Invent. math. 71, 77–129

Inventiones mathematicae

Intersection Homology II

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In [19, 20] we introduced topological invariants $IH_*^{\bar{p}}(X)$ called intersection homology groups for the study of singular spaces X. These groups depend on the choice of a perversity $\bar{p}$: a perversity is a function from $\{2, 3, \ldots\}$ to the non-negative integers such that both $\bar{p}(c)$ and $c-2-\bar{p}(c)$ are positive and increasing functions of c (2.1). The group $IH_i^{\bar{p}}(X)$ is defined for spaces X called pseudomanifolds: a pseudomanifold of dimension n is a space that admits a stratification

$$X = X_n \supset X_{n-2} \supset X_{n-3} \supset \ldots \supset X_1 \supset X_0$$

such that $X_n - X_{n-2}$ is an oriented dense n manifold and $X_i - X_{i-1}$ for $i \leqq n-2$ is an i manifold along which the normal structure of X is locally trivial (§1.1).

The groups $IH_*^{\bar{p}}(X)$ are the total homology groups of a subcomplex $IC_*^{\bar{p}}(X)$ of the ordinary locally finite chains $C_*(X)$. We recall the definition ([20], §1.3) (which uses a fixed stratification of X)

$$IC_i^{\bar{p}}(X) = \left\{ \begin{array}{l} i \text{ chains } c \text{ that intersect each } X_{n-k} \text{ for } k>0 \text{ in a set of} \\ \text{dimension at most } i-k+\bar{p}(k) \text{ and whose boundary } \partial c \\ \text{intersects each } X_{n-k} \text{ for } k>0 \text{ in a set of dimension at} \\ \text{most } i-k-1+\bar{p}(k). \end{array} \right\}$$

Since the conditions $IC_*^{\bar{p}}(X)$ are local, the $IC_*^{\bar{p}}(U)$ for U open in X form a sheaf of chain complexes, denoted $\mathbf{IC}_{\cdot}^{\bar{p}}(X)$. The purpose of this paper is to study this sheaf of chain complexes. Because sheaves of cochain complexes are more familiar, we renumber by $\mathbf{IC}_{\bar{p}}^{-i} = \mathbf{IC}_i^{\bar{p}}(X)$. By studying this complex of intersection chains, $\mathbf{IC}_{\bar{p}}^{\cdot}$, we obtain results about $IH_*^{\bar{p}}(X)$ because the hypercohomology group $\mathscr{H}^{-i}(\mathbf{IC}_{\bar{p}}^{\cdot})$ is $IH_i(X)$.

The change of point of view from the groups $IH_*^{\bar{p}}(X)$ to the sheaves $\mathbf{IC}^{\cdot}$ was suggested to us by Deligne and Verdier. It leads to many advantages, some of which we now list.

* Partially supported by National Science and Engineering Research Council of Canada and the Sloan Foundation

** Partially supported by National Science Foundation

(1) We consider $\mathbf{IC}^{\bullet}$ as an object in the derived category $D^b(X)$ of the category of sheaves of $\mathbb{Z}$ modules on X. This allows us to bring the functorial apparatus of $D^b(X)$ to bear. (This apparatus is reviewed in Chap. 1.)

(2) Motivated by the vanishing properties of the stalk homology of $\mathbf{IC}^{\bullet}$, Deligne gave [14] a second construction of it using the standard operations of sheaf theory. This construction is the key idea in this paper:

Theorem (§ 3.5). *Let* $\tau_{\leqq k}$: $D^b(X) \to D^b(X)$ *be the truncation functor* (§ 1.12) *which kills the stalk homology of degree* $>k$. *Suppose* U_k *is* $X - X_{n-k}$ *and* i_k *is the inclusion* $U_k \to U_{k+1}$. *Then there is a canonical isomorphism in* $D^b(X)$

$$\mathbf{IC}^{\bullet}(X) = \tau_{\leqq p(n)-n} R i_{n*} \ldots \tau_{\leqq p(3)-n} R i_{3*} \tau_{\leqq p(2)-n} R i_{2*} \mathbb{Z}_{U_2}[n].$$

This construction works in any context where both sheaf theory and stratifications have been developed. So it produces intersection homology groups for topological pseudomanifolds (the approach of [20] required a piecewise linear structure) and for algebraic varieties in any characteristic.

(3) There is a stratification free characterization of $\mathbf{IC}^{\bullet}$.

Theorem (§4.1). *For any topological pseudomanifold* X, *there is a constructible complex* (§1.11) $\mathbf{IC}^{\bullet}$ *in* $D^b(X)$ *which is uniquely characterized up to canonical isomorphism in* $D^b(X)$ *by the conditions:*

(a) $\mathbf{IC}^{\bullet} | X - \Sigma = \mathbb{Z}_{X-\Sigma}[n]$ *for some subset* $\Sigma \subset X$ *of dimension* $n-2$.

(b) *The homology of every stalk vanishes in dimension* $< -n$.

(c) $\dim\{x \in X \mid H^m(\mathbf{IC}^{\bullet}_x) \neq 0\} \leqq n - \min\{c \mid p(c) = n+m\}$ *for all* $m \geqq -n+1$.

