REGULAR POINTS IN AFFINE SPRINGER FIBERS

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1. Introduction

Let G be a connected reductive group over $\mathbf{C}$ with Lie algebra $\mathfrak{g}$. We put $F = \mathbf{C}((\epsilon))$ and $\mathcal{O} = \mathbf{C}[[\epsilon]]$. Let $X = X_G$ denote the affine Grassmannian $G(F)/G(\mathcal{O})$. For $u \in \mathfrak{g}(F)$ we write X^u for the affine Springer fiber

$$X^u = \{g \in G(F)/G(\mathcal{O}) : \mathrm{Ad}(g^{-1})(u) \in \mathfrak{g}(\mathcal{O})\}.$$

studied by Kazhdan and Lusztig in [KL88].

For $x = gG(\mathcal{O}) \in X^u$ the $G(\mathcal{O})$-orbit (for the adjoint action) of $\mathrm{Ad}(g^{-1})(u)$ in $\mathfrak{g}(\mathcal{O})$ depends only on x, and its image under $\mathfrak{g}(\mathcal{O}) \twoheadrightarrow \mathfrak{g}(\mathbf{C})$ is a well-defined $G(\mathbf{C})$-orbit in $\mathfrak{g}(\mathbf{C})$. We say that $x \in X^u$ is *regular* if the associated orbit is regular in $\mathfrak{g}(\mathbf{C})$. (Recall that an element of $\mathfrak{g}(\mathbf{C})$ is regular if the nilpotent part of its Jordan decomposition is a principal nilpotent element in the centralizer of the semisimple part of its Jordan decomposition.) We write X^u_{reg} for the (Zariski open) subset of regular elements in X^u.

From now on we assume that u is regular semisimple with centralizer T, a maximal torus in G over F. Assume further that u is *integral*, by which we mean that X^u is non-empty. Kazhdan and Lusztig [KL88] show that X^u is then a locally finite union of projective algebraic varieties, and in Cor. 1 of §4 of [KL88] they show that the open subset X^u_{reg} of X^u is non-empty (and hence dense in at least one irreducible component of X^u). The action of $T(F)$ on X clearly preserves the subsets X^u and X^u_{reg}. Bezrukavnikov [Bez96] proved that X^u_{reg} forms a single orbit under $T(F)$. (Actually Kazhdan-Lusztig and Bezrukavnikov consider only topologically nilpotent elements u, but the general case can be reduced to their special case by using the topological Jordan decomposition of u.)

The goal of this paper is to characterize regular elements in X^u (for integral regular semisimple u as above). When T is elliptic (in other words, F-anisotropic modulo the center of G) the characterization gives no new information. At the other extreme, in the split case, the characterization gives a clear picture of what it means for a point in X^u to be regular.

We will now state our characterization in the split case, leaving the more technical general statement to the next section (see Theorem 1). Fix a split maximal torus $A \subset G$ over $\mathbf{C}$ and denote by $\mathfrak{a}$ its Lie algebra. We identify the affine

Date: August 7, 2003.
1991 *Mathematics Subject Classification.* Primary: 22E50; Secondary: 20G25, 22E67.
Key words and phrases. Affine Springer fiber, fixed point set, affine Grassmannian.
M.G. was partially supported by NSF Grant DMS-0139986.
R.K. was partially supported by NSF Grant DMS-0245639.

Grassmannian $A(F)/A(\mathcal{O})$ for A with the cocharacter lattice $X_*(A)$, the cocharacter μ corresponding to the class of $\mu(\epsilon)$ in $A(F)/A(\mathcal{O})$. For any Borel subgroup $B = AN$ containing A (N denoting the unipotent radical of B) there is a well-known retraction $r_B : X \to X_*(A)$ defined using the Iwasawa decomposition $G(F) = N(F)A(F)G(\mathcal{O})$: the fiber of r_B over $\mu \in X_*(A)$ is $N(F)\mu(\epsilon)G(\mathcal{O})/G(\mathcal{O})$. The family of cocharacters $r_B(x)$ (B ranging through all Borel subgroups containing A) has been studied by Arthur [Art76, Lemma 3.6]; it is the volume of the convex hull of these points that arises as the weight factor for (fully) weighted orbital integrals for elements in $A(F)$. In particular Arthur shows that for $x \in X$ and any pair B, B' of adjacent Borel subgroups containing A, there is a unique non-negative integer $n(x, B, B')$ such that

$$r_B(x) - r_{B'}(x) = n(x, B, B') \cdot \alpha^\vee_{B,B'} \tag{1.0.1}$$

where $\alpha_{B,B'}$ is the unique root of A that is positive for B and negative for B'.

The main result of this paper (in the split case) is that for $x \in X^u$

$$n(x, B, B') \le \operatorname{val} \alpha_{B,B'}(u) \tag{1.0.2}$$

for every pair B, B' of adjacent Borel subgroups containing A, and that $x \in X^u$ is regular if and only if all the inequalities (1.0.2) are actually equalities.

