

## REGULAR POINTS IN AFFINE SPRINGER FIBERS

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## 1. INTRODUCTION

Let  $G$  be a connected reductive group over  $\mathbf{C}$  with Lie algebra  $\mathfrak{g}$ . We put  $F = \mathbf{C}((\epsilon))$  and  $\mathcal{O} = \mathbf{C}[[\epsilon]]$ . Let  $X = X_G$  denote the affine Grassmannian  $G(F)/G(\mathcal{O})$ . For  $u \in \mathfrak{g}(F)$  we write  $X^u$  for the affine Springer fiber

$$X^u = \{g \in G(F)/G(\mathcal{O}) : \text{Ad}(g^{-1})(u) \in \mathfrak{g}(\mathcal{O})\}.$$

studied by Kazhdan and Lusztig in [KL88].

For  $x = gG(\mathcal{O}) \in X^u$  the  $G(\mathcal{O})$ -orbit (for the adjoint action) of  $\text{Ad}(g^{-1})(u)$  in  $\mathfrak{g}(\mathcal{O})$  depends only on  $x$ , and its image under  $\mathfrak{g}(\mathcal{O}) \rightarrow \mathfrak{g}(\mathbf{C})$  is a well-defined  $G(\mathbf{C})$ -orbit in  $\mathfrak{g}(\mathbf{C})$ . We say that  $x \in X^u$  is *regular* if the associated orbit is regular in  $\mathfrak{g}(\mathbf{C})$ . (Recall that an element of  $\mathfrak{g}(\mathbf{C})$  is regular if the nilpotent part of its Jordan decomposition is a principal nilpotent element in the centralizer of the semisimple part of its Jordan decomposition.) We write  $X_{\text{reg}}^u$  for the (Zariski open) subset of regular elements in  $X^u$ .

From now on we assume that  $u$  is regular semisimple with centralizer  $T$ , a maximal torus in  $G$  over  $F$ . Assume further that  $u$  is *integral*, by which we mean that  $X^u$  is non-empty. Kazhdan and Lusztig [KL88] show that  $X^u$  is then a locally finite union of projective algebraic varieties, and in Cor. 1 of §4 of [KL88] they show that the open subset  $X_{\text{reg}}^u$  of  $X^u$  is non-empty (and hence dense in at least one irreducible component of  $X^u$ ). The action of  $T(F)$  on  $X$  clearly preserves the subsets  $X^u$  and  $X_{\text{reg}}^u$ . Bezrukavnikov [Bez96] proved that  $X_{\text{reg}}^u$  forms a single orbit under  $T(F)$ . (Actually Kazhdan-Lusztig and Bezrukavnikov consider only topologically nilpotent elements  $u$ , but the general case can be reduced to their special case by using the topological Jordan decomposition of  $u$ .)

The goal of this paper is to characterize regular elements in  $X^u$  (for integral regular semisimple  $u$  as above). When  $T$  is elliptic (in other words,  $F$ -anisotropic modulo the center of  $G$ ) the characterization gives no new information. At the other extreme, in the split case, the characterization gives a clear picture of what it means for a point in  $X^u$  to be regular.

We will now state our characterization in the split case, leaving the more technical general statement to the next section (see Theorem 1). Fix a split maximal torus  $A \subset G$  over  $\mathbf{C}$  and denote by  $\mathfrak{a}$  its Lie algebra. We identify the affine

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Grassmannian  $A(F)/A(\mathcal{O})$  for  $A$  with the cocharacter lattice  $X_*(A)$ , the cocharacter  $\mu$  corresponding to the class of  $\mu(\epsilon)$  in  $A(F)/A(\mathcal{O})$ . For any Borel subgroup  $B = AN$  containing  $A$  ( $N$  denoting the unipotent radical of  $B$ ) there is a well-known retraction  $r_B : X \rightarrow X_*(A)$  defined using the Iwasawa decomposition  $G(F) = N(F)A(F)G(\mathcal{O})$ : the fiber of  $r_B$  over  $\mu \in X_*(A)$  is  $N(F)\mu(\epsilon)G(\mathcal{O})/G(\mathcal{O})$ . The family of cocharacters  $r_B(x)$  ( $B$  ranging through all Borel subgroups containing  $A$ ) has been studied by Arthur [Art76, Lemma 3.6]; it is the volume of the convex hull of these points that arises as the weight factor for (fully) weighted orbital integrals for elements in  $A(F)$ . In particular Arthur shows that for  $x \in X$  and any pair  $B, B'$  of adjacent Borel subgroups containing  $A$ , there is a unique non-negative integer  $n(x, B, B')$  such that

$$(1.0.1) \quad r_B(x) - r_{B'}(x) = n(x, B, B') \cdot \alpha_{B, B'}^\vee$$

where  $\alpha_{B, B'}$  is the unique root of  $A$  that is positive for  $B$  and negative for  $B'$ .

The main result of this paper (in the split case) is that for  $x \in X^u$

$$(1.0.2) \quad n(x, B, B') \leq \text{val } \alpha_{B, B'}(u)$$

for every pair  $B, B'$  of adjacent Borel subgroups containing  $A$ , and that  $x \in X^u$  is regular if and only if all the inequalities (1.0.2) are actually equalities.

## 2. STATEMENTS

**2.1. Notation.** We write  $\mathfrak{g}$  for the Lie algebra of  $G$  and follow the same convention for groups denoted by other letters.