(d) $\dim\{x \in X \mid H^m_x(\mathbf{IC}^{\bullet}) \neq 0\} \leqq n - \min\{c \mid c-2-p(c) = -m\}$ *for all* $m \leqq -1$.

Here $\mathbf{IC}^{\bullet}_x$ is the stalk and $H^m_x(\mathbf{IC}^{\bullet})$ (§1.7) may be thought of as the compact support hypercohomology of a small open regular neighbourhood around x.

This yields the following axiomatic characterization of intersection homology, which does not depend on derived categories:

If $\mathbf{S}^{\bullet}$ is a constructible complex of fine sheaves satisfying (a) through (d) above, then the cohomology of the complex

$$\ldots \to \Gamma(X; \mathbf{S}^{i-1}) \to \Gamma(X; \mathbf{S}^i) \to \Gamma(X; \mathbf{S}^{i+1}) \ldots$$

is naturally isomorphic to $IH^{\bar{p}}_*(X)$.

This characterization implies the topological invariance of the intersection homology groups. In particular they are independent of the stratification of X.

(4) Sheaf theory allows one to give local (sheaf theoretic) expressions for global facts on hypercohomology. For example the canonical maps of intersection homology theory ($\bar{p}(c) \leqq \bar{q}(c)$ for all c)

$$H^*(X) \xrightarrow{\alpha} IH^{\bar{p}}_*(X) \xrightarrow{\eta} IH^{\bar{q}}_*(X) \xrightarrow{\omega} H_*(x)$$

can be defined locally by giving maps in $D^b(X)$ of the corresponding sheaves of cochain complexes (§5.1, §5.5). Since such a map can be completed to a distinguished triangle, the vanishing of the third term of the triangle gives a local criterion for α, η and ω to be isomorphisms (§5.5, §5.6).

Similarly, the intersection pairing (where $\bar{p}(c)+\bar{q}(c)=c-2$ for all c)

$$IH_*^{\bar{p}}(X) \otimes IH_*^{\bar{q}}(X) \to H_*(X)$$

results from a pairing of the corresponding objects in $D^b(X)$ (§5.2). Poincaré duality also has a local expression:

Theorem (§5.3). *If $\bar{p}(c)+\bar{q}(c)=c-2$ for all c, then if X is oriented*

$$\mathbf{IC}_{\bar{p}}^{\bullet} \otimes \mathbb{Q} \cong R\mathbf{Hom}^{\bullet}(\mathbf{IC}_{\bar{q}}^{\bullet}, \mathbb{D}_X^{\bullet}) \otimes \mathbb{Q}$$

where $\mathbb{D}_X^{\bullet}$ *is the dualizing complex on X.*

The Verdier duality theorem (§1.7) shows that this implies Poincaré duality on the hypercohomology

$$IH_*^{\bar{p}}(X) \otimes \mathbb{Q} \cong \operatorname{Hom}(IH_*^{\bar{q}}(X), \mathbb{Q})$$

provided X is compact.

There is a perversity $\bar{m}$ for which the intersection homology groups (with rational coefficients) are particularly important for the study of a complex analytic variety X. This is the *middle* perversity $\bar{m}(c)=\dfrac{c-2}{2}$. This makes sense since complex analytic varieties admit stratifications with only (real) even dimensional strata.

This paper contains several results on the middle group $IH_*^{\bar{m}}(X;\mathbb{Q})$. Among these are:

(1) Self duality: if $i+j=n$, then

$$IH_i^{\bar{m}}(X) \cong \operatorname{Hom}(IH_j^{\bar{m}}(X), \mathbb{Q})$$

this results from Verdier local duality as explained above.

(2) Kunneth theorem (§6.3):

$$IH_*^{\bar{m}}(X \times Y) \cong IH_*^{\bar{m}}(X) \otimes IH_*^{\bar{m}}(Y).$$

(3) Lefschetz hyperplane theorem:

Theorem (§7.1). *Suppose X is an n dimensional subvariety of complex projective space and H is a generic hyperplane. Then the map*

$$\alpha_*\colon IH_i^{\bar{m}}(X \cap H) \to IH_i^{\bar{m}}(X)$$

(*where α_* is the homomorphism induced by the normally nonsingular inclusion $X \cap H \to X$* (§5.4)) *is an isomorphism for $i<n-1$ and is a surjection for $i=n-1$.*

This paper contains three axiomatic characterizations of the complex of intersection chains $\mathbf{IC}_{\bar{p}}^{\bullet}$. Their ranges of validity are summarized as follows:

1. $[\mathrm{AX1}]_R$(§3.3) uses a (fixed) stratification valid for any perversity $\bar{p}$ and any coefficient ring R.
2. [AX2] (§4.1) stratification independent valid for any perversity $\bar{p}$ and any coefficient ring R.

3. [AX3] (§6.1) stratification independent
valid for middle perversity $\bar{m}$
and field coefficients.