2. Statements

2.1. **Notation.** We write $\mathfrak{g}$ for the Lie algebra of G and follow the same convention for groups denoted by other letters.

Choose an algebraic closure $\overline{F}$ of F and let $\Gamma = \operatorname{Gal}(\overline{F}/F)$. We write G_F for the F-group obtained from G by extension of scalars from $\mathbf{C}$ to F.

As before we use $\mu \mapsto \mu(\epsilon)$ to identify the cocharacter group $X_*(A)$ with $A(F)/A(\mathcal{O})$. By means of this identification the canonical surjection $A(F) \to A(F)/A(\mathcal{O})$ can be viewed as a surjection

$$A(F) \to X_*(A). \tag{2.1.1}$$

Let $\Lambda = \Lambda_G$ denote the quotient of the coweight lattice $X_*(A)$ by the coroot lattice (the subgroup of $X_*(A)$ generated by the coroots of A in G). Up to canonical isomorphism Λ is independent of the choice of A; moreover when defining Λ we could replace A by any maximal torus T in G_F. There is a canonical surjective homomorphism

$$G(F) \to \Lambda, \tag{2.1.2}$$

characterized by the following two properties: it is trivial on the image of $G_{\mathrm{sc}}(F)$ in $G(F)$ (where G_{sc} denotes the simply connected cover of the derived group of G), and its restriction to $A(F)$ coincides with the composition of (2.1.1) and the canonical surjection $X_*(A) \to \Lambda$.

Recall that X denotes the affine Grassmannian $G(F)/G(\mathcal{O})$ for G. The homomorphism (2.1.2) is trivial on $G(\mathcal{O})$ and hence induces a canonical surjection

$$\nu_G : X \to \Lambda, \tag{2.1.3}$$

whose fibers are the connected components of X.

2.2. **Parabolic subgroups.** We will concerned with parabolic subgroups P of G containing A. Such a parabolic subgroup has a unique Levi subgroup M containing A, and we refer to M as *the* Levi component of P.

As usual, by a Levi subgroup of G, we mean a Levi subgroup of some parabolic subgroup of G. Let M be a Levi subgroup of G containing A. We write $\mathcal{P}(M)$ for the set of parabolic subgroups of G that contain A and have Levi component M. Thus any $P \in \mathcal{P}(M)$ can be written as $P = MN$ where $N = N_P$ denotes the unipotent radical of P. As usual there is a notion of adjacency: two parabolic subgroups $P = MN$ and $P' = MN'$ in $\mathcal{P}(M)$ are said to be *adjacent* if there exists (a unique) parabolic subgroup $Q = LU$ containing both P and P' such that the semisimple rank of L is one greater than the semisimple rank of M. Thus $U = N \cap N'$, and, moreover, if L is chosen so that $L \supset A$, then

$$\mathfrak{l} = \mathfrak{m} \oplus (\mathfrak{n} \cap \bar{\mathfrak{n}'}) \oplus (\mathfrak{n}' \cap \bar{\mathfrak{n}})$$

where $\bar{N}$ denotes the unipotent radical of the parabolic subgroup $\bar{P} = M\bar{N}$ opposite to P (and where $\bar{N}'$ is opposite to N').

Given adjacent P, P' in $\mathcal{P}(M)$ we define an element $\beta_{P,P'} \in \Lambda_M$ (the coweight lattice for A modulo the coroot lattice for M) as follows. Consider the collection of elements in Λ_M obtained from coroots $\alpha^\vee$ where α ranges through the set of roots of A in $\mathfrak{n} \cap \bar{\mathfrak{n}}'$. We define $\beta_{P,P'}$ to be the unique element in this collection such that all other members in the collection are positive integral multiples of $\beta_{P,P'}$. Note that although Λ_M may have torsion elements, the elements in our collection lie in the kernel of the canonical map from Λ_M to Λ_G, and this kernel is torsion-free. Thus any member of our collection can be written uniquely as a positive integer times $\beta_{P,P'}$. Note also that $\beta_{P',P} = -\beta_{P,P'}$. In case $M = A$, so that P, P' are Borel subgroups, $\beta_{P,P'}$ is the unique coroot of A that is positive for P and negative for P'.

2.3. **Retractions from X to X_M.** The inclusion of $M(F)$ into $G(F)$ induces an inclusion of the affine Grassmannian X_M for M into the affine Grassmannian X for G. Let $P \in \mathcal{P}(M)$ and let X_P denote the set $P(F)/P(\mathcal{O})$. The canonical inclusion of P in G induces a bijection i from X_P to X, and the canonical surjection $P \to M$ induces a canonical surjective map p (of sets) from X_P to X_M. We define the retraction $r_P = r_P^G : X \to X_M$ as the composed map $p \circ i^{-1}$. Given $x \in X$ we often denote by x_P the image of x under the retraction r_P.