Choose an algebraic closure  $\overline{F}$  of  $F$  and let  $\Gamma = \text{Gal}(\overline{F}/F)$ . We write  $G_F$  for the  $F$ -group obtained from  $G$  by extension of scalars from  $\mathbf{C}$  to  $F$ .

As before we use  $\mu \mapsto \mu(\epsilon)$  to identify the cocharacter group  $X_*(A)$  with  $A(F)/A(\mathcal{O})$ . By means of this identification the canonical surjection  $A(F) \rightarrow A(F)/A(\mathcal{O})$  can be viewed as a surjection

$$(2.1.1) \quad A(F) \rightarrow X_*(A).$$

Let  $\Lambda = \Lambda_G$  denote the quotient of the coweight lattice  $X_*(A)$  by the coroot lattice (the subgroup of  $X_*(A)$  generated by the coroots of  $A$  in  $G$ ). Up to canonical isomorphism  $\Lambda$  is independent of the choice of  $A$ ; moreover when defining  $\Lambda$  we could replace  $A$  by any maximal torus  $T$  in  $G_F$ . There is a canonical surjective homomorphism

$$(2.1.2) \quad G(F) \rightarrow \Lambda,$$

characterized by the following two properties: it is trivial on the image of  $G_{\text{sc}}(F)$  in  $G(F)$  (where  $G_{\text{sc}}$  denotes the simply connected cover of the derived group of  $G$ ), and its restriction to  $A(F)$  coincides with the composition of (2.1.1) and the canonical surjection  $X_*(A) \rightarrow \Lambda$ .

Recall that  $X$  denotes the affine Grassmannian  $G(F)/G(\mathcal{O})$  for  $G$ . The homomorphism (2.1.2) is trivial on  $G(\mathcal{O})$  and hence induces a canonical surjection

$$(2.1.3) \quad \nu_G : X \rightarrow \Lambda,$$

whose fibers are the connected components of  $X$ .

**2.2. Parabolic subgroups.** We will be concerned with parabolic subgroups  $P$  of  $G$  containing  $A$ . Such a parabolic subgroup has a unique Levi subgroup  $M$  containing  $A$ , and we refer to  $M$  as *the* Levi component of  $P$ .

As usual, by a Levi subgroup of  $G$ , we mean a Levi subgroup of some parabolic subgroup of  $G$ . Let  $M$  be a Levi subgroup of  $G$  containing  $A$ . We write  $\mathcal{P}(M)$  for the set of parabolic subgroups of  $G$  that contain  $A$  and have Levi component  $M$ . Thus any  $P \in \mathcal{P}(M)$  can be written as  $P = MN$  where  $N = N_P$  denotes the unipotent radical of  $P$ . As usual there is a notion of adjacency: two parabolic subgroups  $P = MN$  and  $P' = MN'$  in  $\mathcal{P}(M)$  are said to be *adjacent* if there exists (a unique) parabolic subgroup  $Q = LU$  containing both  $P$  and  $P'$  such that the semisimple rank of  $L$  is one greater than the semisimple rank of  $M$ . Thus  $U = N \cap N'$ , and, moreover, if  $L$  is chosen so that  $L \supset A$ , then

$$\mathfrak{l} = \mathfrak{m} \oplus (\mathfrak{n} \cap \bar{\mathfrak{n}}') \oplus (\mathfrak{n}' \cap \bar{\mathfrak{n}})$$

where  $\bar{N}$  denotes the unipotent radical of the parabolic subgroup  $\bar{P} = M\bar{N}$  opposite to  $P$  (and where  $\bar{N}'$  is opposite to  $N'$ ).

Given adjacent  $P, P'$  in  $\mathcal{P}(M)$  we define an element  $\beta_{P,P'} \in \Lambda_M$  (the coweight lattice for  $A$  modulo the coroot lattice for  $M$ ) as follows. Consider the collection of elements in  $\Lambda_M$  obtained from coroots  $\alpha^\vee$  where  $\alpha$  ranges through the set of roots of  $A$  in  $\mathfrak{n} \cap \bar{\mathfrak{n}}'$ . We define  $\beta_{P,P'}$  to be the unique element in this collection such that all other members in the collection are positive integral multiples of  $\beta_{P,P'}$ . Note that although  $\Lambda_M$  may have torsion elements, the elements in our collection lie in the kernel of the canonical map from  $\Lambda_M$  to  $\Lambda_G$ , and this kernel is torsion-free. Thus any member of our collection can be written uniquely as a positive integer times  $\beta_{P,P'}$ . Note also that  $\beta_{P',P} = -\beta_{P,P'}$ . In case  $M = A$ , so that  $P, P'$  are Borel subgroups,  $\beta_{P,P'}$  is the unique coroot of  $A$  that is positive for  $P$  and negative for  $P'$ .

**2.3. Retractions from  $X$  to  $X_M$ .** The inclusion of  $M(F)$  into  $G(F)$  induces an inclusion of the affine Grassmannian  $X_M$  for  $M$  into the affine Grassmannian  $X$  for  $G$ . Let  $P \in \mathcal{P}(M)$  and let  $X_P$  denote the set  $P(F)/P(\mathcal{O})$ . The canonical inclusion of  $P$  in  $G$  induces a bijection  $i$  from  $X_P$  to  $X$ , and the canonical surjection  $P \rightarrow M$  induces a canonical surjective map  $p$  (of sets) from  $X_P$  to  $X_M$ . We define the retraction  $r_P = r_P^G : X \rightarrow X_M$  as the composed map  $p \circ i^{-1}$ . Given  $x \in X$  we often denote by  $x_P$  the image of  $x$  under the retraction  $r_P$ .