We would like to thank J. Cheeger, W. Fulton, O. Gabber, S. Gelfand, C. McCrory, P. Siegel, J.-L. Verdier, and K. Vilonen for valuable conversations. We have especially profited from conversations with P. Deligne whose ideas are fundamental to this paper. A number of people have contributed corrections and improvements to the first draft of this paper. We mention in particular P. Deligne, A. Borel, and J.-L. Verdier. We are grateful to the IHES for their hospitality and support.

Table of Contents

Each chapter $n+1$ begins with a section §n.0 which specifies the assumptions on the space, its stratification (if any), the coefficient ring, and the perversity.

Intersection homology with coefficients in a local system is treated in a series of paragraphs marked $\mathscr{L}$, which are distributed throughout the paper.

Chapter 1. Sheaf Theory

This chapter (except for §1.1 and §1.2) consists of a summary (without proofs) of the theory of derived categories. We have included this material for the convenience of the reader. It is not necessary to absorb all of Chap. 1 before beginning to read this paper. The main references for this chapter are [3, 6, 17, 23, 25, 41, 42, 43]. Additional material on sheaf theory may be found in [4, 5, 9, 38, 39].

We describe a single version of the derived category of the category of sheaves (on the topological spaces which are defined in §1.1) which is closed under the standard operations of derived category theory, and which is rich enough for all the examples and applications we wish to consider.

1.1. *Topological Pseudomanifolds*

Definition. A 0-dimensional topologically stratified Hausdorff space is a countable collection of points with the discrete topology.

An n-dimensional *topological stratification* of a paracompact Hausdorff topological space X is a filtration by closed subsets

$$X=X_n\supset X_{n-1}\supset X_{n-2}\supset\ldots\supset X_1\supset X_0$$

such that for each point $p\in X_i-X_{i-1}$ there exists a distinguished neighborhood N of p in X, a compact Hausdorff space L with an $n-i-1$ dimensional topological stratification,

$$L=L_{n-i-1}\supset\ldots\supset L_1\supset L_0\supset L_{-1}=\phi$$

and a homeomorphism

$$\phi\colon\ \mathbb{R}^i\times\operatorname{cone}^\circ(L)\to N$$

which takes each $\mathbb{R}^i\times\operatorname{cone}^\circ(L_j)$ homeomorphically to $N\cap X_{i+j+1}$. Here, $\operatorname{cone}^\circ(L)$ denotes the open cone, $L\times[0,1)/(l,0)\sim(l',0)$ for all l, $l'\in L$. We use the convention that $L_{-1}=\phi$ and $\operatorname{cone}^\circ(\phi)=$one point.

Thus, if X_i-X_{i-1} is not empty, it is a manifold of dimension i. It is called the i-dimensional *stratum* of X and is denoted S_i.

If X admits a topological stratification then it is locally compact, satisfies conditions *hlc* and *clc* of Borel-Moore [6] and is a *cs* space in the sense of Siebenmann [36]. Every compact topologically stratified space can be embedded in Euclidean space.

A topologically stratified Hausdorff space X is *purely n-dimensional* if $X_n - X_{n-1}$ is dense in X. In this case, all notions of dimension (topological dimension [24], cohomological dimension [5] §1.5 p. 18, Borel-Moore dimension [6] coincide and equal n. A topologically stratified Hausdorff space X is purely n-dimensional if and only if every open subset of X is n-dimensional in any of the above senses.

Definition. A *topological pseudomanifold* of dimension n is a purely n-dimensional stratified paracompact Hausdorff topological space X which admits a stratification

$$X = X_n \supset \ldots \supset X_1 \supset X_0$$

such that $X_{n-1} = X_{n-2}$ (i.e., $S_n = X - X_{n-2}$ is an n-dimensional manifold which is dense in X). We use the symbol Σ to denote the singularity subset X_{n-2}.

The following types of spaces admit topological stratifications: complex algebraic varieties, complex analytic varieties, real analytic varieties, semi-algebraic and semi-analytic sets, subanalytic sets, Whitney stratified sets [40], abstract (or Thom-Mather) stratified sets [30], and piecewise linear spaces.

The following types of spaces are topological pseudomanifolds: irreducible (or equidimensional) complex algebraic or analytic varieties, the locus of points p in a normal n-dimensional real algebraic variety such that every neighborhood of p has topological dimension n, n-dimensional triangulated spaces so that each $n-1$ simplex is contained in exactly two n-simplices.

Throughout this paper we assume that all spaces are topological pseudomanifolds.

1.2. *Stratified Maps*

Let X and Y be stratified topological pseudomanifolds.

Definition. A continuous map $f: X \to Y$ is *stratified* if it satisfies the following two conditions:

(C1) For any connected component S of any stratum $Y_k - Y_{k-1}$, the set $f^{-1}(S)$ is a union of connected components of strata of X.

(C2) For each point $p \in Y_i - Y_{i-1}$ there exists a neighborhood N of p in Y_i, a topologically stratified space

$$F = F_k \supset F_{k-1} \supset \ldots \supset F_{-1} = \phi$$

and a stratum preserving homeomorphism

$$F \times N \to f^{-1}(N)$$

which commutes with the projection to N.