These retractions satisfy the following transitivity property. Suppose that $L \supset M$ are Levi subgroups containing A, and suppose further that $P \in \mathcal{P}(M)$ and $Q \in \mathcal{P}(L)$ satisfy $Q \supset P$. Let P_L denote the parabolic subgroup $P \cap L$ in L. Then

$$r_P^G = r_{P_L}^L \circ r_Q^G. \tag{2.3.1}$$

Moreover, for any $x \in X$ the element $\nu_M(x_P)$ maps to $\nu_L(x_Q)$ under the canonical surjection $\Lambda_M \to \Lambda_L$, and in particular $\nu_M(x_P) \mapsto \nu_G(x)$ under $\Lambda_M \to \Lambda_G$.

2.4. **Definition of $n(x, P, P')$.** A point $x \in X$ determines points $\nu_M(x_P)$ in Λ_M, one for each $P \in \mathcal{P}(M)$. This family of points arises in the definition of the weighted orbital integrals occurring in Arthur's work. A basic fact [Art76] about this family of points is that whenever P, P' are adjacent parabolic subgroups in $\mathcal{P}(M)$, there is a (unique) non-negative integer $n(x, P, P')$ such that

$$\nu_M(x_P) - \nu_M(x_{P'}) = n(x, P, P') \cdot \beta_{P,P'}. \tag{2.4.1}$$

2.5. **Fixed point sets X^u.** Let $u \in \mathfrak{g}(F)$. Define a subset X^u of X by

$$X^u = \{g \in G(F)/G(\mathcal{O}) : \mathrm{Ad}(g^{-1})(u) \in \mathfrak{g}(\mathcal{O})\}.$$

2.6. **Conjugacy classes associated to fixed points.** Let $u \in \mathfrak{g}(F)$. Suppose that the coset $x = gG(\mathcal{O})$ lies in X^u. The image of $\mathrm{Ad}(g^{-1})(u)$ under the canonical surjection $\mathfrak{g}(\mathcal{O}) \to \mathfrak{g}(\mathbf{C})$ gives a well-defined $G(\mathbf{C})$-conjugacy class $\bar{u}_G(x)$ (for the adjoint action) in $\mathfrak{g}(\mathbf{C})$.

As above let M be a Levi subgroup of G and let $P \in \mathcal{P}(M)$. Now suppose that $u \in \mathfrak{m}(F)$ and that $x \in X^u$. Choose $p \in P(F)$ such that $x = pG(\mathcal{O})$; thus x_P is the coset $mM(\mathcal{O})$, where m denotes the image of p under the canonical homomorphism from P onto M. Of course $\mathrm{Ad}(p^{-1})(u)$ lies in $\mathfrak{p}(\mathcal{O})$, and its image in $\mathfrak{p}(\mathbf{C})$ gives a well-defined $P(\mathbf{C})$-conjugacy class $\bar{u}_P(x)$ in $\mathfrak{p}(\mathbf{C})$. It follows that x_P lies in X^u_M (as was first noted by Kazhdan-Lusztig [KL88]), and also that $\bar{u}_P(x)$ maps to $\bar{u}_G(x)$ (respectively, $\bar{u}_M(x_P)$) under the map on conjugacy classes induced by $\mathfrak{p}(\mathbf{C}) \hookrightarrow \mathfrak{g}(\mathbf{C})$ (respectively, $\mathfrak{p}(\mathbf{C}) \twoheadrightarrow \mathfrak{m}(\mathbf{C})$).

2.7. **Review of regular elements.** An element $u \in \mathfrak{g}(\mathbf{C})$ is *regular* if the nilpotent part of its Jordan decomposition is a principal nilpotent element in the centralizer of the semisimple part of its Jordan decomposition, or, equivalently, if the set of Borel subalgebras containing u is finite. It is well-known that the set of regular elements in $\mathfrak{g}(\mathbf{C})$ is open.

As above let M be a Levi subgroup of G and let $P \in \mathcal{P}(M)$. Suppose that u is a regular element in $\mathfrak{g}(\mathbf{C})$ that happens to lie in $\mathfrak{p}(\mathbf{C})$. Then the image u_M of u in $\mathfrak{m}(\mathbf{C})$ is regular in $\mathfrak{m}(\mathbf{C})$.

2.8. **Regular points in X^u.** We say that $x \in X^u$ is *regular* if the associated conjugacy class $\bar{u}_G(x) \in \mathfrak{g}(\mathbf{C})$ consists of regular elements. We denote by X^u_{reg} the set of regular elements in X^u; the subset X^u_{reg} is open in X^u.

