$ of irreducible representations of $\mathbf{G}$ induces a morphism of (complexes of) sheaves,

$$\mathbf{W^pC^\bullet}(\overline{X}, \mathbf{E}_1) \otimes \mathbf{W^qC^\bullet}(\overline{X}, \mathbf{E}_2) \to \mathbf{W^{p+q}C^\bullet}(\overline{X}, \mathbf{E}).$$

Let E be an irreducible representation of $\mathbf{G}$ and let $p \in \chi^*_{\mathbb{Q}}(\mathbf{S_0})$ be a weight profile such that $p|\mathbf{S_G}$ coincides with the character λ_E by which $\mathbf{S_G}$ acts on E. Let E^* be the dual representation and let $q = d - p$ be the "dual" weight profile. Then the resulting morphism

$$\mathbf{W^pC^\bullet}(\overline{X}, \mathbf{E}) \otimes \mathbf{W^qC^\bullet}(\overline{X}, \mathbf{E}^*) \to \mathbf{W^dC^\bullet}(\overline{X}, \mathbb{C}) = \mathbb{D}_{\overline{X}}$$

is a (Borel-Moore-) Verdier dual pairing. In particular, the upper and lower middle weight profiles are dual.

3.8. One may drop the assumption that the representation E is irreducible, requiring instead that in the $\mathbf{S_G}$-isotypical decomposition $E = \bigoplus_\lambda E_\lambda$, each weight $\lambda \in \chi^*_{\mathbb{Q}}(\mathbf{S_G})$ appears at most once and each summand E_λ is irreducible. Sums of this form (with shifts) appear in the restriction to the boundary (Theorem 4.3).

3.9. The singularities of $\overline{X}$ are relatively easy to understand and the weighted cohomology complex on $\overline{X}$ is relatively simple. If X is Hermitian then the identity mapping $X \to X$ has a unique continuous extension $\Phi : \overline{X} \to \widehat{X}$ [Z3]. The pushforward $R\Phi_*(\mathbf{W}^\nu\mathbf{C^\bullet}(\overline{X}, \mathbf{E}))$ of the middle weighted cohomology is canonically isomorphic to the middle intersection complex $\mathbf{IC^\bullet}(\widehat{X}, \mathbf{E})$ which, by the Zucker conjecture ([L], [SS]) is canonically isomorphic to the sheaf of L^2 differential forms with coefficients in $\mathbf{E}$. So the weighted cohomology complex on $\overline{X}$ may be thought of as a sort of (non algebraic) partial resolution of singularities of $\widehat{X}$ together with its sheaf of L^2 differential forms. (In fact, the relation with L^2 cohomology may be described ([N]) completely in terms of $\overline{X}$.)

3.10. **Weighted L^2 cohomology.** For completeness we sketch how the results in this paper may be translated into the language of Lie algebra cohomology using Franke's theory [F] of weighted L^2 cohomology groups.

An element $\lambda \in \chi^*_{\mathbb{Q}}(\mathbf{S_{P_0}})$ determines (via reduction theory) a certain real-valued weight function ([N] §1.6) w_λ on $\Gamma A_G \backslash G$. When $\lambda = 0$ we have $w_\lambda = 1$. For any element D in the universal enveloping algebra of $\mathfrak{g} = \mathrm{Lie}(G)$ there is a left-invariant differential operator R_D which acts on the smooth functions on $\Gamma\backslash G$. (For $D \in \mathfrak{g}$ this is simply the derivative of the right regular representation). Let $\xi \in \chi^*_{\mathbb{Q}}(\mathbf{S_G})$. Let $S_{\lambda,\xi}(\Gamma)$ be the space of smooth $\mathbb{C}$-valued functions f on $\Gamma\backslash G$ satisfying

(1) $f(ag) = \xi(a)f(g)$ for all $a \in A_G$ and
(2) $w_\lambda \xi^{-1} R_D f$ is square-integrable on $\Gamma A_G\backslash G$ for every D.

This space is a $(\mathfrak{g}, K)$-module ([V]). When $\lambda = 0$ it is the familiar space of smooth uniformly L^2 functions. Its $(\mathfrak{g}, K)$-cohomology is often infinite-dimensional.

Let $\epsilon \in \chi^*_{\mathbb{Q}}(\mathbf{S_{P_0}})$ be any dominant weight. Let $S_{\lambda+\log,\xi}(\Gamma) \supset S_{\lambda,\xi}(\Gamma)$ be the module of smooth functions for which

$$w_\lambda \log(w_\epsilon)^m \xi^{-1} R_D f \in L^2(\Gamma A_G\backslash G) \tag{3.10.1}$$

for every positive integer m and for every D. This is also a $(\mathfrak{g}, K)$-module and it always has finite-dimensional cohomology. The main result of [N], extended to reductive groups, gives an isomorphism

$$W^p H^i(\overline{X}, \mathbf{E}) \cong H^i(\mathfrak{g}, K; S_{\lambda+\log,\xi}(\Gamma) \otimes E) \tag{3.10.2}$$

where $\lambda = (-\rho - p)|\mathbf{S_{P_0}}$ and $\xi = -(p|\mathbf{S_G}) = -\lambda_E$. Here ρ is the half-sum of positive roots (see 4.5). In fact, there is a cohomologically constructible complex of sheaves on $\overline{X}$ with (hyper)cohomology equal to the right-hand side of (3.10.2). The isomorphism (3.10.2) is induced from an (explicit and natural) quasi-isomorphism of this object with $\mathbf{W}^p\mathbf{C}^\bullet(\bar{X}, \mathbf{E})$.

4. Computations along boundary strata

4.1. **The boundary stratum.** In this section we restrict the weighted cohomology sheaf to the *closure* of a boundary stratum in the reductive Borel-Serre compactification, and state (in Theorem 4.3) that this restriction decomposes as a direct sum of weighted cohomology sheaves of the closure of the boundary stratum. Throughout this section we fix a weight profile $p \in \chi^*_{\mathbb{Q}}(\mathbf{S_{P_0}})$ and we fix a local system $\mathbf{E}$ on X which corresponds to an irreducible complex representation of $\mathbf{G}$ for which $\mathbf{S_G}$ acts through the character $\lambda_E = p|\mathbf{S_G}$. Fix a boundary stratum X_Q (corresponding to a proper rational parabolic subgroup $\mathbf{Q} \subset \mathbf{G}$). The closure $\overline{X}_Q$ coincides with the reductive Borel-Serre compactification of X_Q which we consider to be the locally symmetric space

$$X_Q = \Gamma_L\backslash L_Q/K_Q A_Q$$

associated to the reductive group $\mathbf{L_Q}$ (notation as in §3.2). The restriction $\mathbf{W^pC^\bullet}(\overline{X},\mathbf{E})|X_Q$ to the interior of the boundary stratum is quasi-isomorphic to the sheaf of differential forms $\mathbf{\Omega^\bullet}(X_Q,\mathbf{H}^*(\mathfrak{N}_Q,E)_{\ge p}))$ (see [GHM] §14.1.2). Here, $\mathbf{H}^*(\mathfrak{N}_Q,E)_{\ge p}$ is the local system on X_Q associated to the $\mathbf{L_Q}$ module $H^*(\mathfrak{N}_Q,E)_{\ge p}$.

4.2. **Weight profile on the boundary stratum.** The boundary stratum X_Q corresponds to a rational parabolic subgroup $\mathbf{Q}\subset\mathbf{G}$ which we may assume to be standard ($\mathbf{Q}\supset\mathbf{P_0}$) and which therefore corresponds to a subset $J\subset\Delta$ of the simple roots, so that

$$\chi^*_{\mathbb{Q}}(\mathbf{S_G})=\ker(\Delta)\subset\chi^*_{\mathbb{Q}}(\mathbf{S_Q})=\ker(J)\subset\chi^*_{\mathbb{Q}}(\mathbf{S_{P_0}})$$

in the notation of §2. The elements of $\Delta-J$ restrict to a linearly independent set $\Delta_Q\subset\chi^*_{\mathbb{Q}}(\mathbf{S_Q})$ and determine a basis of $\chi^*_{\mathbb{Q}}(\mathbf{S'_Q})$ (where $\mathbf{S'_Q}=\mathbf{S_Q}/\mathbf{S_G}$). Let $\{t_\alpha|\ \alpha\in\Delta_Q\}$ denote the dual basis of $\chi^*_{\mathbb{Q}}(\mathbf{S'_Q})$. Let $\beta\in\chi^*_{\mathbb{Q}}(\mathbf{S_Q})_{\ge p}$, that is, $\beta|\mathbf{S_G}=p|\mathbf{S_G}$ and $\langle\beta-p,t_\alpha\rangle\ge 0$ for all $\alpha\in\Delta_Q$. The projection $\nu_Q:\mathbf{Q}\to\mathbf{L_Q}$ determines an identification $\mathbf{S_{P_0}}\cong\mathbf{S_{\overline{P}_0}}$ where

$$\mathbf{\overline{P}_0}=\nu_P(\mathbf{P_0})\subset\mathbf{L_Q}$$

is the corresponding minimal rational parabolic subgroup of $\mathbf{L_Q}$. The elements of $J\subset\Delta$ may be identified with the set $\Delta_{\overline{P}_0}$ of simple rational roots of $\mathbf{S_{\overline{P}_0}}$ occurring in the unipotent radical of $\mathbf{\overline{P}_0}$.