Remark. If $f: X \to Y$ is a subanalytic map between subanalytic sets then there exist stratifications of X and Y such that f is stratified. It is not possible however, to stratify the spaces in an arbitrary diagram of subanalytic (or even complex analytic) maps in such a way that each map is stratified.

1.3. *Complexes of Sheaves*

Let X be a topological pseudomanifold. Throughout this paper, R will denote a regular Noetherian ring with finite Krull dimension. (We shall be mainly concerned with the cases $R=\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{C}$.)

Throughout this paper we shall use the word *sheaf* to mean a topological sheaf of R-modules. Sheaves on X will be denoted $\mathbf{A}$, $\mathbf{B}$, etc., and bounded complexes of sheaves on X will be denoted $\mathbf{A}^{\bullet}$, $\mathbf{B}^{\bullet}$, etc. The constant sheaf on X is denoted $\mathbf{R}_X$. Whenever convenient, we identify a sheaf $\mathbf{A}$ with the complex $\mathbf{A}^{\bullet}$ given by $\mathbf{A}^p=0$ for $p\neq 0$ and $\mathbf{A}^0=\mathbf{A}$.

Let X be a topological pseudomanifold and let $\mathbf{A}^{\bullet}$ be a bounded complex of sheaves of R-modules on X, i.e., a sequence

$$\dots\to\mathbf{A}^{p-1}\xrightarrow{d}\mathbf{A}^{p}\xrightarrow{d}\mathbf{A}^{p+1}\to\dots$$

with $d\circ d=0$ and $\mathbf{A}^p=0$ for $|p|$ sufficiently large. The sheaf of sections associated with $\mathbf{A}^{\bullet}$ assigns to any open set U the chain complex

$$\dots\to\Gamma(U;\mathbf{A}^{p-1})\to\Gamma(U;\mathbf{A}^{p})\to\Gamma(U;\mathbf{A}^{p+1})\to\dots.$$

The p^{th} cohomology sheaf $\mathbf{H}^p(\mathbf{A}^{\bullet})$ associated with $\mathbf{A}^{\bullet}$ is the sheafification of the presheaf whose sections over an open set U is the p^{th} homology group of this chain complex. The stalk at a point $x\in X$ of the sheaves $\mathbf{A}^p$ and $\mathbf{H}^p(\mathbf{A}^{\bullet})$ are denoted $\mathbf{A}^p_x$ and $\mathbf{H}^p(\mathbf{A}^{\bullet})_x$ respectively. In particular $\mathbf{H}^p(\mathbf{A}^{\bullet}_x)\cong\mathbf{H}^p(\mathbf{A}^{\bullet})_x$. The complex $\mathbf{A}[n]$ is defined by $\mathbf{A}[n]^p=\mathbf{A}^{p+n}$. The restriction of $\mathbf{A}^{\bullet}$ to a subspace $Y\subset X$ is denoted $\mathbf{A}^{\bullet}|Y$.

1.4. *Constructible Sheaves*

Definition. A sheaf $\mathbf{S}$ on X is called locally constant if every point $x\in X$ has a neighborhood U such that the restriction maps

$$\mathbf{S}_x\leftarrow\Gamma(U;\mathbf{S})\to\mathbf{S}_y$$

are isomorphisms for all $y\in U$.

A complex of sheaves is called *cohomologically locally constant* (CLC) if the associated local cohomology sheaves are locally constant.

Let $X_0\subset X_1\subset\dots\subset X_n=X$ be a filtration by closed subsets. A complex of sheaves $\mathbf{A}^{\bullet}$ on X is said to be *constructible* with respect to this filtration if, for each j, $\mathbf{A}^{\bullet}|(X_j-X_{j-1})$ is CLC, and has finitely generated stalk cohomology.

If X has a subanalytic structure, then the complex $\mathbf{A}^{\bullet}$ is said to be *subanalytically constructible* if it is constructible with respect to some filtration of X by closed subanalytic subsets. One defines *PL*-constructible and algebraically constructible complexes of sheaves similarly. $\mathbf{A}^{\bullet}$ is *topologically constructible* if it is bounded and is constructible with respect to some topological stratification of X.

Note. All complexes of sheaves considered in this paper will be topologically constructible.

If $\mathbf{A}^\bullet$ is a complex of sheaves on X which is constructible with respect to a given stratification and if $f\colon X\to Y$ is a stratified map then $f_*\mathbf{A}^\bullet$ is constructible with respect to the given stratification of Y. A similar remark holds for f^*.

Theorem. *Suppose X is a topological pseudomanifold and $\mathbf{A}^\bullet$ is a topologically constructible complex of sheaves on X. Then $\mathbf{A}^\bullet$ is* perfect *in the sense of* [3] *Exp.* 9, cohomologically constructible *in the sense of Verdier* [42, 45], *and satisfies* condition (P, Q) of Wilder ([47, 6]). *Therefore $\mathscr{H}^i(X;\mathbf{A}^\bullet)$ is finitely generated, when X is compact.* ([6] *Prop.* 6.8.)