As above let M be a Levi subgroup of G and let $P \in \mathcal{P}(M)$. Suppose that $u \in \mathfrak{m}(F)$. We have already seen that r_P maps X^u into X^u_M, and that the conjugacy class in $\mathfrak{g}(\mathbf{C})$ associated to $x \in X^u$ is compatible with the conjugacy class in $\mathfrak{m}(\mathbf{C})$ associated to the retracted point $x_P \in X^u_M$, compatible in the sense that there is a conjugacy class in $\mathfrak{p}(\mathbf{C})$ that maps to both of them. Therefore x_P is regular in X^u_M if x is regular in X^u.

2.9. **Set-up for the main result.** As before let M denote a Levi subgroup of G containing A. We now assume that u is an integral regular semisimple element of $\mathfrak{g}(F)$ that happens to lie in $\mathfrak{m}(F)$. (It is equivalent to assume that the centralizer T of u is contained in M_F.) For each pair $P = MN$, $P' = MN'$ of adjacent parabolic subgroups in $\mathcal{P}(M)$ we are going to define a non-negative integer $n(u, P, P')$. This collection of integers measures how far X^u sticks out from X^u_M.

As before we need the parabolic subgroups $\bar{P} = M\bar{N}$ and $\bar{P}' = M\bar{N}'$ opposite to P and P' respectively. Let α be a root of T in $N \cap \bar{N}'$. Since T, N and N' are defined over F, the group $\mathrm{Gal}(\overline{F}/F)$ preserves the set of roots of T in $N \cap \bar{N}'$. Let F_α denote the field of definition of α, so that $\mathrm{Gal}(\overline{F}/F_\alpha)$ is the stabilizer of α in $\mathrm{Gal}(\overline{F}/F)$. For any finite extension F' of F (*e.g.* F_α) we normalize the valuation $\mathrm{val}_{F'}$ on F' so that a uniformizing element in F' has valuation 1, or, equivalently, so that ϵ has valuation $[F' : F]$. There exists a unique positive integer m_α such that the image of the element $\alpha^\vee$ in Λ_M is equal to $m_\alpha \cdot \beta_{P,P'}$, where $\beta_{P,P'}$ is the element of Λ_M defined above. Note that m_α depends only on the orbit of α under the Galois group; here we use that the Galois group acts on the cocharacter group of T through the Weyl group of M, so that any two elements in the Galois orbit of $\alpha^\vee$ have the same image in Λ_M. Finally we define $n(u,P,P')$ as the sum

$$n(u,P,P') = \sum \operatorname{val}_{F_\alpha}(\alpha(u)) \cdot m_\alpha, \tag{2.9.1}$$

where the sum is taken over a set of representatives α of the orbits of $\operatorname{Gal}(\overline{F}/F)$ on the set of roots of T in $N \cap \bar{N}'$. In the special case that $M = A$ (and hence $T = A$) $n(u,P,P')$ is equal to $\operatorname{val}_F(\alpha(u))$, where α is the unique root of A that is positive for P and negative for P'.

Theorem 1. *Let M and u be as above, and let $x \in X^u$. Recall that $x_P \in X^u_M$ for all $P \in \mathcal{P}(M)$.*

(a) *For every pair $P, P' \in \mathcal{P}(M)$ of adjacent parabolic subgroups*
$$n(x,P,P') \le n(u,P,P').$$
(b) *The point x is regular in X^u if and only if the following two conditions hold:*
  (i) *the point x_P is regular in X^u_M for all $P \in \mathcal{P}(M)$, and*
  (ii) *for every pair $P, P' \in \mathcal{P}(M)$ of adjacent parabolic subgroups*
$$n(x,P,P') = n(u,P,P').$$

3. Proofs

3.1. **The case of $SL(2)$.** The key step in proving our main theorem is to verify it for $SL(2)$, where it reduces to a computation that can be found in [Lan80]. To keep things self-contained we reproduce the calculation here. Let A, B, $\bar{B}$ denote the diagonal, upper triangular and lower triangular subgroups of $SL(2)$ respectively, and let α be the unique root of A that is positive for B. Of course $\beta_{B,\bar{B}} = \alpha^\vee$. Let $x \in X$ and let $u = \begin{bmatrix} c & 0 \\ 0 & -c \end{bmatrix}$ for non-zero $c \in \mathcal{O}$. Note that $n(u,B,\bar{B}) = \operatorname{val}_F(c)$. We will show that $x \in X^u$ if and only if $n(x,B,\bar{B}) \le n(u,B,\bar{B})$, and that $x \in X^u_{\mathrm{reg}}$ if and only if $n(x,B,\bar{B}) = n(u,B,\bar{B})$.