As in §2 define

$$p\boxplus\beta\in\chi^*_{\mathbb{Q}}(\mathbf{S_{\overline{P}_0}})\tag{4.2.1}$$

to be the unique rational (quasi-) character such that $p\boxplus\beta(y+z)=\beta(y)+p(z)$ for all $y\in\chi_*^{\mathbb{Q}}(\mathbf{S_Q})$ and all $z\in\ker(\Delta-J)$. We may consider $p\boxplus\beta$ to be a weight profile for the reductive Borel-Serre compactification $\overline{X}_Q$ of the locally symmetric space corresponding to the reductive group $\mathbf{L_Q}$. It satisfies $(p\boxplus\beta)|\mathbf{S_Q}=\beta$. The proof of the following theorem will appear in §5.

4.3. **Theorem.** *Let $p\in\chi^*_{\mathbb{Q}}(\mathbf{S_0})$ be a weight profile. The restriction of the weighted cohomology complex to the closure of the boundary stratum X_Q is quasi-isomorphic to the (finite) direct sum of weighted cohomology sheaves:*

$$\mathbf{W^pC^\bullet}(\overline{X},\mathbf{E})|\overline{X}_Q\cong\bigoplus_{\beta\in\chi^*_{\mathbb{Q}}(\mathbf{S_Q})_{\ge p}}\bigoplus_i\mathbf{W^{p\boxplus\beta}C^\bullet}(\overline{X}_Q,\mathbf{H}^i(\mathfrak{N}_Q,E)_\beta)[-i]\tag{4.3.1}$$

where $[-i]$ denotes a shift in degree: $\mathbf{C}^k[-i]=\mathbf{C}^{k-i}$.

4.4. The following remarks will be needed for the proof of Theorem 4.3. Suppose $\mathbf{P}\subset\mathbf{Q}$ is another standard rational parabolic subgroup. Then $\mathbf{S_G}\subset\mathbf{S_Q}\subset\mathbf{S_P}\subset\mathbf{S_{P_0}}$. Let $p\in\chi^*_{\mathbb{Q}}(\mathbf{S_{P_0}})$, $\beta\in\chi^*_{\mathbb{Q}}(\mathbf{S_Q})$, and suppose $\beta|\mathbf{S_G}=p|\mathbf{S_G}$ as above. We claim that

$$(p\boxplus\beta)|\mathbf{S_P}=(p|\mathbf{S_P})\boxplus\beta\tag{4.4.1}$$

where, for the sake of notational simplicity, we confuse the restriction to $\mathbf{S_P}$ with the restriction to $\chi_*^{\mathbb{Q}}(\mathbf{S_P})$. The right hand side of this equation needs some explanation.

Let Δ_P be the simple roots for $\mathbf{P}$. They determine a basis of $\chi^*_{\mathbb{Q}}(\mathbf{S_P}/\mathbf{S_G})$. Let $J' \subset \Delta_P$ be the subset corresponding to $\mathbf{Q}$, that is,

$$\chi_*^{\mathbb{Q}}(\mathbf{S_Q}) = \ker(J'). \tag{4.4.2}$$

Here, we consider the elements of J' to be linear functionals on the vectorspace

$$X = \chi_*^{\mathbb{Q}}(\mathbf{S_P}). \tag{4.4.3}$$

Then we may apply the construction of §2 with

$$Y = \chi_*^{\mathbb{Q}}(\mathbf{S_Q}) = \ker(J')$$
$$Z = \ker(\Delta_P - J') \subset \chi^*_{\mathbb{Q}}(\mathbf{S_P})$$

to obtain an element $(p|\mathbf{S_P}) \boxplus \beta \in \chi^*_{\mathbb{Q}}(\mathbf{S_P})$ such that

$$(p|\mathbf{S_P}) \boxplus \beta(y+z) = \beta(y) + p(z)$$

for all $y \in Y$ and all $z \in Z$. This defines the right hand side of (4.4.1).

The proof of the claim is a straightforward matter of bookkeeping. The sets $J' \subset \Delta_P$ may be considered as subsets of Δ, that is, as characters of the larger torus $\mathbf{S_{P_0}}$. Set

$$\Delta_P = \Delta - I \text{ so } \chi_*^{\mathbb{Q}}(\mathbf{S_P}) = \ker(I)$$
$$\Delta_Q = \Delta - J \text{ so } \chi_*^{\mathbb{Q}}(\mathbf{S_Q}) = \ker(J).$$

It follows that $J = J' \cup I$ (disjoint union) and that

$$Y = \ker(J') \cap \chi^*_{\mathbb{Q}}(\mathbf{S_P}) \text{ and } Z = \ker(\Delta_P - J') \cap \chi^*_{\mathbb{Q}}(\mathbf{S_P}).$$

To verify the claim we have to check that the subspaces Y and Z used to define the right hand side of (4.4.1) agree with the subspaces (say, Y' and Z') used to define $(p \boxplus \beta)|\mathbf{S_P}$ on the left hand side. Clearly, $Y = Y' = \chi_*^{\mathbb{Q}}(\mathbf{S_Q})$. Moreover,

$$Z' = \ker(\Delta - J) \cap \chi_*^{\mathbb{Q}}(\mathbf{S_P}) = \ker(\Delta - I - J') \cap \chi_*^{\mathbb{Q}}(\mathbf{S_P}) = \ker(\Delta_P - J) \cap \chi_*^{\mathbb{Q}}(\mathbf{S_P})$$

which completes the proof of (4.4.1).

4.5. **Kostant's theorem.** Fix a Cartan subgroup and a Borel subgroup, $\mathbf{H}(\mathbb{C}) \subset \mathbf{B}(\mathbb{C}) \subset \mathbf{L_Q}(x_0)(\mathbb{C})$. Let $\Phi^+ = \Phi^+(\mathbf{H}(\mathbb{C}), \mathbf{G}(\mathbb{C}))$ be the resulting system of positive roots for $\mathbf{G}$ and let $W_Q = W(\mathbf{H}(\mathbb{C}), \mathbf{L_Q}(\mathbb{C}))$ be the Weyl group for $\mathbf{L_Q}(\mathbb{C})$. For each $w \in W_Q$ set

$$\Phi^+(w) = \left\{ \alpha \in \Phi^+ \;\middle|\; w^{-1}\alpha \in \Phi^- \right\}.$$

Then $|\Phi^+(w)| = \ell(w)$ is the length of w. The set

$$W_Q^1 = \left\{ w \in W_Q \;\middle|\; \Phi^+(w) \subset \Phi(\mathbf{H}, \mathfrak{N}_Q(\mathbb{C})) \right\} \tag{4.5.1}$$

consists of the unique element of minimal length from each of the cosets $W_Q x \in W_Q \backslash W$ ([Sp] §10.2, [V] §3.2.1). (Here, $\Phi(\mathbf{H}, \mathfrak{N}_Q(\mathbb{C}))$ denotes the set of (positive) roots which occur in the nilradial $\mathfrak{N}_Q(\mathbb{C})$.) Let Λ be the highest weight of the irreducible representation $\psi : \mathbf{G} \to \mathbf{GL}(E)$ and let V_μ be the irreducible $\mathbf{L_Q}(\mathbb{C})$ module with highest weight μ. Then

Kostant's theorem ([K] §5.14, [V] §3.2.16) states that, as a representation of $\mathbf{L_Q}$, the Lie algebra cohomology $H^*(\mathfrak{N}_Q, E)$ is given by

$$H^*(\mathfrak{N}_Q, E) \cong \bigoplus_{w \in W_Q^1} V_{w(\Lambda+\rho)-\rho}[-\ell(w)]$$

where $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$ is one-half the sum of the positive roots of $\mathbf{G}$. (In this formula, we view $V_{w(\Lambda+\rho)-\rho}$ as a trivial complex concentrated in degree 0, so that $V_{w(\Lambda+\rho)-\rho}[-\ell(w)]$ is concentrated in degree $\ell(w)$. We use H^* rather than $H^\bullet$ to indicate that the cohomology is viewed as a complex with trivial differential.)

It follows that the expression (4.3.1) may be rewritten as

$$\bigoplus_{w \in W_Q^p(E)} \mathbf{W}^{\mathbf{p} \boxplus \boldsymbol{\beta}(\mathbf{w})} \mathbf{C}^\bullet(\overline{X}_Q, \mathbf{V}_{w(\Lambda+\rho)-\rho})[\ell(w)] \tag{4.5.2}$$

where the symbol $\mathbf{V}_\alpha$ denotes the local system $V_\alpha \times_{\Gamma_L} (Q/K_Q A_Q) \to X_Q$ which arises from an irreducible representation V_α of $\mathbf{L_Q}$ with highest weight α, and where

$$\beta(w) = (w(\Lambda+\rho)-\rho)|\mathbf{S_Q} \in \chi^*_{\mathbb{Q}}(\mathbf{S_Q}) \tag{4.5.3}$$

$$W_Q^p(E) = \left\{ w \in W_Q^1 \,\middle|\, \beta(w) \in \chi^*_{\mathbb{Q}}(\mathbf{S_Q})_{\geq p} \right\}. \tag{4.5.4}$$

See [GHM] §11.5, §11.7. This gives the following

4.6. **Corollary.** *Suppose E is an irreducible representation of $\mathbf{G}$ with highest weight Λ. Then the restriction of the weighted cohomology sheaf to the closure $\overline{X}_Q$ of the boundary stratum X_Q decomposes as the sum*

$$\mathbf{W}^{\mathbf{p}}\mathbf{C}^\bullet(\overline{X}, \mathbf{E})|\overline{X}_Q \cong \bigoplus_{w \in W_Q^p(E)} \mathbf{W}^{p\boxplus\beta(w)}\mathbf{C}^{\bullet-\ell(w)}(\overline{X}_Q, \mathbf{V}_{w(\Lambda+\rho)-\rho}).$$

5. Proof of Theorem 4.3

5.1. **Special differential forms.** Let $\mathbf{Q} \subset \mathbf{P}$ be a rational parabolic subgroup, let $e_Q = Q/K_Q A_Q$ be the Borel-Serre boundary component and let $Y_Q = \Gamma_Q \backslash e_Q$ be the corresponding stratum in the Borel-Serre compactification $\widetilde{X}$ of X, with $\tau : e_Q \to Y_Q$ the projection. Let $\mathbf{E} = E \times_{\Gamma_Q} e_Q$ be the local system on Y_Q arising from some irreducible representation $\mathbf{G} \to \mathbf{GL}(E)$. Recall ([GHM] §12.3) that a differential form $\omega_Q \in \Omega^i(Y_Q, \mathbf{E})$ is "invariant" if its pullback $\tau^*(\omega_Q) \in \Omega^i(e_Q, \mathbf{E})$ is invariant under $\mathcal{U}_Q$. The invariant differential forms give rise to a complex of sheaves $\mathbf{\Omega}^\bullet_{\mathrm{inv}}(Y_Q, \mathbf{E})$ on Y_Q.