These retractions satisfy the following transitivity property. Suppose that  $L \supset M$  are Levi subgroups containing  $A$ , and suppose further that  $P \in \mathcal{P}(M)$  and  $Q \in \mathcal{P}(L)$  satisfy  $Q \supset P$ . Let  $P_L$  denote the parabolic subgroup  $P \cap L$  in  $L$ . Then

$$(2.3.1) \quad r_P^G = r_{P_L}^L \circ r_Q^G.$$

Moreover, for any  $x \in X$  the element  $\nu_M(x_P)$  maps to  $\nu_L(x_Q)$  under the canonical surjection  $\Lambda_M \rightarrow \Lambda_L$ , and in particular  $\nu_M(x_P) \mapsto \nu_G(x)$  under  $\Lambda_M \rightarrow \Lambda_G$ .

**2.4. Definition of  $n(x, P, P')$ .** A point  $x \in X$  determines points  $\nu_M(x_P)$  in  $\Lambda_M$ , one for each  $P \in \mathcal{P}(M)$ . This family of points arises in the definition of the weighted orbital integrals occurring in Arthur's work. A basic fact [Art76] about this family of points is that whenever  $P, P'$  are adjacent parabolic subgroups in  $\mathcal{P}(M)$ , there is a (unique) non-negative integer  $n(x, P, P')$  such that

$$(2.4.1) \quad \nu_M(x_P) - \nu_M(x_{P'}) = n(x, P, P') \cdot \beta_{P,P'}.$$

**2.5. Fixed point sets  $X^u$ .** Let  $u \in \mathfrak{g}(F)$ . Define a subset  $X^u$  of  $X$  by

$$X^u = \{g \in G(F)/G(\mathcal{O}) : \text{Ad}(g^{-1})(u) \in \mathfrak{g}(\mathcal{O})\}.$$

**2.6. Conjugacy classes associated to fixed points.** Let  $u \in \mathfrak{g}(F)$ . Suppose that the coset  $x = gG(\mathcal{O})$  lies in  $X^u$ . The image of  $\text{Ad}(g^{-1})(u)$  under the canonical surjection  $\mathfrak{g}(\mathcal{O}) \rightarrow \mathfrak{g}(\mathbf{C})$  gives a well-defined  $G(\mathbf{C})$ -conjugacy class  $\bar{u}_G(x)$  (for the adjoint action) in  $\mathfrak{g}(\mathbf{C})$ .

As above let  $M$  be a Levi subgroup of  $G$  and let  $P \in \mathcal{P}(M)$ . Now suppose that  $u \in \mathfrak{m}(F)$  and that  $x \in X^u$ . Choose  $p \in P(F)$  such that  $x = pG(\mathcal{O})$ ; thus  $x_P$  is the coset  $mM(\mathcal{O})$ , where  $m$  denotes the image of  $p$  under the canonical homomorphism from  $P$  onto  $M$ . Of course  $\text{Ad}(p^{-1})(u)$  lies in  $\mathfrak{p}(\mathcal{O})$ , and its image in  $\mathfrak{p}(\mathbf{C})$  gives a well-defined  $P(\mathbf{C})$ -conjugacy class  $\bar{u}_P(x)$  in  $\mathfrak{p}(\mathbf{C})$ . It follows that  $x_P$  lies in  $X_M^u$  (as was first noted by Kazhdan-Lusztig [KL88]), and also that  $\bar{u}_P(x)$  maps to  $\bar{u}_G(x)$  (respectively,  $\bar{u}_M(x_P)$ ) under the map on conjugacy classes induced by  $\mathfrak{p}(\mathbf{C}) \hookrightarrow \mathfrak{g}(\mathbf{C})$  (respectively,  $\mathfrak{p}(\mathbf{C}) \rightarrow \mathfrak{m}(\mathbf{C})$ ).

**2.7. Review of regular elements.** An element  $u \in \mathfrak{g}(\mathbf{C})$  is *regular* if the nilpotent part of its Jordan decomposition is a principal nilpotent element in the centralizer of the semisimple part of its Jordan decomposition, or, equivalently, if the set of Borel subalgebras containing  $u$  is finite. It is well-known that the set of regular elements in  $\mathfrak{g}(\mathbf{C})$  is open.

As above let  $M$  be a Levi subgroup of  $G$  and let  $P \in \mathcal{P}(M)$ . Suppose that  $u$  is a regular element in  $\mathfrak{g}(\mathbf{C})$  that happens to lie in  $\mathfrak{p}(\mathbf{C})$ . Then the image  $u_M$  of  $u$  in  $\mathfrak{m}(\mathbf{C})$  is regular in  $\mathfrak{m}(\mathbf{C})$ .

**2.8. Regular points in  $X^u$ .** We say that  $x \in X^u$  is *regular* if the associated conjugacy class  $\bar{u}_G(x) \in \mathfrak{g}(\mathbf{C})$  consists of regular elements. We denote by  $X_{\text{reg}}^u$  the set of regular elements in  $X^u$ ; the subset  $X_{\text{reg}}^u$  is open in  $X^u$ .