The difference $\nu_A(x_B) - \nu_A(x_{\bar{B}})$ and the sets X^u and X^u_{reg} are invariant under the action of $A(F)$ on X, so it is enough to consider x of the form $x = gG(\mathcal{O})$ with $g = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$. (Note that for this reason our calculations apply just as well to any group whose semisimple rank is 1.) For such x we have $\nu_A(x_{\bar{B}}) = 0$. If $t \in \mathcal{O}$, then $\nu_A(x_B) = 0$. If $t \notin \mathcal{O}$, then $\begin{bmatrix} 0 & -1 \\ 1 & t^{-1} \end{bmatrix} \in G(\mathcal{O})$ and thus

$$\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} = \begin{bmatrix} t^{-1} & 1 \\ 0 & t \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & t^{-1} \end{bmatrix} \in \begin{bmatrix} t^{-1} & 1 \\ 0 & t \end{bmatrix} \cdot G(\mathcal{O}),$$

which shows that $\nu_A(x_B) = \operatorname{val}_F(t^{-1}) \cdot \alpha^\vee$. We conclude that $n(x,B,\bar{B})$ equals 0 if $t \in \mathcal{O}$ and equals $\operatorname{val}_F(t^{-1})$ if $t \notin \mathcal{O}$. In any case $n(x,B,\bar{B})$ is a non-negative integer.

For x, u as above we have

$$\operatorname{Ad}(g^{-1})u = \begin{bmatrix} c & 0 \\ -2ct & -c \end{bmatrix}.$$

Therefore $x \in X^u \iff ct \in \mathcal{O} \iff n(x,B,\bar{B}) \le n(u,B,\bar{B})$. Moreover $x \in X^u_{\mathrm{reg}}$ $\iff ct \in \mathcal{O}^\times$ or ($c \in \mathcal{O}^\times$ and $t \in \mathcal{O}$) $\iff n(x,B,\bar{B}) = n(u,B,\bar{B})$.

3.2. **Review of $n(x,P,P')$.** We need to review Arthur's proof of the existence of the non-negative integers $n(x,P,P')$. We begin with the case $M = A$. Let $x \in X$. We must check that for any two adjacent Borel subgroups P, $P' \in \mathcal{P}(A)$ there is a (unique) non-negative integer $n(x,P,P')$ such that

$$\nu_A(x_P) - \nu_A(x_{P'}) = n(x,P,P') \cdot \alpha^\vee,$$

where α is the unique root of A that is positive for P and negative for P'. For this we consider the unique parabolic subgroup Q containing P and P' whose Levi component L has semisimple rank 1. By transitivity of retractions we have

$$\nu_A(x_P) - \nu_A(x_{P'}) = \nu_A(y_B) - \nu_A(y_{\bar{B}}) \tag{3.2.1}$$

where $y = x_Q$ and $B = L \cap P$, $\bar{B} = L \cap P'$. This reduces us to the case in which G has semisimple rank 1, which has already been done. For future use we note that (3.2.1) can be reformulated as the equality

$$n(x,P,P') = n(y,B,\bar{B}).$$

Again let $x \in X$. Now we check that for any Levi subgroup $M \supset A$ and any adjacent parabolic subgroups $P = MN$, $P' = MN'$ in $\mathcal{P}(M)$ there is a (unique) non-negative integer $n(x,P,P')$ such that

$$\nu_M(x_P) - \nu_M(x_{P'}) = n(x,P,P') \cdot \beta_{P,P'}.$$

Fix a Borel subgroup B_M in M and let B (respectively, B') be the inverse image of B_M under $P \to M$ (respectively, $P' \to M$); thus B, B' are Borel subgroups containing A.

Now choose a minimal gallery of Borel subgroups $B = B_0, B_1, B_2, \dots, B_l = B'$ joining B to B', and for $i = 1, \dots, l$ let α_i be the unique root of A that is positive for B_{i-1} and negative for B_i. Then

$$\nu_A(x_B) - \nu_A(x_{B'}) = \sum_{i=1}^{l} n(x, B_{i-1}, B_i) \cdot \alpha_i^\vee.$$

Note that $\{\alpha_1, \dots, \alpha_l\}$ is precisely the set of roots of A in $\mathfrak{n} \cap \bar{\mathfrak{n}}'$ and that for each i there exists a (unique) positive integer m_i such that the image of $\alpha_i^\vee$ in Λ_M is equal to $m_i \cdot \beta_{P,P'}$. Applying the canonical surjection $\Lambda_A \to \Lambda_M$ to the previous equation, we find that (see 2.3)

$$\nu_M(x_P) - \nu_M(x_{P'}) = n(x,P,P') \cdot \beta_{P,P'}.$$

where $n(x,P,P')$ is the non-negative integer

$$\sum_{i=1}^{l} m_i \cdot n(x, B_{i-1}, B_i).$$

3.3. **Proof of part of the main theorem in case $A = T$.** Let $u \in \mathfrak{a}(\mathcal{O})$ and assume that u is regular in $\mathfrak{g}(F)$. Let $x \in X^u$.