Recall ([GHM] §13) that a differential i-form ω on $X = \Gamma\backslash D$ with values in $\mathbf{E}$ is called *special* if for each stratum Y_Q of the Borel-Serre compactification $\widetilde{X}$, there exists a neighborhood of Y_Q in $\widetilde{X}$ (which depends on the differential form ω), such that in this neighborhood, the following two conditions hold:

1. the differential form ω is the pull-up of a differential form $\omega_Q \in \Omega^i(Y_Q, \mathbf{E})$ from the boundary stratum, via the geodesic retraction, and
2. the form ω_Q is $\mathcal{U}_Q$-invariant, i.e. $\omega_Q \in \Omega^i_{\text{inv}}(Y_Q, \mathbf{E})$.

We denote by $\mathbf{\Omega}^\bullet_{\text{sp}}$ the complex of pre-sheaves of special differential forms on X, whose sections over an open set $U \subset X$ are

$$\Gamma(U; \mathbf{\Omega}^\bullet_{\text{sp}}) = \left\{ \omega \in \Omega^\bullet(U, \mathbf{E}) \,\middle|\, \begin{array}{c} \omega \text{ is the restriction to } U \\ \text{of a special differential form} \end{array} \right\}$$

Let Sh denote the sheafification functor, let $j : X \hookrightarrow \widetilde{X}$ be the inclusion of X into its Borel-Serre compactification, and let $\pi : \widetilde{X} \to \overline{X}$ denote the projection from the Borel-Serre compactification to the reductive Borel-Serre compactification. Then

$$\widetilde{\mathbf{\Omega}}^\bullet_{\text{sp}}(\widetilde{X}, \mathbf{E}) = Sh(j_* \mathbf{\Omega}^\bullet_{\text{sp}}) \tag{5.1.1}$$

is the complex of sheaves of special differential forms on $\widetilde{X}$, and

$$\overline{\mathbf{\Omega}}^\bullet_{\text{sp}}(\overline{X}, \mathbf{E}) = \pi_*(\widetilde{\mathbf{\Omega}}^\bullet_{\text{sp}}(\widetilde{X}, \mathbf{E})) \tag{5.1.2}$$

is the complex of sheaves of special differential forms on $\overline{X}$ ([GHM] §13.8).

For any boundary stratum $Y_Q \subset \widetilde{X}$ the restriction $\pi | Y_Q : Y_Q \to X_Q$ is a smooth fiber bundle with nilmanifold fiber $\pi^{-1}(x) \cong N_Q = (\Gamma \cap \mathcal{U}_Q) \backslash \mathcal{U}_Q$. The complex $\mathbf{C}^\bullet(N_Q, \mathbf{E})$ of $\mathcal{U}_Q$-invariant differential forms along the fibers of π constitute a complex of flat vectorbundles over the stratum X_Q which is associated (see [GHM] §12.5) to the adjoint representation of $\mathbf{L_Q}$ on the (Koszul) complex

$$C^\bullet(\mathfrak{N}_Q, E) = \operatorname{Hom}_{\mathbb{R}}(\textstyle\bigwedge^\bullet \mathfrak{N}_Q, E) = \operatorname{Hom}_{\mathbb{C}}(\textstyle\bigwedge^\bullet \mathfrak{N}_Q(\mathbb{C}), E). \tag{5.1.3}$$

In fact, the choice of basepoint $x_0 \in D$ determines an isomorphism ([GHM] §12.13)

$$\mathbf{C}^\bullet(N_Q, \mathbf{E}) \cong C^\bullet(\mathfrak{N}_Q, E) \times_{\Gamma_L} (L_Q / K_Q A_Q). \tag{5.1.4}$$

The theorem of Nomizu and van Est identifies the cohomology of this complex with the flat vectorbundle on X_Q which is determined by the representation of L_Q on $H^*(N_Q, \mathbb{E})$.

5.2. **Lemma.** The restriction $\overline{\mathbf{\Omega}}^\bullet_{\text{sp}}(\overline{X}, \mathbf{E}) | \overline{X}_Q$ of this sheaf to the closure of the boundary stratum X_Q decomposes as a direct sum

$$\overline{\mathbf{\Omega}}^\bullet_{\text{sp}}(\overline{X}, \mathbf{E}) | \overline{X}_Q \cong \bigoplus_{q+r=\bullet} \overline{\mathbf{\Omega}}^q_{\text{sp}}(\overline{X}_Q, \mathbf{C}^r(N_Q, \mathbf{E}))$$

where the differential is given by the differential of the double complex.

5.3. **Remark.** In [GHM] §12.6, it is shown that the integrable connection on the vector-bundle $\mathbf{C}^q(N_Q, \mathbf{E})$ determines an isomorphism

$$\mathbf{\Omega}^{\bullet}_{\mathrm{sp}}(X, \mathbf{E})|X_Q \cong \bigoplus_{q+r=\bullet} \mathbf{\Omega}^q(X_Q, \mathbf{C}^r(N_Q, \mathbf{E}))$$

over the interior of the stratum X_Q. The content of Lemma 5.2 is that this decomposition extends over the closure of the stratum X_Q.

5.4. **Proof of Lemma 5.2.** Restricting (5.1.2) to the closure $\overline{X}_Q$ of the boundary stratum, we have

$$\overline{\mathbf{\Omega}}^{\bullet}_{\mathrm{sp}}(\overline{X}, \mathbf{E})|\overline{X}_Q = \pi_*(\widetilde{\mathbf{\Omega}}^{\bullet}_{\mathrm{sp}}(\widetilde{X}, \mathbf{E})|\widetilde{Y}_Q). \tag{5.4.1}$$

where $\widetilde{Y}_Q$ is the closure in $\widetilde{X}$ of the Borel-Serre stratum Y_Q. The sheaf $\widetilde{\mathbf{\Omega}}^{\bullet}_{\mathrm{sp}}(\widetilde{X}, \mathbf{E})|\widetilde{Y}_Q$ may be canonically identified with the sheaf $\widetilde{\mathbf{\Omega}}^{\bullet}_{\mathrm{sp,inv}}(\widetilde{Y}_Q, \mathbf{E})$ of $\mathcal{U}_Q$-invariant differential forms ω_Q on Y_Q (with values in $\mathbf{E}$) which are "special" near each boundary stratum $Y_P \subset \widetilde{Y}_Q$ and for which the resulting differential form $\omega_P \in \Omega^{\bullet}(Y_P, \mathbf{E})$ is $\mathcal{U}_P$-invariant. The map $\pi : \widetilde{Y}_Q \to \widetilde{X}_Q$ (from the closure of the Borel-Serre boundary stratum to the closure of the reductive Borel-Serre boundary stratum) factors through the Borel-Serre compactification $\widetilde{X}_Q$ of X_Q as the composition,

$$\widetilde{Y}_Q \xrightarrow{\alpha} \widetilde{X}_Q \xrightarrow{\beta} \overline{X}_Q. \tag{5.4.2}$$

The map α is a fibration with fiber N_Q and the integrable connection on $\alpha|Y_Q = \pi|Y_Q : Y_Q \to X_Q$ extends to an integrable connection on the closure, $\widetilde{Y}_Q \to \widetilde{X}_Q$. It may be verified that for any boundary stratum $Y_P \subset \widetilde{Y}_Q$, the geodesic retraction to Y_P preserves the flat connection on Y_Q. It follows that the isomorphism [GHM] §(12.6),

$$\mathbf{T}^{\bullet}_Q = \pi_*\mathbf{\Omega}^{\bullet}_{\mathrm{inv}}(Y_Q, \mathbf{E}) \cong \bigoplus_{q+r=\bullet} \mathbf{\Omega}^q(X_Q, \mathbf{C}^r(N_Q, \mathbf{E})) \tag{5.4.3}$$

extends uniquely to an isomorphism

$$\alpha_*\widetilde{\mathbf{\Omega}}^{\bullet}_{\mathrm{sp,inv}}(\widetilde{Y}_Q, \mathbf{E}) \cong \bigoplus_{q+r=\bullet} \widetilde{\mathbf{\Omega}}^q_{\mathrm{sp}}(\widetilde{X}_Q, \mathbf{C}^r(N_Q, \mathbf{E})). \tag{5.4.4}$$