Proof. We claim that for any $x\in X$ there is a neighborhood basis $U_1\supset U_2\supset U_3\supset\dots$ such that for each i and m, the restriction map

$$\mathscr{H}^i(U_m;\mathbf{A}^\bullet)\to\mathscr{H}^i(U_{m+1};\mathbf{A}^\bullet)$$

is an isomorphism. It follows that $\mathscr{H}^i(U_m;\mathbf{A}^\bullet)\cong\mathbf{H}^i(\mathbf{A}^\bullet)_x$, but $\mathbf{H}^i(\mathbf{A}^\bullet)_x$ is a finitely generated R-module.

By [3] Exp. 9 §5.1 this (plus the fact that R is a regular Noetherian ring) will imply $\mathbf{A}^\bullet$ is perfect.

By [6] Prop. 6.8 this will imply $\mathbf{A}^\bullet$ satisfies the Wilder condition, which implies $\mathscr{H}^i(X,\mathbf{A}^\bullet)$ is finitely generated for compact X.

By [V] Theorem 8, this will imply $\mathbf{A}^\bullet$ satisfies condition CC of Verdier.

Proof of Claim. Fix $x\in X$ and let N be a distinguished neighborhood $N\cong\mathbb{R}^i\times\operatorname{cone}^\circ(L)$ as in §1.1. Let Y be the join $S^{i-1}*L$, stratified in the obvious way (i.e., $S^{i-1}\subset S^{i-1}*L$ is a stratum but $L\subset S^{i-1}*L$ is not a union of strata unless $i=0$). This determines a stratification of the open cone,

$$\operatorname{cone}^\circ(Y)=Y\times[0,1)/(y,0)\sim(y',0)\qquad\text{for all } y,y'\in Y.$$

Choose a stratum preserving homeomorphism $\psi\colon\operatorname{cone}^\circ(Y)\to N$, with $\psi(\text{vertex})=x$. Let $U_m\subset N$ be the smaller neighborhood $U_m=\psi\left(Y\times\left[0,\frac{1}{m}\right)\right)$ we will now verify the claim for $m=1$, the other cases being very similar. Define a 1-parameter family of stratum preserving stretching embeddings,

$$G\colon U_2\times[\tfrac12,1]\to U_1$$

$$G((y,t),s)=(y,t/s).$$

For any $s\in[\frac12,1]$ let $i_s\colon U_2\to U_2\times[\frac12,1]$ be the inclusion at the level s. We must show that

$$\mathscr{H}^*(U_2;i_1^*G^*\mathbf{A}^\bullet)\cong\mathscr{H}^*(U_2;i_{\frac12}^*G^*\mathbf{A}^\bullet).$$

In fact, $i_1^*G^*\mathbf{A}^\bullet$ and $i_{\frac12}^*G^*\mathbf{A}^\bullet$ are both quasi-isomorphic to $R\pi_*G^*\mathbf{A}^\bullet$ where $\pi\colon U_2\times[\frac12,1]\to U_2$ is the projection to the second factor. To see this, consider the effect on stalk cohomology of the natural map

$$R\pi_*G^*\mathbf{A}^\bullet\to R\pi_*i_{s*}i_s^*G^*\mathbf{A}^\bullet=i_s^*G^*\mathbf{A}^\bullet.$$

For any $x'=(y,t)\in U_2$ the stalk cohomology of $i_s^* G^* \mathbf{A}^\bullet$ is simply $\mathbf{H}^\bullet(\mathbf{A}^\bullet)_{(g,t/s)}$. However, by [17] 4.17.1 the stalk cohomology of $R\pi_* G^* \mathbf{A}^\bullet$ is $\mathscr{H}^*(\pi^{-1}(x'); G^* \mathbf{A}^\bullet)$. Since $\pi^{-1}(x')$ is an interval, the following lemma now implies that this natural map is an isomorphism on stalk cohomology (where $\mathbf{B}^\bullet = G^* \mathbf{A}^\bullet$).

Lemma. *Let $\mathbf{B}^\bullet$ be a complex of sheaves on the unit interval I. Suppose the cohomology sheaves $\mathbf{H}^q(\mathbf{B}^\bullet)$ are locally constant. Then for any $t\in I$, the restriction map*

$$\mathscr{H}^q(I;\mathbf{B}^\bullet)\rightarrow H^q(\mathbf{B}^\bullet)_t$$

is an isomorphism.

Proof of Lemma. The sheaves $\mathbf{H}^q(\mathbf{B}^\bullet)$ are constant on I. The spectral sequence for the hypercohomology of $\mathbf{B}^\bullet$ collapses, since

$$E_2^{p,q} = H^p(I;\mathbf{H}^q(\mathbf{B}^\bullet)) = H^p(I;\mathbf{H}^q(\mathbf{B}^\bullet)_t) = 0$$

unless $p=0$ and $E_2^{0,q} = \mathbf{H}^q(\mathbf{B}^\bullet)_t$.