Let M be a Levi subgroup of G containing A. We are now going to prove the first main assertion in our theorem, namely that for any pair of adjacent $P, P' \in \mathcal{P}(M)$ there is an inequality

$$n(x, P, P') \leq n(u, P, P').$$

Let $B, B', B_0, \dots, B_l$ and α_i, m_i $(i = 1, \dots, l)$ be as in 3.2. Then by definition

$$n(u, P, P') = \sum_{i=1}^{l} m_i \cdot \mathrm{val}_F(\alpha_i(u)).$$

Let M_i be the Levi subgroup containing A whose root system is $\{\pm\alpha_i\}$, and let B'_{i-1}, B'_i denote the Borel subgroups in M_i obtained by intersecting B_{i-1}, B_i with M_i. Let Q_i be the unique parabolic subgroup in $\mathcal{P}(M_i)$ such that Q_i contains B_{i-1} and B_i. We showed in 3.2 that

$$n(x, P, P') = \sum_{i=1}^{l} m_i \cdot n(x, B_{i-1}, B_i)$$

and that

$$n(x, B_{i-1}, B_i) = n(y_i, B'_{i-1}, B'_i),$$

where $y_i = x_{Q_i} \in X^u_{M_i}$. Since M_i has semisimple rank 1, we know that

$$n(y_i, B'_{i-1}, B'_i) \leq \mathrm{val}_F(\alpha_i(u)).$$

This completes the proof of the first main assertion.

Now suppose that x is regular in X^u. Then each point $y_i \in X^u_{M_i}$ above is regular in $X^u_{M_i}$, and therefore from the rank 1 case (see 3.1) we know that

$$n(y_i, B'_{i-1}, B'_i) = \mathrm{val}_F(\alpha_i(u)).$$

We conclude that if x is regular in X^u, then

$$n(x, P, P') = n(u, P, P'),$$

which is another of the assertions in our theorem.

3.4. **Proof of the rest of the main theorem in case $M = A = T$.** We continue with $u \in \mathfrak{a}(\mathcal{O})$ and $x \in X^u$ as before, but for the moment we only consider the case $M = A$. We assume that

$$n(x, P, P') = \mathrm{val}_F(\alpha_{P,P'}(u)) \tag{3.4.1}$$

for all adjacent Borel subgroups $P, P' \in \mathcal{P}(A)$, where $\alpha_{P,P'}$ is the unique root of A that is positive for P and negative for P'. We want to prove that x is regular in X^u. To do so we must first select a suitable Borel subgroup $B \in \mathcal{P}(A)$.

Let $u_0 \in \mathfrak{a}(\mathbf{C})$ denote the image of u under $\mathfrak{a}(\mathcal{O}) \to \mathfrak{a}(\mathbf{C})$, and let M denote the centralizer of u_0 in G. Thus M is a Levi subgroup of G containing A, and we choose $P \in \mathcal{P}(M)$. Then we obtain a suitable Borel subgroup by taking any $B \in \mathcal{P}(A)$ such that $B \subset P$. For any B-simple root α we denote by B_α the unique Borel subgroup in $\mathcal{P}(A)$ that is adjacent to B and for which α is negative, and we write P_α for the unique parabolic subgroup containing B and B_α such that the semisimple rank of the Levi component M_α of P_α is 1. Consider the element (well-defined up to $B(\mathbf{C})$-conjugacy) $v := \bar{u}_B(x) \in \mathfrak{b}(\mathbf{C})$ defined in 2.6. The equation (3.4.1) plus the semisimple rank 1 theory implies that the points $x_{P_\alpha} \in X^u_{M_\alpha}$ are regular, and this in turn implies (see 2.6) that for every B-simple root α the image of the element v under $\mathfrak{b}(\mathbf{C}) \hookrightarrow \mathfrak{p}_\alpha(\mathbf{C}) \twoheadrightarrow \mathfrak{m}_\alpha(\mathbf{C})$ is regular in $\mathfrak{m}_\alpha(\mathbf{C})$. Moreover it is evident that the image of v under the canonical surjection $\mathfrak{b}(\mathbf{C}) \twoheadrightarrow \mathfrak{a}(\mathbf{C})$ is equal to u_0. Using only these facts, we now check that v is regular in $\mathfrak{g}(\mathbf{C})$ (and hence that x is regular in X^u).