As above let  $M$  be a Levi subgroup of  $G$  and let  $P \in \mathcal{P}(M)$ . Suppose that  $u \in \mathfrak{m}(F)$ . We have already seen that  $r_P$  maps  $X^u$  into  $X_M^u$ , and that the conjugacy class in  $\mathfrak{g}(\mathbf{C})$  associated to  $x \in X^u$  is compatible with the conjugacy class in  $\mathfrak{m}(\mathbf{C})$  associated to the retracted point  $x_P \in X_M^u$ , compatible in the sense that there is a conjugacy class in  $\mathfrak{p}(\mathbf{C})$  that maps to both of them. Therefore  $x_P$  is regular in  $X_M^u$  if  $x$  is regular in  $X^u$ .

**2.9. Set-up for the main result.** As before let  $M$  denote a Levi subgroup of  $G$  containing  $A$ . We now assume that  $u$  is an integral regular semisimple element of  $\mathfrak{g}(F)$  that happens to lie in  $\mathfrak{m}(F)$ . (It is equivalent to assume that the centralizer  $T$  of  $u$  is contained in  $M_F$ .) For each pair  $P = MN$ ,  $P' = MN'$  of adjacent parabolic subgroups in  $\mathcal{P}(M)$  we are going to define a non-negative integer  $n(u, P, P')$ . This collection of integers measures how far  $X^u$  sticks out from  $X_M^u$ .

As before we need the parabolic subgroups  $\bar{P} = M\bar{N}$  and  $\bar{P}' = M\bar{N}'$  opposite to  $P$  and  $P'$  respectively. Let  $\alpha$  be a root of  $T$  in  $N \cap \bar{N}'$ . Since  $T$ ,  $N$  and  $N'$  are defined over  $F$ , the group  $\text{Gal}(\bar{F}/F)$  preserves the set of roots of  $T$  in  $N \cap \bar{N}'$ . Let  $F_\alpha$  denote the field of definition of  $\alpha$ , so that  $\text{Gal}(\bar{F}/F_\alpha)$  is the stabilizer of  $\alpha$  in  $\text{Gal}(\bar{F}/F)$ . For any finite extension  $F'$  of  $F$  (e.g.  $F_\alpha$ ) we normalize the valuation  $\text{val}_{F'}$  on  $F'$  so that a uniformizing element in  $F'$  has valuation 1, or, equivalently, so that  $\epsilon$  has valuation  $[F' : F]$ . There exists a unique positive integer  $m_\alpha$  such that the image of the element  $\alpha^\vee$  in  $\Lambda_M$  is equal to  $m_\alpha \cdot \beta_{P, P'}$ , where  $\beta_{P, P'}$  is the element of  $\Lambda_M$  defined above. Note that  $m_\alpha$  depends only on the orbit of  $\alpha$  under

the Galois group; here we use that the Galois group acts on the cocharacter group of  $T$  through the Weyl group of  $M$ , so that any two elements in the Galois orbit of  $\alpha^\vee$  have the same image in  $\Lambda_M$ . Finally we define  $n(u, P, P')$  as the sum

$$(2.9.1) \quad n(u, P, P') = \sum \text{val}_{F_\alpha}(\alpha(u)) \cdot m_\alpha,$$

where the sum is taken over a set of representatives  $\alpha$  of the orbits of  $\text{Gal}(\overline{F}/F)$  on the set of roots of  $T$  in  $N \cap \overline{N}'$ . In the special case that  $M = A$  (and hence  $T = A$ )  $n(u, P, P')$  is equal to  $\text{val}_F(\alpha(u))$ , where  $\alpha$  is the unique root of  $A$  that is positive for  $P$  and negative for  $P'$ .

**Theorem 1.** *Let  $M$  and  $u$  be as above, and let  $x \in X^u$ . Recall that  $x_P \in X_M^u$  for all  $P \in \mathcal{P}(M)$ .*

(a) *For every pair  $P, P' \in \mathcal{P}(M)$  of adjacent parabolic subgroups*

$$n(x, P, P') \leq n(u, P, P').$$

(b) *The point  $x$  is regular in  $X^u$  if and only if the following two conditions hold:*

- (i) *the point  $x_P$  is regular in  $X_M^u$  for all  $P \in \mathcal{P}(M)$ , and*
- (ii) *for every pair  $P, P' \in \mathcal{P}(M)$  of adjacent parabolic subgroups*

$$n(x, P, P') = n(u, P, P').$$

### 3. PROOFS

**3.1. The case of  $SL(2)$ .** The key step in proving our main theorem is to verify it for  $SL(2)$ , where it reduces to a computation that can be found in [Lan80]. To keep things self-contained we reproduce the calculation here. Let  $A, B, \bar{B}$  denote the diagonal, upper triangular and lower triangular subgroups of  $SL(2)$  respectively, and let  $\alpha$  be the unique root of  $A$  that is positive for  $B$ . Of course  $\beta_{B, \bar{B}} = \alpha^\vee$ . Let  $x \in X$  and let  $u = \begin{bmatrix} c & 0 \\ 0 & -c \end{bmatrix}$  for non-zero  $c \in \mathcal{O}$ . Note that  $n(u, B, \bar{B}) = \text{val}_F(c)$ . We will show that  $x \in X^u$  if and only if  $n(x, B, \bar{B}) \leq n(u, B, \bar{B})$ , and that  $x \in X_{\text{reg}}^u$  if and only if  $n(x, B, \bar{B}) = n(u, B, \bar{B})$ .