Now apply β_* to obtain an isomorphism,

$$\begin{aligned} \overline{\mathbf{\Omega}}^{\bullet}_{\mathrm{sp}}(\overline{X}, \mathbf{E})|\overline{X}_Q &\cong \beta_*\alpha_*(\widetilde{\mathbf{\Omega}}^{\bullet}_{\mathrm{sp,inv}}(\widetilde{Y}_Q, \mathbf{E})) \\ &\cong \bigoplus_{q+r=\bullet} \beta_*\widetilde{\mathbf{\Omega}}^q_{\mathrm{sp}}(\widetilde{X}_Q, \mathbf{C}^r(N_Q, \mathbf{E})) \quad \text{by (5.4.4)} \\ &\cong \bigoplus_{q+r=\bullet} \overline{\mathbf{\Omega}}^q_{\mathrm{sp}}(\overline{X}_Q, \mathbf{C}^r(N_Q, \mathbf{E})) \quad \text{by (5.1.2).} \end{aligned}$$

This completes the proof of Lemma 5.2. ☐

5.5. The flat bundle $\mathbf{C}^{\bullet}(N_Q, \mathbf{E})$ decomposes ([GHM] §12.8.1) as a sum of flat subbundles,

$$\mathbf{C}^{\bullet}(N_Q, \mathbf{E}) \cong \bigoplus_{\beta \in \chi^*_{\mathbb{Q}}(\mathbf{S_Q})} \mathbf{C}^{\bullet}(N_Q, \mathbf{E})_{\beta} \tag{5.5.1}$$

according to the weights of $\mathbf{S_Q}$. The weight subbundle ([GHM] §12.9) is defined to be

$$\mathbf{C}^{\bullet}(N_Q, \mathbf{E})_{\geq p} = \bigoplus_{\beta \in \chi^*_{\mathbb{Q}}((\mathbf{S_Q})_{\geq p}} \mathbf{C}^{\bullet}(N_Q, \mathbf{E})_{\beta} \tag{5.5.2}$$

(where p denotes the weight profile chosen in Theorem 4.3).

Recall ([GHM] §14) that the weighted cohomology sheaf is defined to be the subsheaf of $\overline{\mathbf{\Omega}}^{\bullet}_{\mathrm{sp}}(\overline{X}, \mathbf{E})$ (5.1.2) which is obtained by truncating with respect to the weight profile p on $\mathbf{G}$. In other words, it is the unique subsheaf such that for every boundary stratum X_P,

$$\mathbf{W}^{\mathbf{p}}\mathbf{C}^{\bullet}(\overline{X}, \mathbf{E})|X_P = (\mathbf{T}^{\bullet}{}_P)_{\geq p} = \bigoplus_{q+r=\bullet} \mathbf{\Omega}^q(X_P, \mathbf{C}^r(\mathfrak{N}_P, \mathbf{E})_{\geq p}), \tag{5.5.3}$$

the identification being determined by the choice of basepoint $x_0 \in D$. If $\beta \in \chi^*_{\mathbb{Q}}(\mathbf{S_Q})$ then the weight profile $p \boxplus \beta$ (4.2.1) is defined on $\mathbf{L_Q}$.

5.6. **Definition.** Using Lemma 5.2, define the subsheaf $\mathbf{\Theta}^{\bullet} \subset \mathbf{\Omega}^{\bullet}_{\mathrm{sp}}(\overline{X}, \mathbf{E})|\overline{X}_Q$ as follows:

$$\mathbf{\Theta}^{\bullet} = \bigoplus_{\beta \in \chi^*_{\mathbb{Q}}(\mathbf{S_Q})_{\geq p}} \bigoplus_{q+r=\bullet} \mathbf{W}^{p\boxplus\beta}\mathbf{C}^q(\overline{X}_Q, \mathbf{C}^r(N_Q, \mathbf{E})_{\beta}) \subset \bigoplus_{q+r=\bullet} \overline{\mathbf{\Omega}}^q_{\mathrm{sp}}(\overline{X}_Q, \mathbf{C}^r(N_Q, \mathbf{E})) \tag{5.6.1}$$

5.7. **Lemma.** For any boundary stratum $X_P \subseteq \overline{X}_Q$ the restriction

$$\mathbf{\Theta}^{\bullet}|X_P \subset \overline{\mathbf{\Omega}}^{\bullet}_{\mathrm{sp}}(\overline{X}, \mathbf{E})|X_P$$

coincides with the subsheaf

$$(\mathbf{T}^{\bullet}_P)_{\geq p} = \bigoplus_{q+r=\bullet} \mathbf{\Omega}^q(X_P; \mathbf{C}^r(N_P, \mathbf{E})_{\geq p}).$$

Consequently,

$$\mathbf{\Theta}^{\bullet} = \mathbf{W}^p\mathbf{C}^{\bullet}(\overline{X}, \mathbf{E})|\overline{X}_Q. \tag{5.7.1}$$

5.8. **Proof.** First consider the case $\mathbf{P} = \mathbf{Q}$. The restriction $\mathbf{\Theta}^{\bullet}|X_Q$ is given by

$$\mathbf{\Theta}^{\bullet}|X_Q = \bigoplus_{\beta \in \chi^*_{\mathbb{Q}}(\mathbf{S_Q})_{\geq p}} \bigoplus_{q+r=\bullet} \mathbf{\Omega}^q(X_Q, \mathbf{C}^r(N_Q, \mathbf{E})_{\beta}) \tag{5.8.1}$$

since the weight conditions at the edge of X_Q do not affect the sections of this sheaf over the interior of X_Q. But this is precisely the weight subcomplex $(T^{\bullet}_Q)_{\geq p}$.

Now suppose that $\mathbf{P} \subset \mathbf{Q}$ is a proper parabolic subgroup. As in §3.3, set $\overline{\mathbf{P}} = \nu_Q(\mathbf{P})$. Consider the restriction to X_P of a single weighted cohomology sheaf $\mathbf{W}^{\mathbf{p}\boxplus\beta}\mathbf{C}^q(\overline{X}_Q, \mathbf{C}^r(N_Q, \mathbf{E})_{\beta})$

which occurs in (5.6.1). The choice of basepoint $x_0 \in D$ determines an isomorphism (5.5.3) between this restriction and

$$\bigoplus_{a+b=r} \bigoplus_{\gamma \in \chi^*_{\mathbb{Q}}(\mathbf{S}_{\overline{\mathbf{P}}})_{\geq p \boxplus \beta}} \mathbf{\Omega}^q(X_P, \mathbf{C}^a(N_{\overline{P}}, \mathbf{C}^b(N_Q, \mathbf{E})_\beta)_\gamma).$$

Thus we obtain an identification

$$\mathbf{\Theta}^\bullet | X_P \cong \bigoplus_{\beta \in \chi^*_{\mathbb{Q}}(\mathbf{S_Q})_{\geq p}} \bigoplus_{q+r=\bullet} \bigoplus_{a+b=r} \bigoplus_{\gamma \in \chi^*_{\mathbb{Q}}(\mathbf{S}_{\overline{\mathbf{P}}})_{\geq p \boxplus \beta}} \mathbf{\Omega}^q(X_P; C^a(N_{\overline{P}}; C^b(N_Q, \mathbf{E})_\beta)_\gamma) \tag{5.8.2}$$

Let us compare this with $(T^\bullet_P)_{\geq p}$. It suffices to show that the coefficient subbundles coincide. This amounts (by (5.1.4)) to comparing the following two representations of $\mathbf{L}_{\overline{\mathbf{P}}}$:

$$\Theta = \bigoplus_{a+b=r} \bigoplus_{\beta \in \chi^*_{\mathbb{Q}}(\mathbf{S_Q})_{\geq p}} \bigoplus_{\gamma \in \chi^*_{\mathbb{Q}}(\mathbf{S}_{\overline{\mathbf{P}}})_{\geq p \boxplus \beta}} C^a(\mathfrak{N}_{\overline{P}}, C^b(\mathfrak{N}_Q, \mathbf{E})_\beta)_\gamma) \tag{5.8.3}$$

and

$$C^r(\mathfrak{N}_P, E)_{\geq p}. \tag{5.8.4}$$

The basepoint also determines a splitting $i : \mathcal{U}_{\overline{P}} \to \mathcal{U}_P$ of the sequence

$$1 \to \mathcal{U}_Q \to \mathcal{U}_P \to \mathcal{U}_{\overline{P}} \to 1$$

and hence a decomposition $\mathfrak{N}_P \cong \mathfrak{N}_{\overline{P}} \oplus \mathfrak{N}_Q$. Therefore, we obtain an isomorphism

$$C^\bullet(\mathfrak{N}_P, E) = \mathrm{Hom}(\wedge^\bullet \mathfrak{N}_P, E) \cong \bigoplus_{a+b=\bullet} \mathrm{Hom}(\wedge^a \mathfrak{N}_{\overline{P}}, \mathrm{Hom}(\wedge^b \mathfrak{N}_Q, E)) \tag{5.8.5}$$

and hence an isomorphism

$$C^r(\mathfrak{N}_P, E)_{\geq p} \cong \bigoplus_{a+b=r} C^a(\mathfrak{N}_{\overline{P}}; C^b(\mathfrak{N}_Q, E))_{\geq p} \tag{5.8.6}$$

where "$\geq p$" indicates the direct sum of those $\mathbf{S_P}$-isotypical components with weights $\alpha \in \chi^*_{\mathbb{Q}}(\mathbf{S_P})_{\geq p}$.