Let $v = v_s + v_n$ be the Jordan decomposition of v, with v_s semisimple and v_n nilpotent. Since it is harmless to replace v by any $B(\mathbf{C})$-conjugate, we may assume without loss of generality that $v_s \in \mathfrak{a}(\mathbf{C})$. Then, since $v_s \mapsto u_0$ under $\mathfrak{b}(\mathbf{C}) \twoheadrightarrow \mathfrak{a}(\mathbf{C})$, it follows that $v_s = u_0$. Since v_n commutes with $v_s = u_0$, it lies in $\mathfrak{m}(\mathbf{C})$, and we must check that v_n is a principal nilpotent element in $\mathfrak{m}(\mathbf{C})$. As v_n lies in the Borel subalgebra $(\mathfrak{b} \cap \mathfrak{m})(\mathbf{C})$ of $\mathfrak{m}(\mathbf{C})$, it is enough to check that the projection of v_n into each simple root space of $(\mathfrak{b} \cap \mathfrak{m})(\mathbf{C})$ is non-zero, and this follows from the statement (proved above) that the image of v under $\mathfrak{b}(\mathbf{C}) \hookrightarrow \mathfrak{p}_\alpha(\mathbf{C}) \twoheadrightarrow \mathfrak{m}_\alpha(\mathbf{C})$ is regular in $\mathfrak{m}_\alpha(\mathbf{C})$ for every simple root α of A in M.

3.5. **End of the proof of the main theorem in case $A = T$.** We continue with $u \in \mathfrak{a}(\mathcal{O})$ and $x \in X^u$ as above. Let M be any Levi subgroup containing A. It remains to prove that if x_P is regular in X^u_M for all $P \in \mathcal{P}(M)$ and if

$$n(x, P, P') = n(u, P, P') \tag{3.5.1}$$

for every adjacent pair $P, P' \in \mathcal{P}(M)$, then x is regular in X^u. We have already proved this in case $M = A$, and now we want to reduce the general case to this special case.

The equality (3.5.1) is equivalent to the equality

$$\nu_M(x_P) - \nu_M(x_{P'}) = n(u, P, P') \cdot \beta_{P,P'}. \tag{3.5.2}$$

Fix $P \in \mathcal{P}(M)$ and sum (3.5.2) over the set of neighboring pairs in a minimal gallery joining P to its opposite $\bar{P} \in \mathcal{P}(M)$. Doing this yields the equality

$$\nu_M(x_P) - \nu_M(x_{\bar{P}}) = \sum_{\alpha \in R_N} \mathrm{val}_F(\alpha(u)) \cdot \pi_M(\alpha^\vee), \tag{3.5.3}$$

where $\pi_M : X_*(A) \to \Lambda_M$ is the canonical surjection and R_N is the set of roots of A in $\mathfrak{n}$.

Fix a Borel subgroup B_M in M containing A and let B (respectively, B_1) be the Borel subgroups in $\mathcal{P}(A)$ obtained as the inverse image of B_M under $P \to M$ (respectively, $\bar{P} \to M$). Then (3.5.3) implies (see 2.3) that

$$\nu_A(x_B) - \nu_A(x_{B_1}) \equiv \sum_{\alpha \in R_N} \mathrm{val}_F(\alpha(u)) \cdot \alpha^\vee$$

modulo the coroot lattice for M. Since R_N is also the set of roots that are positive on B and negative on B_1, it follows that

$$\nu_A(x_B) - \nu_A(x_{B_1}) = \sum_{\alpha \in R_N} j_\alpha \cdot \alpha^\vee$$

for some integers j_α such that $0 \le j_\alpha \le \mathrm{val}_F(\alpha(u))$. (To prove this pick a minimal gallery joining B to B_1 and use the inequality stated in the main theorem for each neighboring pair in the gallery.) Comparing this equality with the congruence, we see that the linear combination

$$\sum_{\alpha \in R_N} (\mathrm{val}_F(\alpha(u)) - j_\alpha) \cdot \alpha^\vee \tag{3.5.4}$$

maps to 0 in Λ_M.

We get a basis for $\Lambda_M \otimes \mathbf{R}$ by taking the elements $\beta_{P,P'}$ as P' varies through the set of parabolic subgroups in $\mathcal{P}(M)$ adjacent to P. Moreover for any $\alpha \in R_N$ the image $\pi_M(\alpha^\vee)$ of $\alpha^\vee$ in Λ_M is a non-negative linear combination of basis elements $\beta_{P,P'}$ (with at least one non-zero coefficient). Therefore the fact that (3.5.4) maps to 0 in Λ_M means that

$$\nu_A(x_B) - \nu_A(x_{B_1}) = \sum_{\alpha \in R_N} \operatorname{val}_F(\alpha(u)) \cdot \alpha^\vee \tag{3.5.5}$$