The difference  $\nu_A(x_B) - \nu_A(x_{\bar{B}})$  and the sets  $X^u$  and  $X_{\text{reg}}^u$  are invariant under the action of  $A(F)$  on  $X$ , so it is enough to consider  $x$  of the form  $x = gG(\mathcal{O})$  with  $g = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$ . (Note that for this reason our calculations apply just as well to any group whose semisimple rank is 1.) For such  $x$  we have  $\nu_A(x_B) = 0$ . If  $t \in \mathcal{O}$ , then  $\nu_A(x_B) = 0$ . If  $t \notin \mathcal{O}$ , then  $\begin{bmatrix} 0 & -1 \\ 1 & t^{-1} \end{bmatrix} \in G(\mathcal{O})$  and thus

$$\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} = \begin{bmatrix} t^{-1} & 1 \\ 0 & t \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & t^{-1} \end{bmatrix} \in \begin{bmatrix} t^{-1} & 1 \\ 0 & t \end{bmatrix} \cdot G(\mathcal{O}),$$

which shows that  $\nu_A(x_B) = \text{val}_F(t^{-1}) \cdot \alpha^\vee$ . We conclude that  $n(x, B, \bar{B})$  equals 0 if  $t \in \mathcal{O}$  and equals  $\text{val}_F(t^{-1})$  if  $t \notin \mathcal{O}$ . In any case  $n(x, B, \bar{B})$  is a non-negative integer.

For  $x, u$  as above we have

$$\text{Ad}(g^{-1})u = \begin{bmatrix} c & 0 \\ -2ct & -c \end{bmatrix}.$$

Therefore  $x \in X^u \iff ct \in \mathcal{O} \iff n(x, B, \bar{B}) \leq n(u, B, \bar{B})$ . Moreover  $x \in X_{\text{reg}}^u \iff ct \in \mathcal{O}^\times$  or  $(c \in \mathcal{O}^\times \text{ and } t \in \mathcal{O}) \iff n(x, B, \bar{B}) = n(u, B, \bar{B})$ .

**3.2. Review of  $n(x, P, P')$ .** We need to review Arthur's proof of the existence of the non-negative integers  $n(x, P, P')$ . We begin with the case  $M = A$ . Let  $x \in X$ . We must check that for any two adjacent Borel subgroups  $P, P' \in \mathcal{P}(A)$  there is a (unique) non-negative integer  $n(x, P, P')$  such that

$$\nu_A(x_P) - \nu_A(x_{P'}) = n(x, P, P') \cdot \alpha^\vee,$$

where  $\alpha$  is the unique root of  $A$  that is positive for  $P$  and negative for  $P'$ . For this we consider the unique parabolic subgroup  $Q$  containing  $P$  and  $P'$  whose Levi component  $L$  has semisimple rank 1. By transitivity of retractions we have

$$(3.2.1) \quad \nu_A(x_P) - \nu_A(x_{P'}) = \nu_A(y_B) - \nu_A(y_{\bar{B}})$$

where  $y = x_Q$  and  $B = L \cap P, \bar{B} = L \cap P'$ . This reduces us to the case in which  $G$  has semisimple rank 1, which has already been done. For future use we note that (3.2.1) can be reformulated as the equality

$$n(x, P, P') = n(y, B, \bar{B}).$$

Again let  $x \in X$ . Now we check that for any Levi subgroup  $M \supset A$  and any adjacent parabolic subgroups  $P = MN, P' = MN'$  in  $\mathcal{P}(M)$  there is a (unique) non-negative integer  $n(x, P, P')$  such that

$$\nu_M(x_P) - \nu_M(x_{P'}) = n(x, P, P') \cdot \beta_{P, P'}.$$

Fix a Borel subgroup  $B_M$  in  $M$  and let  $B$  (respectively,  $B'$ ) be the inverse image of  $B_M$  under  $P \rightarrow M$  (respectively,  $P' \rightarrow M$ ); thus  $B, B'$  are Borel subgroups containing  $A$ .

Now choose a minimal gallery of Borel subgroups  $B = B_0, B_1, B_2, \dots, B_l = B'$  joining  $B$  to  $B'$ , and for  $i = 1, \dots, l$  let  $\alpha_i$  be the unique root of  $A$  that is positive for  $B_{i-1}$  and negative for  $B_i$ . Then

$$\nu_A(x_B) - \nu_A(x_{B'}) = \sum_{i=1}^l n(x, B_{i-1}, B_i) \cdot \alpha_i^\vee.$$

Note that  $\{\alpha_1, \dots, \alpha_l\}$  is precisely the set of roots of  $A$  in  $\mathfrak{n} \cap \bar{\mathfrak{n}}'$  and that for each  $i$  there exists a (unique) positive integer  $m_i$  such that the image of  $\alpha_i^\vee$  in  $\Lambda_M$  is equal to  $m_i \cdot \beta_{P, P'}$ . Applying the canonical surjection  $\Lambda_A \rightarrow \Lambda_M$  to the previous equation, we find that (see 2.3)

$$\nu_M(x_P) - \nu_M(x_{P'}) = n(x, P, P') \cdot \beta_{P, P'}.$$

where  $n(x, P, P')$  is the non-negative integer

$$\sum_{i=1}^l m_i \cdot n(x, B_{i-1}, B_i).$$

**3.3. Proof of part of the main theorem in case  $A = T$ .** Let  $u \in \mathfrak{a}(\mathcal{O})$  and assume that  $u$  is regular in  $\mathfrak{g}(F)$ . Let  $x \in X^u$ .