Now it is important to keep track of which simple roots are involved in the weight truncations. The parabolic subgroup $\mathbf{Q}$ corresponds to a subset $J \subset \Delta_P$ of the simple roots with $\chi_*^{\mathbb{Q}}(\mathbf{S_Q}) = \ker(J)$. The elements of $\Delta_P - J$ restrict to a linearly independent set $\Delta_Q \subset \chi^*_{\mathbb{Q}}(\mathbf{S_Q})$ and determine a basis of $\chi^*_{\mathbb{Q}}(\mathbf{S'_Q})$. The elements of J may be identified with the set $\Delta_{\overline{P}}$ of simple (rational) roots of $\mathbf{S}_{\overline{\mathbf{P}}}$ occurring in the unipotent radical of $\overline{\mathbf{P}}$.

Let $\alpha \in \chi^*_{\mathbb{Q}}(\mathbf{S_Q})$ and let $\beta = \alpha | \mathbf{S_Q}$. By taking $X = \chi_*^{\mathbb{Q}}(\mathbf{S_P})$, $Y = \ker(J) = \chi_*^{\mathbb{Q}}(\mathbf{S_Q})$ and $Z = \ker(\Delta_P - J)$, and by forming the restriction $p|\mathbf{S_P} \in \chi^*_{\mathbb{Q}}(\mathbf{S_P})$, the construction of §2 defines an element $(p|\mathbf{S_P}) \boxplus \beta \in \chi^*_{\mathbb{Q}}(\mathbf{S}_{\overline{\mathbf{P}}})$ such that $(p|\mathbf{S_P}) \boxplus \beta(y+z) = \beta(y) + p(z)$ for all $y \in Y$ and $z \in Z$. By (4.4.1), this agrees with the restriction $(p \boxplus \beta)|\mathbf{S_P}$ of the weight profile $p \boxplus \beta \in \chi^*_{\mathbb{Q}}(\mathbf{S_{P_0}})$ defined in §4.2, so we may refer to it simply as $p \boxplus \beta$. Let $\gamma \in \chi^*_{\mathbb{Q}}(\mathbf{S}_{\overline{\mathbf{P}}})$

be the element corresponding to $\alpha \in \chi^*_{\mathbb{Q}}(\mathbf{S_P})$) under the canonical isomorphism $\mathbf{S_P} \cong \mathbf{S_{\overline{P}}}$. Then Proposition 2.2 says:

$$\boxed{\alpha \in \chi^*_{\mathbb{Q}}(\mathbf{S_P})_{\geq p(\Delta_P)} \text{ iff } \beta \in \chi^*_{\mathbb{Q}}(\mathbf{S_Q})_{\geq p(\Delta_Q)} \text{ and } \gamma \in \chi^*_{\mathbb{Q}}(\mathbf{S_{\overline{P}}})_{\geq p \boxplus \beta(\Delta_{\overline{P}})}.} \tag{5.8.7}$$

Let us regard the double complex (5.8.6) as a module over $L_{\overline{P}}$ and decompose it into $\mathbf{S_{\overline{P}}}$-isotypical components. We obtain

$$C^r(\mathfrak{N}_P, E)_{\geq p(\Delta_P)} \cong \bigoplus_{a+b=r} \bigoplus_{\beta} \bigoplus_{\gamma} C^a(\mathfrak{N}_{\overline{P}}, C^b(\mathfrak{N}_Q, E)_\beta)_\gamma$$

where the second sum is over those $\beta \in \chi^*_{\mathbb{Q}}(\mathbf{S_Q})_{\geq p(\Delta_Q)}$ and the third sum is over those $\gamma \in \chi^*_{\mathbb{Q}}(\mathbf{S_{\overline{P}}})_{\geq p \boxplus \beta(\Delta_{\overline{P}})}$. This is exactly (5.8.3).

It is easy to see that the isomorphisms and splittings are compatible, so that the representations (5.8.3) and (5.8.4) are actually the same subspace of $C^r(\mathfrak{N}_P, E)$. In fact, as in [GHM] §10.4, these splittings do not depend on the choice of basepoint. This completes the proof of Lemma 5.7. □

5.9. **Proof of Theorem 4.3.** By [GHM] §12.15 the complex of local systems $\mathbf{C}^\bullet(N_Q, \mathbf{E})_\beta$ is quasi-isomorphic to its cohomology sheaves, $\mathbf{H}^*(N_Q, \mathbf{E})_\beta$ (where we use * rather than $^\bullet$ to indicate that this is to be considered a complex of sheaves with trivial differentials). From (5.6.1) this determines a quasi-isomorphism

$$\mathbf{\Theta}^\bullet \cong \bigoplus_{\beta \in \chi^*_{\mathbb{Q}}(\mathbf{S_Q})_{\geq p}} \bigoplus_{q+r=\bullet} \mathbf{W}^{\mathbf{p}\boxplus\beta} \mathbf{C}^q(\overline{X}_Q, \mathbf{H}^r(N_Q, \mathbf{E})_\beta). \tag{5.9.1}$$

On the other hand, (5.7.1) identifies $\mathbf{\Theta}^\bullet$ with the desired weighted cohomology complex. □

6. Local intersection cohomology

6.1. **Baily-Borel compactification.** In this section we suppose (until §6.11) that $\mathbf{G}$ is a semi-simple $\mathbb{Q}$-algebraic group, and that $D = G/K$ is a Hermitian symmetric space (where $K \subset G$ is a maximal compact subgroup corresponding to a choice of basepoint $x_0 \in D$). Let $\Phi : \overline{X} \to \widehat{X}$ be the projection from the reductive Borel-Serre compactification of $X = \Gamma \backslash G/K$ to the Baily-Borel Satake compactification ([Z3], [GHM] §22). Fix a boundary stratum $F \subset \widehat{X}$. Then F corresponds to a Γ-conjugacy class of proper maximal rational parabolic subgroups of $\mathbf{G}$, from which we choose one and denote it by $\mathbf{Q}$. Let us consider the reductive Borel-Serre stratum X_Q to be a locally symmetric space associated to the connected reductive algebraic group $\mathbf{L_Q}$ (3.2. By [AMRT] III §4 (see also [LR] §6.1) the group $\mathbf{L_Q}$ decomposes as an almost direct product,

$$\mathbf{L_Q} = \mathbf{Q}_h \mathbf{Q}_\ell$$

of commuting algebraic subgroups with finite intersection. The compact factors (if any) of $\mathbf{L_Q}$ may be distributed among $\mathbf{Q_h}$ and $\mathbf{Q}_\ell$ so that both $\mathbf{Q_h}$ and $\mathbf{Q}_\ell$ are defined over $\mathbb{Q}$. The group $\mathbf{Q}_h$ acts by holomorphic automorphisms of the boundary component F. It contains no rational anisotropic subgroup of positive dimension. The group $\mathbf{Q}_\ell$ is reductive with split center $\mathbf{S_Q}$, and it acts by linear automorphisms on a certain self-adjoint homogeneous symmetric cone. Set

$$\begin{aligned} K_Q &= K \cap Q \subset K_\ell \subset L_Q(x_0) & \Gamma_L &= \nu_Q(\Gamma \cap Q) \\ K_\ell &= K_Q \cap Q_\ell & K_h &= \mu(K_Q) \\ \Gamma_\ell &= \Gamma_L \cap Q_\ell & \Gamma_h &= \mu(\Gamma_L) \end{aligned}$$

where $\nu_Q : \mathbf{Q} \to \mathbf{L_Q}$ and $\mu : \mathbf{L_Q} \to \mathbf{Q_h}$ are the projections. Then the boundary stratum F is diffeomorphic to $\Gamma_h \backslash Q_h / K_h$. For any $x \in F$, a choice of $q \in Q_h$ which projects to x determines a stratum preserving homeomorphism ([GHM] §22.6),

$$\overline{f}_q : \Phi^{-1}(x) \to \overline{X_\ell} \tag{6.1.1}$$

(which is smooth on each stratum) between the fiber $\Phi^{-1}(x)$ and the reductive Borel-Serre compactification of the locally symmetric space

$$X_\ell = \Gamma_\ell \backslash Q_\ell / A_Q K_\ell. \tag{6.1.2}$$

The restriction $(\Phi|X_Q) : X_Q \to F$ agrees with the projection determined by μ.

6.2. **Weight profiles.** A weight profile for $\mathbf{L_Q} = \mathbf{Q_h Q}_\ell$ determines weight profiles for $\mathbf{Q_h}$ and $\mathbf{Q}_\ell$. In fact, if $\mathbf{S_{0h}} \subset \mathbf{Q_h}$ and $\mathbf{S_{0\ell}} \subset \mathbf{Q}_\ell$ are maximal $\mathbb{Q}$-split tori then their product defines a maximal $\mathbb{Q}$-split torus $\mathbf{S_{0L}}$ in $\mathbf{L_Q}$ and a canonical isomorphism $\chi^*_{\mathbb{Q}}(\mathbf{S_{0L}}) \cong \chi^*_{\mathbb{Q}}(\mathbf{S_{0h}}) \oplus \chi^*_{\mathbb{Q}}(\mathbf{S_{0\ell}})$. So for any weight profile p for $\mathbf{L_Q}$ we may speak of the weight profile p which is obtained by restriction to the linear factor $\mathbf{Q}_\ell$.

Now, fix an algebraic irreducible representation $G \to GL(E)$ with highest weight Λ. It determines a local system $\mathbf{E}$ on X and by restriction, a local system (which we also denote by $\mathbf{E}$) on X_ℓ. An irreducible $\mathbf{L_Q}$-module V_α of highest weight α may be considered, by restriction, as a module over $\mathbf{Q}_\ell$ and hence determines a local system $\mathbf{V}_\alpha$ over the space X_ℓ. The following is the second main result in this paper.