1.5. *Quasi-Isomorphisms and Injective Resolutions*

A sheaf map ϕ: $\mathbf{A}^\bullet \rightarrow \mathbf{B}^\bullet$ which commutes with the differentials, is called a quasi-isomorphism if the induced map $\mathbf{H}^p(\phi)$: $\mathbf{H}^p(\mathbf{A}^\bullet)\rightarrow \mathbf{H}^p(\mathbf{B}^\bullet)$ is an isomorphism for each p. If ϕ is a quasi-isomorphism and if each $\mathbf{B}^p$ is injective, then $\mathbf{B}^\bullet$ is called an injective resolution of $\mathbf{A}^\bullet$. Injective resolutions exist for any complex of sheaves of R-modules and are uniquely determined up to chain homotopy.

We now recall the "canonical" bounded injective resolution of a bounded complex of sheaves ([6] §1.3, [4] p. 32, [17] *I* §1.4, *II* §7.1). First we describe the canonical resolution of a sheaf $\mathbf{B}$.

For each $x\in X$ let $\mathbf{B}_x \rightarrow I(x)$ be the canonical embedding of the stalk of $\mathbf{B}$ into an injective R-module, as described in [4] p. 32 and [17] I §1.4. (If R is a field, $\mathbf{B}_x$ will already be injective, so we may simplify the construction by taking $I(x)=\mathbf{B}_x$.) We obtain a canonical embedding of $\mathbf{B}$ into the injective sheaf $\mathbf{I}^0$ where

$$\mathbf{I}^0(U) = \prod_{x\in U} I(x).$$

Similarly the cokernel of $\mathbf{B}\rightarrow \mathbf{I}^0$ has a canonical embedding into an injective sheaf $\mathbf{I}^1$. Continuing this way gives a resolution

$$\mathbf{B}\rightarrow \mathbf{I}^0 \xrightarrow{d^0} \mathbf{I}^1 \rightarrow \dots.$$

By [44] and [3] Exp. 2 Theorem 4.3, the sheaf $\mathbf{ker}\, d^{p+n+1}$ is injective (where $n = \dim(X)$ and $p=\dim(R)$). Define the canonical resolution of $\mathbf{B}$ to be

$$0\rightarrow \mathbf{I}^0 \rightarrow \mathbf{I}^1 \rightarrow \dots \rightarrow \mathbf{I}^{p+n} \rightarrow \mathbf{ker}\, d^{p+n+1} \rightarrow 0.$$

This construction is functorial in $\mathbf{B}$.

For any bounded complex of sheaves $\mathbf{A}^\bullet$, let

$$0 \to \mathbf{J}^{m,0} \to \mathbf{J}^{m,1} \to \dots \to \mathbf{J}^{m,p+n+1} \to 0$$

be the canonical injection resolution of $\mathbf{A}^m$. These $\mathbf{J}^{m,r}$ form a double complex with differentials $\mathbf{J}^{m,r} \to \mathbf{J}^{m+1,r}$ induced from the differential $\mathbf{A}^m \to \mathbf{A}^{m+1}$. The *canonical injective resolution* $\mathbf{A}^\bullet \to \mathbf{I}^\bullet$ is the single complex

$$\mathbf{I}^p = \bigoplus_{m+r=p} \mathbf{J}^{m,r}$$

which is associated to this double complex [17] I §2.6.

1.6. *Hypercohomology*

The p^{th} *hypercohomology group* $\mathscr{H}^p(X;\mathbf{A}^\bullet)$ of a complex of sheaves $\mathbf{A}^\bullet$ is defined to be the p^{th} cohomology group of the cochain complex

$$\dots \to \Gamma(X;\mathbf{I}^{p-1}) \to \Gamma(X;\mathbf{I}^p) \to \Gamma(X;\mathbf{I}^{p+1}) \to \dots$$

where $\mathbf{I}^\bullet$ is the canonical injective resolution of $\mathbf{A}^\bullet$. This group is naturally isomorphic to the p^{th} cohomology group of the single complex which is associated to the double complex $C^p(X;\mathbf{A}^q)$ [17] II §4.6.

The double complex $C^p(X;\mathbf{A}^q)$ gives rise to a spectral sequence for hypercohomology, with

$$E^2_{pq} = H^p(X;\mathbf{A}^q) \to \mathscr{H}^{p+q}(X;\mathbf{A}^\bullet).$$

1.7. *Sheaf* Hom

If $\mathbf{A}$ and $\mathbf{B}$ are sheaves on X, let $\mathrm{Hom}(\mathbf{A},\mathbf{B})$ denote the abelian group of all (global) sheaf maps $\mathbf{A} \to \mathbf{B}$. Let $\mathbf{Hom}(\mathbf{A},\mathbf{B})$ be the sheaf whose sections over an open set U are $\Gamma(U;\mathbf{Hom}(\mathbf{A},\mathbf{B})) = \mathrm{Hom}(\mathbf{A}|U,\mathbf{B}|U)$. If $\mathbf{A}^\bullet$ and $\mathbf{B}^\bullet$ are complexes of sheaves, let $\mathbf{Hom}^\bullet(\mathbf{A}^\bullet,\mathbf{B}^\bullet)$ be the single complex of sheaves which is obtained from the double complex $\mathbf{Hom}^{p,q}(\mathbf{A}^\bullet,\mathbf{B}^\bullet) = \mathbf{Hom}(\mathbf{A}^p,\mathbf{B}^q)$ in the usual way.