Let  $M$  be a Levi subgroup of  $G$  containing  $A$ . We are now going to prove the first main assertion in our theorem, namely that for any pair of adjacent  $P, P' \in \mathcal{P}(M)$  there is an inequality

$$n(x, P, P') \leq n(u, P, P').$$

Let  $B, B', B_0, \dots, B_l$  and  $\alpha_i, m_i$  ( $i = 1, \dots, l$ ) be as in 3.2. Then by definition

$$n(u, P, P') = \sum_{i=1}^l m_i \cdot \text{val}_F(\alpha_i(u)).$$

Let  $M_i$  be the Levi subgroup containing  $A$  whose root system is  $\{\pm\alpha_i\}$ , and let  $B'_{i-1}, B'_i$  denote the Borel subgroups in  $M_i$  obtained by intersecting  $B_{i-1}, B_i$  with  $M_i$ . Let  $Q_i$  be the unique parabolic subgroup in  $\mathcal{P}(M_i)$  such that  $Q_i$  contains  $B_{i-1}$  and  $B_i$ . We showed in 3.2 that

$$n(x, P, P') = \sum_{i=1}^l m_i \cdot n(x, B_{i-1}, B_i)$$

and that

$$n(x, B_{i-1}, B_i) = n(y_i, B'_{i-1}, B'_i),$$

where  $y_i = x_{Q_i} \in X_{M_i}^u$ . Since  $M_i$  has semisimple rank 1, we know that

$$n(y_i, B'_{i-1}, B'_i) \leq \text{val}_F(\alpha_i(u)).$$

This completes the proof of the first main assertion.

Now suppose that  $x$  is regular in  $X^u$ . Then each point  $y_i \in X_{M_i}^u$  above is regular in  $X_{M_i}^u$ , and therefore from the rank 1 case (see 3.1) we know that

$$n(y_i, B'_{i-1}, B'_i) = \text{val}_F(\alpha_i(u)).$$

We conclude that if  $x$  is regular in  $X^u$ , then

$$n(x, P, P') = n(u, P, P'),$$

which is another of the assertions in our theorem.

**3.4. Proof of the rest of the main theorem in case  $M = A = T$ .** We continue with  $u \in \mathfrak{a}(\mathcal{O})$  and  $x \in X^u$  as before, but for the moment we only consider the case  $M = A$ . We assume that

$$(3.4.1) \quad n(x, P, P') = \text{val}_F(\alpha_{P, P'}(u))$$

for all adjacent Borel subgroups  $P, P' \in \mathcal{P}(A)$ , where  $\alpha_{P, P'}$  is the unique root of  $A$  that is positive for  $P$  and negative for  $P'$ . We want to prove that  $x$  is regular in  $X^u$ . To do so we must first select a suitable Borel subgroup  $B \in \mathcal{P}(A)$ .

Let  $u_0 \in \mathfrak{a}(\mathbf{C})$  denote the image of  $u$  under  $\mathfrak{a}(\mathcal{O}) \rightarrow \mathfrak{a}(\mathbf{C})$ , and let  $M$  denote the centralizer of  $u_0$  in  $G$ . Thus  $M$  is a Levi subgroup of  $G$  containing  $A$ , and we choose  $P \in \mathcal{P}(M)$ . Then we obtain a suitable Borel subgroup by taking any  $B \in \mathcal{P}(A)$  such that  $B \subset P$ . For any  $B$ -simple root  $\alpha$  we denote by  $B_\alpha$  the unique Borel subgroup in  $\mathcal{P}(A)$  that is adjacent to  $B$  and for which  $\alpha$  is negative, and we write  $P_\alpha$  for the unique parabolic subgroup containing  $B$  and  $B_\alpha$  such that the semisimple rank of the Levi component  $M_\alpha$  of  $P_\alpha$  is 1. Consider the element (well-defined up to  $B(\mathbf{C})$ -conjugacy)  $v := \bar{u}_B(x) \in \mathfrak{b}(\mathbf{C})$  defined in 2.6. The equation (3.4.1) plus the semisimple rank 1 theory implies that the points  $x_{P_\alpha} \in X_{M_\alpha}^u$  are

regular, and this in turn implies (see 2.6) that for every  $B$ -simple root  $\alpha$  the image of the element  $v$  under  $\mathfrak{b}(\mathbf{C}) \hookrightarrow \mathfrak{p}_\alpha(\mathbf{C}) \twoheadrightarrow \mathfrak{m}_\alpha(\mathbf{C})$  is regular in  $\mathfrak{m}_\alpha(\mathbf{C})$ . Moreover it is evident that the image of  $v$  under the canonical surjection  $\mathfrak{b}(\mathbf{C}) \twoheadrightarrow \mathfrak{a}(\mathbf{C})$  is equal to  $u_0$ . Using only these facts, we now check that  $v$  is regular in  $\mathfrak{g}(\mathbf{C})$  (and hence that  $x$  is regular in  $X^u$ ).