By hypothesis $x_{\bar{P}}$ is regular. Therefore (transitivity of retractions plus the part of our theorem we have already proved) for all adjacent Borel subgroups $B_1, B_2 \in \mathcal{P}(A)$ such that $B_1, B_2 \subset \bar{P}$ we have

$$\nu_A(x_{B_1}) - \nu_A(x_{B_2}) = \operatorname{val}_F(\alpha_{B_1,B_2}(u)) \cdot \alpha^\vee_{B_1,B_2},$$

where α_{B_1,B_2} denotes the unique root that is positive on B_1 and negative on B_2. Summing these equalities over neighboring pairs in a minimal gallery joining B_1 to $\bar{B}$, we find that

$$\nu_A(x_{B_1}) - \nu_A(x_{\bar{B}}) = \sum_{\alpha \in R_M^+} \operatorname{val}_F(\alpha(u)) \cdot \alpha^\vee,$$

where R_M^+ denotes the set of roots of A in B_M. Adding this last equality to (3.5.5), we see that

$$\nu_A(x_B) - \nu_A(x_{\bar{B}}) = \sum_{\alpha \in R^+} \operatorname{val}_F(\alpha(u)) \cdot \alpha^\vee. \tag{3.5.6}$$

Now consider any minimal gallery $B = B_0, B_1, \dots, B_l = \bar{B}$ joining B to $\bar{B}$. Then

$$\nu_A(x_B) - \nu_A(x_{\bar{B}}) = \sum_{i=1}^{l} n(x, B_{i-1}, B_i) \cdot \alpha_i^\vee, \tag{3.5.7}$$

where α_i is the unique root that is positive for B_{i-1} and negative for B_i. We know that $n(x, B_{i-1}, B_i) \le \operatorname{val}_F(\alpha_i(u))$ for all i. Subtracting (3.5.7) from (3.5.6), we find that 0 is a non-negative linear combination of positive roots; therefore each coefficient in this linear combination is 0, which means that

$$n(x, B_{i-1}, B_i) = \operatorname{val}_F(\alpha_i(u))$$

for $i = 1, \dots, l$.

Now consider any pair B', B'' of adjacent Borel subgroups in $\mathcal{P}(A)$. After reversing the order of B', B'' if necessary we can find a minimal gallery as above and an index i such that $(B_{i-1}, B_i) = (B', B'')$. Therefore

$$n(x, B', B'') = \operatorname{val}_F(\alpha(u)), \tag{3.5.8}$$

where α is the unique root that is positive on B' and negative on B''. Since both sides of (3.5.8) remain unchanged when B', B'' are switched, we see that (3.5.8) holds for any adjacent pair B', B''. By what we have already done, it follows that x is regular in X^u.

3.6. **Proof of the main theorem in general.** Now let M be any Levi subgroup of G containing A, and let u be an integral regular semisimple element of $\mathfrak{g}(F)$ that happens to lie in $\mathfrak{m}(F)$. Let $T = \mathrm{Cent}_{G_F}(u)$, a maximal torus in M_F. We choose a finite extension F'/F that splits T.

We normalize the valuation $\mathrm{val}_{F'}$ on F' so that uniformizing elements in F' have valuation 1. Thus $\mathrm{val}_{F'}(\epsilon) = [F' : F]$. We write X' for the set $G(F')/G(\mathcal{O}_{F'})$. The inclusion $G(F) \hookrightarrow G(F')$ induces a canonical injection $X \hookrightarrow X'$.

For any $P \in \mathcal{P}(M)$ the diagram

$$\begin{array}{ccc} X & \xrightarrow{r_P} & X_M \\ \downarrow & & \downarrow \\ X' & \xrightarrow{r'_P} & X'_M \end{array}$$

commutes, where the horizontal maps are retractions and the vertical maps are the canonical injections. Moreover the diagram

$$\begin{array}{ccc} X & \xrightarrow{\nu_G} & \Lambda_G \\ \downarrow & & \downarrow \\ X' & \xrightarrow{\nu'_G} & \Lambda_G \end{array}$$

commutes, where the left vertical map is the canonical injection and the right vertical map is multiplication by $e := [F' : F]$.

For any $x \in X^u$ the image of x in X' lies in $(X')^u$, and x is regular in X^u if and only if x is regular in $(X')^u$. Indeed the conjugacy class $\bar{u}_G(x)$ attached to u and x is the same for X and X'.

The torus T is conjugate under $M(F')$ to A, so our theorem is true for T over F'. Therefore for $x \in X^u$ and adjacent $P = MN, P' = MN' \in \mathcal{P}(M)$

$$e \cdot n(x, P, P') \leq \sum_{\alpha \in R_N \cap R_{\bar{N}'}} \mathrm{val}_{F'}(\alpha(u)) \cdot m_\alpha, \tag{3.6.1}$$

and x is regular in X^u if and only if all of these inequalities are equalities. (As before R_N denotes the set of roots of A in $\mathfrak{n}$; the positive integers m_α were defined in 2.9.) Dividing by e, and noting that the term indexed by α depends only on the Γ-orbit of α, we find that (3.6.1) is equivalent to the inequality

$$n(x, P, P') \leq n(u, P, P').$$

This completes the proof of the theorem.

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