A cocycle $\xi \in \Gamma(X;\mathbf{Hom}^k(\mathbf{A}^\bullet,\mathbf{B}^\bullet))$ is a chain map from $\Gamma(X;\mathbf{A}^\bullet)$ to $\Gamma(X;\mathbf{B}^\bullet[k])$ which commutes with the differentials of $\mathbf{A}^\bullet$ and $\mathbf{B}^\bullet$. ξ is a coboundary if it is chain homotopic to 0.

If $\mathbf{S}_1$ and $\mathbf{S}_2$ are sheaves, we may consider them to be complexes in degree 0 (i.e. $S_1^p = 0$ for $p \neq 0$) with $d=0$. Then the sheaf $\mathbf{Ext}^i(\mathbf{S}_1,\mathbf{S}_2)$ is equal to the i^{th} cohomology sheaf associated to $\mathbf{Hom}^\bullet(\mathbf{S}_1,\mathbf{I}^\bullet)$ where $\mathbf{I}^\bullet$ is an injective resolution of $\mathbf{S}_2$.

1.8. *The Constructible Derived Category* [42, 43, 23]

Let $K(X)$ denote the category whose objects $\mathbf{A}^\bullet$ are topologically constructible bounded complexes of sheaves on X and whose morphisms ϕ: $\mathbf{A}^\bullet \to \mathbf{B}^\bullet$ are homotopy classes of sheaf maps which commute with the differentials. Let

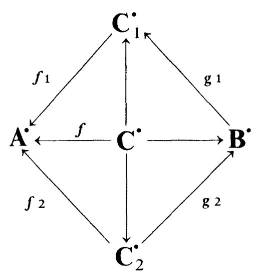
$INJ(X)$ denote the category of constructible bounded complexes of injective sheaves and chain homotopy classes of sheaf maps. There is an equivalence of categories between $INJ(X)$ and the constructible *derived* category $D^b(X)$, induced by the canonical functors

$$INJ(X) \to K(X) \to D^b(X).$$

An object in $D^b(X)$ is a doubly bounded complex of topologically constructible sheaves. A morphism in $D^b(X)$ from $\mathbf{A}^\bullet$ to $\mathbf{B}^\bullet$ is an equivalence class of diagrams of chain maps $\mathbf{A}^\bullet \leftarrow \mathbf{C}^\bullet \to \mathbf{B}^\bullet$ where $\mathbf{A}^\bullet \leftarrow \mathbf{C}^\bullet$ is a quasi-isomorphism. Two such diagrams

$$\mathbf{A}^\bullet \xleftarrow{f_1} \mathbf{C}_1^\bullet \xrightarrow{g_1} \mathbf{B}^\bullet, \qquad \mathbf{A}^\bullet \xleftarrow{f_2} \mathbf{C}_2^\bullet \xrightarrow{g_2} \mathbf{B}^\bullet$$

are considered equivalent if there is a homotopy commutative diagram



where f is a quasi-isomorphism.

Two complexes $\mathbf{A}^\bullet$ and $\mathbf{B}^\bullet$ are isomorphic in $D^b(X)$ if there is a morphism $\mathbf{A}^\bullet \leftarrow \mathbf{C}^\bullet \to \mathbf{B}^\bullet$ where both arrows are quasi-isomorphisms. In this case we say $\mathbf{A}^\bullet$ and $\mathbf{B}^\bullet$ are different *incarnations* of the same isomorphism class of objects in $D^b(X)$.

If $\mathbf{B}^\bullet$ is injective, then

$$\operatorname{Hom}_{D^b(X)}(\mathbf{A}^\bullet, \mathbf{B}^\bullet) = H^0(X; \mathbf{Hom}^\bullet(\mathbf{A}^\bullet, \mathbf{B}^\bullet)).$$

Remark. Although a chain map ϕ: $\mathbf{A}^\bullet \to \mathbf{B}^\bullet$ which induces isomorphisms on the associated cohomology sheaves becomes an isomorphism in $D^b(X)$, there exist sheaf maps which induce the 0 map on cohomology but which are not 0 in $D^b(X)$. (However, see §1.13.)

1.9. *Derived Functors*

A covariant additive functor T from complexes of sheaves to an abelian category gives rise to its right derived functor RT defined on $D^b(X)$ by the formula

$$RT(\mathbf{A}^\bullet) = T(\mathbf{I}^\bullet)$$

where $\mathbf{I}^\bullet$ is the canonical injective resolution of $\mathbf{A}^\bullet$ (see § 1.5).

This procedure applies to the functors $\mathbf{Hom}(\mathbf{B}^{\bullet},*)$, Γ (global sections), Γ_c (global sections with compact support), f_* (direct image), and $f_!$ (direct image with proper supports), where $f\colon X \to Y$ is a continuous map. For a closed subspace $Z \subset X$, the functor Γ_Z assigns to any complex of sheaves $\mathbf{S}^{\bullet}$ the complex of global sections of $\mathbf{S}^{\bullet}$ which vanish on $X - Z$. The i^{th} cohomology group of $R\Gamma_Z(\mathbf{S}^{\bullet})$ is denoted $\mathscr{H}^i_Z(\mathbf{S}^{\bullet})$.