Let  $v = v_s + v_n$  be the Jordan decomposition of  $v$ , with  $v_s$  semisimple and  $v_n$  nilpotent. Since it is harmless to replace  $v$  by any  $B(\mathbf{C})$ -conjugate, we may assume without loss of generality that  $v_s \in \mathfrak{a}(\mathbf{C})$ . Then, since  $v_s \mapsto u_0$  under  $\mathfrak{b}(\mathbf{C}) \twoheadrightarrow \mathfrak{a}(\mathbf{C})$ , it follows that  $v_s = u_0$ . Since  $v_n$  commutes with  $v_s = u_0$ , it lies in  $\mathfrak{m}(\mathbf{C})$ , and we must check that  $v_n$  is a principal nilpotent element in  $\mathfrak{m}(\mathbf{C})$ . As  $v_n$  lies in the Borel subalgebra  $(\mathfrak{b} \cap \mathfrak{m})(\mathbf{C})$  of  $\mathfrak{m}(\mathbf{C})$ , it is enough to check that the projection of  $v_n$  into each simple root space of  $(\mathfrak{b} \cap \mathfrak{m})(\mathbf{C})$  is non-zero, and this follows from the statement (proved above) that the image of  $v$  under  $\mathfrak{b}(\mathbf{C}) \hookrightarrow \mathfrak{p}_\alpha(\mathbf{C}) \twoheadrightarrow \mathfrak{m}_\alpha(\mathbf{C})$  is regular in  $\mathfrak{m}_\alpha(\mathbf{C})$  for every simple root  $\alpha$  of  $A$  in  $M$ .

**3.5. End of the proof of the main theorem in case  $A = T$ .** We continue with  $u \in \mathfrak{a}(\mathcal{O})$  and  $x \in X^u$  as above. Let  $M$  be any Levi subgroup containing  $A$ . It remains to prove that if  $x_P$  is regular in  $X_M^u$  for all  $P \in \mathcal{P}(M)$  and if

$$(3.5.1) \quad n(x, P, P') = n(u, P, P')$$

for every adjacent pair  $P, P' \in \mathcal{P}(M)$ , then  $x$  is regular in  $X^u$ . We have already proved this in case  $M = A$ , and now we want to reduce the general case to this special case.

The equality (3.5.1) is equivalent to the equality

$$(3.5.2) \quad \nu_M(x_P) - \nu_M(x_{P'}) = n(u, P, P') \cdot \beta_{P, P'}.$$

Fix  $P \in \mathcal{P}(M)$  and sum (3.5.2) over the set of neighboring pairs in a minimal gallery joining  $P$  to its opposite  $\bar{P} \in \mathcal{P}(M)$ . Doing this yields the equality

$$(3.5.3) \quad \nu_M(x_P) - \nu_M(x_{\bar{P}}) = \sum_{\alpha \in R_N} \text{val}_F(\alpha(u)) \cdot \pi_M(\alpha^\vee),$$

where  $\pi_M : X_*(A) \rightarrow \Lambda_M$  is the canonical surjection and  $R_N$  is the set of roots of  $A$  in  $\mathfrak{n}$ .

Fix a Borel subgroup  $B_M$  in  $M$  containing  $A$  and let  $B$  (respectively,  $B_1$ ) be the Borel subgroups in  $\mathcal{P}(A)$  obtained as the inverse image of  $B_M$  under  $P \rightarrow M$  (respectively,  $\bar{P} \rightarrow M$ ). Then (3.5.3) implies (see 2.3) that

$$\nu_A(x_B) - \nu_A(x_{B_1}) \equiv \sum_{\alpha \in R_N} \text{val}_F(\alpha(u)) \cdot \alpha^\vee$$

modulo the coroot lattice for  $M$ . Since  $R_N$  is also the set of roots that are positive on  $B$  and negative on  $B_1$ , it follows that

$$\nu_A(x_B) - \nu_A(x_{B_1}) = \sum_{\alpha \in R_N} j_\alpha \cdot \alpha^\vee$$

for some integers  $j_\alpha$  such that  $0 \leq j_\alpha \leq \text{val}_F(\alpha(u))$ . (To prove this pick a minimal gallery joining  $B$  to  $B_1$  and use the inequality stated in the main theorem for each neighboring pair in the gallery.) Comparing this equality with the congruence, we see that the linear combination

$$(3.5.4) \quad \sum_{\alpha \in R_N} (\text{val}_F(\alpha(u)) - j_\alpha) \cdot \alpha^\vee$$

maps to 0 in  $\Lambda_M$ .

We get a basis for  $\Lambda_M \otimes \mathbf{R}$  by taking the elements  $\beta_{P,P'}$  as  $P'$  varies through the set of parabolic subgroups in  $\mathcal{P}(M)$  adjacent to  $P$ . Moreover for any  $\alpha \in R_N$  the image  $\pi_M(\alpha^\vee)$  of  $\alpha^\vee$  in  $\Lambda_M$  is a non-negative linear combination of basis elements  $\beta_{P,P'}$  (with at least one non-zero coefficient). Therefore the fact that (3.5.4) maps to 0 in  $\Lambda_M$  means that

$$(3.5.5) \quad \nu_A(x_B) - \nu_A(x_{B_1}) = \sum_{\alpha \in R_N} \text{val}_F(\alpha(u)) \cdot \alpha^\vee$$