6.3. **Theorem.** *Let p be a weight profile for the reductive Borel-Serre compactification, $\overline{X}$ of $X = \Gamma \backslash G/K$. Let $R\Phi_* \mathbf{W}^p\mathbf{C}^\bullet(\overline{X}, \mathbf{E})$ denote the pushforward of the weighted cohomology sheaf to the Baily-Borel compactification $\widehat{X}$. Let $F \subset \widehat{X}$ be a boundary component, corresponding to a maximal rational parabolic subgroup $\mathbf{Q} \subset \mathbf{G}$. Then the stalk cohomology $H^k_x(R\Phi_* \mathbf{W}^p\mathbf{C}^\bullet)$*

of this sheaf at a point $x \in F \subset \widehat{X}$ is given by

$$\bigoplus_{\beta \in \chi^*_{\mathbb{Q}}(\mathbf{S_Q})_{\geq p}} \bigoplus_i W^{p \boxplus \beta} H^{k-i}(\overline{X_\ell}; \mathbf{H}^i(\mathfrak{N}_Q, E)_\beta) \tag{6.3.1}$$

$$\cong \bigoplus_{w \in W^p_Q(E)} W^{p \boxplus \beta(w)} H^{k-\ell(w)}(\overline{X_\ell}; \mathbf{V}_{w(\Lambda+\rho)-\rho}) \tag{6.3.2}$$

where $\chi^*_{\mathbb{Q}}(\mathbf{S_Q})_{\geq p}$ is given by (3.5.1), $\beta(w)$ is given by (4.5.3), and $W^p_Q(E)$ is given by (4.5.4).

6.4. **Proof.** Since $\Phi : \overline{X} \to \widehat{X}$ is proper, the stalk cohomology of $R\Phi_* \mathbf{W^p C^\bullet}$) at a point $x \in \widehat{X}$ is canonically isomorphic to the cohomology of the fiber, $H^*(\Phi^{-1}(x); \mathbf{W}^p\mathbf{C}^\bullet(\mathbf{E}))$. If F denotes the stratum of $\widehat{X}$ which contains x and if $\mathbf{Q}$ denotes a corresponding maximal rational parabolic subgroup of $\mathbf{G}$, then $\Phi^{-1}(F)$ is a union of strata in $\overline{X_Q}$ and the restriction $\Phi : \Phi^{-1}(F) \to F$ is a fiber bundle. Using the Poincaré lemma, it is not hard to show that the restriction of the weighted cohomology sheaf $\mathbf{W}^p\mathbf{C}^\bullet(\overline{X}_Q; \mathbf{E}')$ to a fiber $\Phi^{-1}(x)$ is quasi-isomorphic to the weighted cohomology sheaf $\mathbf{W}^p\mathbf{C}^\bullet(\overline{X_\ell}; \mathbf{E})$ (where p also denotes the restriction of the weight profile to the linear factor.) Thus,

$$\begin{aligned} H^k(\Phi^{-1}(x); \mathbf{W}^p\mathbf{C}^\bullet(\mathbf{E})) &\cong H^k(\Phi^{-1}(x); \mathbf{W}^p\mathbf{C}^\bullet(\mathbf{E})|\overline{X}_Q) \\ &\cong H^k(\Phi^{-1}(x); \bigoplus_{w \in W^p_Q(E)} \mathbf{W}^{p \boxplus \beta(w)} \mathbf{C}^\bullet(\overline{X}_Q; V_{w(\Lambda+\rho)-\rho}))[\ell(w)] \\ &\cong \bigoplus_{w \in W^p_Q(E)} W^{p \boxplus \beta(w)} H^{k-\ell(w)}(\Phi^{-1}(x), V_{w(\Lambda+\rho)-\rho}) \end{aligned}$$

by (4.5.2), as desired. □

6.5. **Corollary.** *Let $p = \mu$ or $p = \nu$ denote the upper middle or lower middle weight profile, respectively (3.6). Let $\rho_Q = \rho|\mathbf{S_Q}$ be the restriction of ρ. Then the stalk cohomology of the intersection cohomology, at a point $x \in F \subset \widehat{X}$ is given by Theorem 6.3. In other words,*

$$IH^k_x(\widehat{X}, \mathbf{E}) \cong \bigoplus_{\beta \geq -\rho_Q} \bigoplus_i \mathbf{W}^{p \boxplus \beta} H^{k-i}(\overline{\Gamma_\ell \backslash Q_\ell / A_Q K_\ell}; \mathbf{H}^i(\mathfrak{N}_Q, E)_\beta). \tag{6.5.1}$$

The first sum may be replaced by $\bigoplus_{\beta > -\rho_Q}$.

This expression may also be evaluated as in (6.3.2) using Kostant's theorem and it may be translated as in §3.10 into Lie algebra cohomology.

6.6. **Proof.** The proof follows by combining Theorem 6.3 above and Theorem 23.2 of [GHM] which constructs a canonical isomorphism

$$R\Phi_* \mathbf{W}^\mu \mathbf{C}^\bullet(\mathbf{E}) \cong R\Phi_* \mathbf{W}^\nu \mathbf{C}^\bullet(\mathbf{E}) \cong \mathbf{IC}^\bullet(\widehat{X}, \mathbf{E}) \tag{6.6.1}$$

between the pushforward of the middle weighted cohomology on $\overline{X}$ with the intersection cohomology of $\widehat{X}$. $\square$

Taking $p = -\infty$ gives the following well known result [LR], the étale version of which is proven in [P1], [P2]:

6.7. **Corollary.** *Let $\hat{\imath} : X \hookrightarrow \widehat{X}$ be the inclusion of the locally symmetric space into the Baily-Borel compactification. Then the stalk cohomology of the sheaf $R\hat{\imath}_*(\mathbf{E})$ is given by*

$$H^k_x(R\hat{\imath}_*(\mathbf{E})) \cong \bigoplus_{w \in W^1_Q} H^{k-\ell(w)}(\Gamma_\ell; V_{w(\Lambda+\rho)-\rho}).$$

Here, W^1_Q is given by (4.5.1), and V_α is the irreducible $\mathbf{L_Q}$-module with highest weight α. It is considered as a module over Γ_ℓ by way of the inclusion $\Gamma_\ell \subset \Gamma_L = \nu_Q(\Gamma_Q) \subset L_Q$.

6.8. **Proof.** In Theorem 6.3, take $p = -\infty$ to be the weight profile which involves no truncation. Then $W^p_Q(E) = W^1_Q$. The weighted cohomology $W^pH^i(\overline{X_\ell}; V_\alpha)$ is equal to the ordinary cohomology of X_ℓ, (with coefficients in V_α) which (since Γ is neat) in turn coincides with the group cohomology $H^i(\Gamma_\ell, V_\alpha)$. $\square$

6.9. **Remarks.** The vanishing of the stalk cohomology of the intersection cohomology, and more generally, the purity theorem of Looijenga (i.e. that the stalk cohomology in degree i of the intersection cohomology has weight $\leq i$) may be translated into vanishing theorems for certain weighted cohomology groups of the fibers $\Phi^{-1}(x)$ using Theorem 6.3. A general framework for such vanishing theorems has been developed in [S]. See [B2] for a related vanishing theorem for the L^2 cohomology of linear locally symmetric spaces.

6.10. **Mixed Hodge weights.** By [LR], the direct sum (6.5.1) over $\beta \in \chi^*_{\mathbb{Q}}(\mathbf{S_Q})$ is a splitting of the weight filtration (on the stalk cohomology) which comes from Saito's theory of mixed Hodge modules.

6.11. **The L^2 Euler characteristic.** When G/K is Hermitian, the Zucker conjecture ([L],[SS]) implies that the L^2 cohomology Euler characteristic $L^2\chi(\Gamma, E)$ is equal to the intersection cohomology Euler characteristic of the Baily-Borel compactification, i.e. $L^2\chi(\Gamma, E) = I\chi(\widehat{X}, \mathbf{E}) = \sum_i (-1)^i \dim IH^i(\widehat{X}, \mathbf{E})$. By (6.6.1) this is equal to the Euler characteristic of the weighted cohomology complex of the reductive Borel-Serre compactification,

$$L^2\chi(\Gamma, E) = \sum_i (-1)^i \dim W^\mu H^i(\overline{X}, \mathbf{E}) \tag{6.11.1}$$

In fact this relation holds more generally. Suppose only that $\mathbf{G}$ is a reductive $\mathbb{Q}$-algebraic group such that the derived group of G possesses a compact Cartan subgroup. Then $W^\mu H^*(\overline{X}, \mathbf{E})$ is isomorphic to the L^2 cohomology of Γ by [N], and therefore (6.11.1) holds.