By hypothesis  $x_{\bar{P}}$  is regular. Therefore (transitivity of retractions plus the part of our theorem we have already proved) for all adjacent Borel subgroups  $B_1, B_2 \in \mathcal{P}(A)$  such that  $B_1, B_2 \subset \bar{P}$  we have

$$\nu_A(x_{B_1}) - \nu_A(x_{B_2}) = \text{val}_F(\alpha_{B_1, B_2}(u)) \cdot \alpha_{B_1, B_2}^\vee,$$

where  $\alpha_{B_1, B_2}$  denotes the unique root that is positive on  $B_1$  and negative on  $B_2$ . Summing these equalities over neighboring pairs in a minimal gallery joining  $B_1$  to  $\bar{B}$ , we find that

$$\nu_A(x_{B_1}) - \nu_A(x_{\bar{B}}) = \sum_{\alpha \in R_M^+} \text{val}_F(\alpha(u)) \cdot \alpha^\vee,$$

where  $R_M^+$  denotes the set of roots of  $A$  in  $B_M$ . Adding this last equality to (3.5.5), we see that

$$(3.5.6) \quad \nu_A(x_B) - \nu_A(x_{\bar{B}}) = \sum_{\alpha \in R^+} \text{val}_F(\alpha(u)) \cdot \alpha^\vee.$$

Now consider any minimal gallery  $B = B_0, B_1, \dots, B_l = \bar{B}$  joining  $B$  to  $\bar{B}$ . Then

$$(3.5.7) \quad \nu_A(x_B) - \nu_A(x_{\bar{B}}) = \sum_{i=1}^l n(x, B_{i-1}, B_i) \cdot \alpha_i^\vee,$$

where  $\alpha_i$  is the unique root that is positive for  $B_{i-1}$  and negative for  $B_i$ . We know that  $n(x, B_{i-1}, B_i) \leq \text{val}_F(\alpha_i(u))$  for all  $i$ . Subtracting (3.5.7) from (3.5.6), we find that 0 is a non-negative linear combination of positive roots; therefore each coefficient in this linear combination is 0, which means that

$$n(x, B_{i-1}, B_i) = \text{val}_F(\alpha_i(u))$$

for  $i = 1, \dots, l$ .

Now consider any pair  $B', B''$  of adjacent Borel subgroups in  $\mathcal{P}(A)$ . After reversing the order of  $B', B''$  if necessary we can find a minimal gallery as above and an index  $i$  such that  $(B_{i-1}, B_i) = (B', B'')$ . Therefore

$$(3.5.8) \quad n(x, B', B'') = \text{val}_F(\alpha(u)),$$

where  $\alpha$  is the unique root that is positive on  $B'$  and negative on  $B''$ . Since both sides of (3.5.8) remain unchanged when  $B', B''$  are switched, we see that (3.5.8) holds for any adjacent pair  $B', B''$ . By what we have already done, it follows that  $x$  is regular in  $X^u$ .

**3.6. Proof of the main theorem in general.** Now let  $M$  be any Levi subgroup of  $G$  containing  $A$ , and let  $u$  be an integral regular semisimple element of  $\mathfrak{g}(F)$  that happens to lie in  $\mathfrak{m}(F)$ . Let  $T = \text{Cent}_{G_F}(u)$ , a maximal torus in  $M_F$ . We choose a finite extension  $F'/F$  that splits  $T$ .

We normalize the valuation  $\text{val}_{F'}$  on  $F'$  so that uniformizing elements in  $F'$  have valuation 1. Thus  $\text{val}_{F'}(\epsilon) = [F' : F]$ . We write  $X'$  for the set  $G(F')/G(\mathcal{O}_{F'})$ . The inclusion  $G(F) \hookrightarrow G(F')$  induces a canonical injection  $X \hookrightarrow X'$ .

For any  $P \in \mathcal{P}(M)$  the diagram

$$\begin{array}{ccc} X & \xrightarrow{r_P} & X_M \\ \downarrow & & \downarrow \\ X' & \xrightarrow{r'_P} & X'_M \end{array}$$

commutes, where the horizontal maps are retractions and the vertical maps are the canonical injections. Moreover the diagram

$$\begin{array}{ccc} X & \xrightarrow{\nu_G} & \Lambda_G \\ \downarrow & & \downarrow \\ X' & \xrightarrow{\nu'_G} & \Lambda_G \end{array}$$

commutes, where the left vertical map is the canonical injection and the right vertical map is multiplication by  $e := [F' : F]$ .

For any  $x \in X^u$  the image of  $x$  in  $X'$  lies in  $(X')^u$ , and  $x$  is regular in  $X^u$  if and only if  $x$  is regular in  $(X')^u$ . Indeed the conjugacy class  $\bar{u}_G(x)$  attached to  $u$  and  $x$  is the same for  $X$  and  $X'$ .

The torus  $T$  is conjugate under  $M(F')$  to  $A$ , so our theorem is true for  $T$  over  $F'$ . Therefore for  $x \in X^u$  and adjacent  $P = MN, P' = MN' \in \mathcal{P}(M)$

$$(3.6.1) \quad e \cdot n(x, P, P') \leq \sum_{\alpha \in R_N \cap R_{N'}} \text{val}_{F'}(\alpha(u)) \cdot m_\alpha,$$

and  $x$  is regular in  $X^u$  if and only if all of these inequalities are equalities. (As before  $R_N$  denotes the set of roots of  $A$  in  $\mathfrak{n}$ ; the positive integers  $m_\alpha$  were defined in 2.9.) Dividing by  $e$ , and noting that the term indexed by  $\alpha$  depends only on the  $\Gamma$ -orbit of  $\alpha$ , we find that (3.6.1) is equivalent to the inequality

$$n(x, P, P') \leq n(u, P, P').$$

This completes the proof of the theorem.

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