As in [GM1], (6.11.1) is equal to a sum over strata of $\overline{X}$, with the contribution from a single stratum X_P being given by the compactly supported Euler characteristic $\chi_c(X_P)$ of the stratum X_P times the stalk Euler characteristic of the weighted cohomology at any point $x \in X_P$. Moreover, $\chi_c(X_P) = \chi(X_P)$. Let $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_l$ be a collection of representatives, one from each $\Gamma-$conjugacy class of proper rational parabolic subgroups of $\mathbf{G}$. For each representative $\mathbf{P}_j$, let $\Gamma_{L_j} = \nu(\Gamma \cap P_j)$ be the projection of $\Gamma \cap P_j$ to the Levi quotient. Since Γ is neat, the Euler characteristic $\chi(X_{P_j}) = \chi(\Gamma_{L_j})$ is equal to the Euler characteristic of the discrete group Γ_{L_j}. Let $V_\alpha^{(j)}$ be the irreducible L_j-module with highest weight α. Let us suppose that $\mathbf{E}$ is the local system associated to an irreducible representation of $\mathbf{G}$ with highest weight Λ. Evaluating (3.5.3) using Kostant's theorem gives the following formula (which is in [GHM] §17.9, [S] for Hermitian X):

6.12**. Theorem.** *Let* $\mathbf{G}$ *be a reductive* $\mathbb{Q}$*-algebraic group and suppose the derived group of* G *has a compact Cartan subgroup. Them the Euler characteristic of the* L^2 *cohomology of the locally symmetric space* $X = \Gamma \backslash G/A_G K$ *is given by*

$$L^2\chi(\Gamma, E) = \chi(\Gamma)\dim(E) + \sum_{j=1}^{l} \chi(\Gamma_{L_j}) \cdot \sum_{w \in W^{\mu}_{P_j}} (-1)^{\ell(w)} \dim(V^{(j)}_{w(\Lambda+\rho)-\rho})$$

7. Computations for $\mathbf{Sp_4}$

7.1. **The symmetric space.** Throughout this section we fix a prime $p \geq 3$ and we take $\Gamma = \Gamma(p)$ to be the principal congruence subgroup of $\mathbf{Sp_4}(\mathbb{Z})$ consisting of matrices which are congruent to the identity modulo p. Define $X = \Gamma \backslash \mathbf{Sp_4}(\mathbb{R})/\mathbf{U_2}$ to be the associated (real) 6-dimensional Hermitian locally symmetric space. The Baily-Borel compactification $\widehat{X}$ has boundary strata of real dimension 2 and of real dimension 0. In this section we will compute the intersection Euler characteristic of $\widehat{X}$ and also the local L^2 (or intersection) cohomology (with constant coefficients) at a most singular point (i.e. at a 0-dimensional stratum) of $\widehat{X}$. Denote by $\Phi : \overline{X} \to \widehat{X}$ the projection from the reductive Borel-Serre compactification to the Baily-Borel compactification. Let $\mathfrak{h}$ denote the upper half plane.

7.2. **Parabolic subgroups.** There are three types of rational proper parabolic subgroups $\mathbf{P}$ of $\mathbf{G}$.

- (A) P is the stabilizer of a rational 1-dimensional (and hence isotropic) subspace $F^1 \subset \mathbb{Q}^4$. Such a parabolic subgroup is maximal with Hermitian Levi factor $L = P_h$. The associated reductive Borel-Serre stratum X_P is (real) 2-dimensional, in fact it is a modular curve.
- (B) P is the stabilizer of a rational Lagrangian subspace $F^2 \subset \mathbb{Q}^4$. Such a parabolic subgroup is maximal with linear Levi factor $L = P_\ell$. The associated reductive Borel-Serre stratum X_P is (real) 2-dimensional and is diffeomorphic to a modular curve.
- (C) P is a rational Borel subgroup: it is the stabilizer of a rational isotropic flag $F^1 \subset F^2 \subset \mathbb{Q}^4$. The associated reductive Borel-Serre stratum X_P is a point. Such a point

is simultaneously a cusp for a single type A stratum and a single type B stratum in the reductive Borel-Serre compactification.

The projection Φ takes each type A stratum $X_P \subset \overline{X}$ isomorphically to a (real) two dimensional stratum $Y_P \subset \widehat{X}$. The projection Φ collapses each type B stratum X_P to a single point in $\widehat{X}$. The preimage $\Phi^{-1}(x)$ of such a point is the reductive Borel-Serre compactification of the type B stratum X_P, and it is obtained by adding type C strata as cusps of X_P.

7.3. Consider the case of the trivial local system $\mathbf{E} = \mathbb{C}$. By Corollary 6.5, the L^2 cohomology, or the intersection cohomology of $\widehat{X}$ is isomorphic to either of the middle weighted cohomology groups of $\overline{X}$. In fact, the upper and lower middle weight profiles μ, ν give rise to the same weighted cohomology sheaf on $\overline{X}$ since in this case, the middle weight does not appear in the $\mathfrak{N}_P$ cohomology for any parabolic subgroup P.

7.4. First we consider the weighted Euler characteristic. Let n_i be the number of $\Gamma(p)$-conjugacy classes of rational parabolic subgroups of type i (for $i = A, B, C$). Let Γ_i denote the projection of Γ into the Levi quotient $L = P/\mathcal{U}_P$ for each parabolic subgroup P of type $i = A, B, C$. Let

$$\chi_i = \sum_{w \in W_P^{\mu}} (-1)^{\ell(w)} \dim(V_{w\rho-\rho})$$

be the factor which appears in Theorem (6.12) and which arises from a parabolic subgroup P of type $i = A, B, C$. Then the Euler characteristic of the intersection cohomology of $\widehat{X}$ is given by

$$I\chi(\widehat{X}) = \chi(\Gamma) + n_A \chi(\Gamma_A)\chi_A + n_B \chi(\Gamma_B)\chi_B + n_C \chi(\Gamma_C)\chi_C \qquad (7.4.1)$$

7.5. These constants will now be evaluated. From the root system for $\mathbf{Sp_4}$ we find

$$\chi_A = -1, \quad \chi_B = -2, \quad \chi_C = -1. \qquad (7.5.1)$$

The groups Γ_A and Γ_B both turn out to be $\Gamma(p) \subset SL_2(\mathbb{Z})$, i.e. the principal congruence subgroup consisting of elements which are congruent to the identity modulo p. The modular curve $\Gamma(p)\backslash\mathfrak{h}$ has $(p^2-1)/2$ cusps ([Sh] Lemma 1.42) and Euler characteristic

$$\chi(\Gamma_A) = \chi(\Gamma_B) = -\frac{1}{2}\binom{p+1}{3} \qquad (7.5.2)$$

by [Sh] (1.6.4). From this we conclude that

$$\dim(H^1(\Gamma_A)) = \frac{1}{2}\binom{p+1}{3} + 1 \qquad (7.5.3)$$

Each type C boundary component is simultaneously a cusp of a type A boundary component and of a type B boundary component of the reductive Borel-Serre compactification so $n_A =$

n_B and $n_C = n_A(p^2-1)/2$. Moreover, n_C is the number of double cosets $B(\mathbb{Z})\backslash\mathbf{Sp_4}(\mathbb{Z})/\Gamma$ which is $(p^4-1)(p^2-1)/4$ (where B denotes the standard Borel subgroup), hence

$$n_A = \frac{p^4-1}{2} \quad n_B = \frac{p^4-1}{2} \quad n_C = \frac{(p^4-1)(p^2-1)}{4} \tag{7.5.4}$$

Finally, Harder's Gauss-Bonnet formula ([H], p.453) gives $\chi(\Gamma) = \zeta(-1)\zeta(-3)\cdot|\mathbf{Sp_4}(\mathbb{F}_p)|$ (where ζ is Riemann's zeta function), which is

$$\frac{-p^4(p-1)(p+1)(p^4-1)}{2^5\cdot 3^2\cdot 5}. \tag{7.5.5}$$

(Note that this an integer precisely when $p\geq 3$, i.e. when Γ is neat.) From (7.4.1), (7.5.1), (7.5.4) and (7.5.5), we obtain

7.6. **Theorem.** *The Euler characteristic of the intersection cohomology of the Baily-Borel compactification is*

$$I\chi(\widehat{X}) = \frac{(p^4-1)(p^2-1)}{2^3}\left(\frac{-p^4}{2^2\cdot 3^2\cdot 5}+p-2\right) \tag{7.6.1}$$

7.7. **Stalk cohomology.** Now consider the stalk cohomology of the intersection cohomology at a 0-dimensional boundary stratum in $\widehat{X}$. By Corollary 6.6, this is a sum of two weighted cohomology groups of a boundary stratum of type B,

$$IH^i_x \cong WH^i(\overline{\Gamma(p)\backslash\mathfrak{h}};H^0(\mathfrak{N}_P))\oplus WH^{i-1}(\overline{\Gamma(p)\backslash\mathfrak{h}};H^1(\mathfrak{N}_P)) \tag{7.7.1}$$

A calculation with the root system for $\mathbf{Sp_4}$ shows that the weight profile for the first factor (with coefficients in the trivial local system $H^0(\mathfrak{N}_P)$) involves no cutoff at all, while the weight profile for the second factor cuts off all the stalk cohomology in degree 1 at each cusp point, so

$$IH^i_x \cong H^i(\Gamma(p)\backslash\mathfrak{h};\mathbb{C})\oplus IH^{i-1}(\overline{\Gamma(p)\backslash\mathfrak{h}};H^1(\mathfrak{N}_P)) \tag{7.7.2}$$

The cohomology groups of the first term were computed in (7.5.3). We will use two tricks to evaluate the second factor. First, the local system $H^1(\mathfrak{N}_P)$ has weight χ^{-2}, where χ is the canonical positive generator of the character group of A_P. In fact, it is the irreducible 3-dimensional representation of SL_2. By Looijenga's purity theorem, weight 2 classes cannot occur in the stalk of the intersection cohomology except in degree 2 or more. Therefore $IH^0(\overline{\Gamma(p)\backslash\mathfrak{h}};H^1(\mathfrak{N}_P)) = 0$. Also, $IH^2(\overline{\Gamma(p)\backslash\mathfrak{h}};H^1(\mathfrak{N}_P)) = 0$ since $IH^3_x = 0$ by the usual vanishing property for intersection cohomology. Therefore the second factor in (7.7.2) is completely determined by the Euler characteristic of this (compactified) modular curve, which we now calculate: the stalk of the intersection cohomology at each cusp point has cohomology $\mathbb{C}$ in degree 0, and 0 in all other degrees. So each cusp contributes 1 to the Euler characteristic while the interior contributes $\dim(H^1(\mathfrak{N}_P)).\chi(\Gamma(p))$. Hence,

$$I\chi(\overline{\Gamma(p)\backslash\mathfrak{h}};H^1(\mathfrak{N}_P)) = -\frac{3}{2}\binom{p+1}{3}+\frac{p^2-1}{2} \tag{7.7.3}$$

7.8**. Theorem.** *The Betti numbers of the stalk intersection cohomology at a 0-dimensional stratum of* $\widehat{X}$ *are given by:*

$$\dim(IH^0_x) = 1$$
$$\dim(IH^1_x) = \frac{1}{2}\binom{p+1}{3} + 1$$
$$\dim(IH^2_x) = \frac{3}{2}\binom{p+1}{3} - \frac{(p^2-1)}{2}$$

7.9. **Remark.** The same result holds with p replaced by an arbitrary integer N, provided N is sufficiently large.

7.10. **Remark.** For $p = 3$ we obtain the local intersection cohomology Betti numbers $I\beta^0 = 1$, $I\beta^1 = 3$, $I\beta^2 = 2$. These numbers agree with the intensive computations which were carried out by M. McConnell on the Symbolics computer at Brown University in 1986.

7.11. **Discrete series multiplicities.** The computation of the intersection Euler characteristic (7.6.1) gives some information about multiplicities. The group $\mathbf{Sp_4}(\mathbb{R})$ has two discrete series representations with nonzero $(\mathfrak{g}, K)$-cohomology (see [T] for a list of all representations with cohomology and for other facts used below). Let us denote these π^H (belonging to the holomorphic discrete series) and π^W (having a Whittaker model). Via the isomorphisms

$$IH^i(\widehat{X}) \cong H^i_{(2)}(X) \cong H^i(\mathfrak{g}, K; L^2_{dis}(\Gamma(p)\backslash G))$$

both of these contribute to IH^3 (in bidegrees $(3,0), (0,3)$ and $(2,1), (1,2)$ respectively). Here L^2_{dis} stands for the discrete spectrum of the L^2 space. The other representations of $\mathbf{Sp_4}(\mathbb{R})$ with cohomology contribute in degrees $0, 2, 4, 6$. Using the decomposition of the discrete spectrum we write

$$I\chi(\widehat{X}) = \sum_\pi m_{dis}(\pi, p) \sum_i (-1)^i \dim\, H^i(\mathfrak{g}, K; \pi) = \sum_\pi m_{dis}(\pi, p)\chi(\pi)$$

where $m_{dis}(\pi, p)$ is the multiplicity of π in the discrete spectrum of $\Gamma(p)$ and $\chi(\pi)$ is the $(\mathfrak{g}, K)$-Euler characteristic. If π is not either π^H or π^W then $\chi(\pi)$ is positive while $\chi(\pi^H) = \chi(\pi^W) = -2$. It follows that

$$2(m_{dis}(\pi^H, p) + m_{dis}(\pi^W, p))$$

is at least as large as the negative of (7.6.1). Since π^H and π^W are tempered, an observation of Wallach [W] implies that $m_{dis}(\pi^H, p) = m_{cusp}(\pi^H, p)$ (multiplicity in the cuspidal spectrum) and similarly for π^W. So

$$m_{cusp}(\pi^H, p) + m_{cusp}(\pi^W, p)$$

is at least

$$\frac{(p^4-1)(p^2-1)}{2^4}\left(\frac{p^4}{2^2 \cdot 3^2 \cdot 5} - p + 2\right).$$

(This is positive for $p > 3$.) Hence, for $p > 3$, there are nonzero cusp forms with infinity type either π^W or π^H.

In particular, in the adelic situation, there is an irreducible admissible representation π of the group $\mathbf{Sp_4}(\mathbb{A}_f)$ of finite adeles such that $m_{cusp}(\pi \otimes \pi^H) + m_{cusp}(\pi \otimes \pi^H) > 0$ (where m_{cusp} now stands for multiplicity in $L^2_{cusp}(\mathbf{Sp_4}(\mathbb{Q})\backslash\mathbf{Sp_4}(\mathbb{A}))$). To such a π Taylor [T] has associated a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ occurring in degree 3 ℓ-adic cohomology of the associated Shimura variety. (Under the added hypothesis that both multiplicities $m_{cusp}(\pi \otimes \pi^H)$ and $m_{cusp}(\pi \otimes \pi^H)$ are positive, Taylor shows it to have the correct characteristic polynomial of Frobenius at many primes. However one would need a Hodge (p, q) analog of (7.6.1) in order to verify this hypothesis by evaluating these multiplicities.)

References

[AMRT] A. Ash, D. Mumford, M. Rapoport, and Y.Tai, **Smooth compactifications of locally symmetric varieties.** Math. Sci. Press, Brookline MA. 1975.

[BB] W. Baily and A. Borel, Compactifications of arithmetic quotients of bounded symmetric domains. Ann. Math. **84** (1966), pp. 442–528.

[B1] A. Borel. **Introduction aux groupes arithmétiques**. Hermann, Paris (1968).

[B2] A. Borel, A vanishing theorem in relative Lie algebra cohomology, in **Algebraic Groups Utrecht 1986**. Lecture Notes in Math. **1271**, Springer-Verlag (1987).

[BS] A. Borel and J.P. Serre, Corners and arithmetic groups. Comm. Math. Helv. **48** (1973), pp. 436–491.

[Est] W. T. van Est, A generalization of the Cartan-Leray spectral sequence I,II. Indag. Math. **XX** (1958), pp. 399–413.

[F] J. Franke, Harmonic analysis in weighted L_2 spaces. Ann. Sci. École Norm. Sup. **31** (1998), pp. 181–279.

[GKM] M. Goresky, R. Kottwitz, and R. MacPherson, Discrete series characters and the Lefschetz formula for Hecke operators. Duke Math J. **89** (1997), pp. 477–554.

[GM1] M. Goresky and R. MacPherson, Local contribution to the Lefschetz fixed point formula. Inv. Math **111** (1993), pp. 1-33.

[GM2] M. Goresky and R. MacPherson, The topological trace formula, to appear.

[GHM] M. Goresky, G. Harder, and R. MacPherson, Weighted cohomology. Inv. Math **116** (1994), pp. 139-213.

[H] G. Harder, A Gauss-Bonnet formula for discrete arithmetically defined groups. Ann. Sci. Ecole Norm. Sup. **4** (1971), pp. 409–455.

[K] B. Kostant, Lie algebra cohomology and the generalized Borel-Weil theorem. Ann. Math. **74** (1961), pp. 329–387.

[L] E. Looijenga, L^2-cohomology of locally symmetric varieties. Comp. Math. **67** (1988), pp. 3–20.

[LR] E. Looijenga and M. Rapoport, Weights in the local cohomology of a Baily-Borel compactification, in **Complex Geometry and Lie Theory**, Proc. Symp. Pure Math. **53** (1991). Amer. Math. Soc. Providence R. I., pp. 223-260.

[MM] R. MacPherson and M. McConnell, Classical projective geometry and modular varieties, in **Algebraic analysis, geometry, and number theory**. Proc. of J.A.M.I. Inaugural Conference, J. J. Igusa, ed. Johns Hopkins University Press, Baltimore MD (1989).

[N] A. Nair, Weighted cohomology of arithmetic groups. Ann. Math. **150** (1999), pp. 1–31.

[No] K. Nomizu, On the cohomology of compact homogeneous spaces of nilpotent Lie groups. Ann. Math. **59** (1954), pp. 531–538.

[P1] R. Pink, **Arithmetical compactification of mixed Shimura varieties**. Dissertation, Univ. of Bonn, Bonn (1989).

[P2] R. Pink, On ℓ-adic sheaves on Shimura varieties and their higher direct images in the Baily-Borel compactification. Math. Ann. **292** (1992), pp. 197-240.

[S] L. Saper, $\mathcal{L}$-modules and the conjecture of rapoport and Goresky-MacPherson, preprint, Duke Univ. 2001.

[SS] L. Saper and M. Stern, L_2 cohomology of arithmetic varieties. Ann. Math. **132** (1990), pp. 1-69.

[Sh] G. Shimura, **Introduction to the Arithmetic Theory of Automorphic Forms**. Math. Soc. of Japan and Princeton University Press, Princeton NJ (1971).

[Sp] T. A. Springer, **Linear algebraic groups**. Birkhauser Boston, Boston MA (1981).

[S] M. Stern, Lefschetz formula for arithmetic varieties. Inv. Math **115** (1994), pp. 241-296.

[T] R. Taylor, On the ℓ-adic cohomology of Siegel threefolds. Invent. Math. **114** (1993), pp. 289-310.

[V] D. Vogan, **Representations of Real Reductive Lie Groups**. Birkhauser Boston, Boston MA (1981).

[W] N. Wallach, On the constant term of a square-integrable automorphic form, in **Operator Algebras and Group Representations II** (Mono. Stud. Math. **18**). Pitman, Boston MA (1984), pp. 227-237.

[Z1] S. Zucker, L_2 cohomology of warped products and arithmetic groups. Inv. Math. **70** (1982), pp. 169–218.

[Z2] S. Zucker, L_2-cohomology and intersection homology of locally symmetric varieties II. Comp. Math. **59** (1986), pp. 339-398.

[Z3] S. Zucker, Satake compactifications. Comm. Math. Helv. **58** (1983), pp. 312–343.