In certain cases we may substitute a simpler resolution for $\mathbf{I}^{\bullet}$ (Hartshorne [23]). If $T = \Gamma$ then $RT(\mathbf{A}^{\bullet}) = T(\mathbf{J}^{\bullet})$ where $\mathbf{J}^{\bullet}$ is a flabby or a fine resolution of $\mathbf{A}^{\bullet}$. If $T = \mathbf{Hom}^{\bullet}(\mathbf{B}^{\bullet},*)$ we may take $\mathbf{I}^{\bullet}$ to be any flabby or fine resolution of $\mathbf{A}^{\bullet}$ by sheaves of *injective* R-modules. Since f^* is exact we have $Rf^*(\mathbf{A}^{\bullet}) \cong Lf^*(\mathbf{A}^{\bullet}) \cong f^*(\mathbf{A}^{\bullet})$. If f is an inclusion of a subspace then $f_!$ is exact so $Rf_!(\mathbf{A}^{\bullet}) \cong f_!(\mathbf{A}^{\bullet})$.

Define $\mathbf{A}^{\bullet} \otimes \mathbf{B}^{\bullet}$ to be the single complex which is associated to the double complex $\mathbf{A}^p \otimes \mathbf{B}^q$. Define the derived functor $\mathbf{A}^{\bullet} \overset{L}{\otimes} ?$ by the formula

$$\mathbf{A}^{\bullet} \overset{L}{\otimes} \mathbf{B}^{\bullet} = \mathbf{A}^{\bullet} \otimes \mathbf{J}^{\bullet}$$

where $\mathbf{J}^{\bullet} \to \mathbf{B}^{\bullet}$ is a resolution of $\mathbf{B}^{\bullet}$ whose stalks are flat R-modules. If R is a field then $\mathbf{A}^{\bullet} \overset{L}{\otimes} \mathbf{B}^{\bullet} \cong \mathbf{A}^{\bullet} \otimes \mathbf{B}^{\bullet}$.

To verify that these functors are defined on the constructible derived category, we need the following

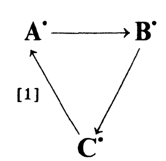
Proposition. *If $\mathbf{A}^{\bullet}$ and $\mathbf{B}^{\bullet} \in D^b(X)$ are constructible with respect to a given stratification of X, then so are*

$$R\mathbf{Hom}^{\bullet}(\mathbf{A}^{\bullet}, \mathbf{B}^{\bullet}) \quad \text{and} \quad \mathbf{A}^{\bullet} \overset{L}{\otimes} \mathbf{B}^{\bullet}.$$

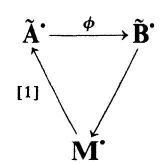
Furthermore $Rf_* \mathbf{A}^{\bullet}$ and $Rf_! \mathbf{A}^{\bullet}$ are constructible with respect to a given stratification of Y, whenever $f\colon X \to Y$ is stratified with respect to these stratifications.

1.10. *Triangles* ([23] p. 20, 32)

$D^b(X)$ is not an abelian category, but it has "distinguished triangles" as a replacement for exact sequences. A triangle of morphisms in $D^b(X)$,



is called distinguished if it is isomorphic (in $D^b(X)$) to a diagram of sheaf maps



where $\mathbf{M}^\bullet$ is the (algebraic) mapping cone of ϕ and $\tilde{\mathbf{B}}^\bullet \to \mathbf{M}^\bullet \to \tilde{\mathbf{A}}^\bullet[1]$ are the canonical maps.

For example, a short exact sequence of complexes of sheaves becomes a distinguished triangle in $D^b(X)$.

Any edge of a distinguished triangle determines the third member up to (non canonical) isomorphism in $D^b(X)$.

A distinguished triangle determines a long exact sequence on the associated cohomology sheaves,

$$\to \mathbf{H}^p(\mathbf{A}^\bullet) \to \mathbf{H}^p(\mathbf{B}^\bullet) \to \mathbf{H}^p(\mathbf{C}^\bullet) \to \mathbf{H}^{p+1}(\mathbf{A}^\bullet) \to \mathbf{H}^{p+1}(\mathbf{B}^\bullet) \to$$

as well as a long exact sequence on hypercohomology.

If $\mathbf{F}^\bullet$ is a complex of sheaves, and

$$\begin{array}{ccc} \mathbf{A}^\bullet & \longrightarrow & \mathbf{B}^\bullet \\ {\scriptstyle [1]}\nwarrow & & \swarrow \\ & \mathbf{C}^\bullet & \end{array}$$

is a distriangle in $D^b(X)$, then we have distinguished triangles

$$\begin{array}{ccc} R\mathbf{Hom}^\bullet(\mathbf{F}^\bullet, \mathbf{A}^\bullet) & \longrightarrow & R\mathbf{Hom}^\bullet(\mathbf{F}^\bullet, \mathbf{B}^\bullet) \\ {\scriptstyle [1]}\nwarrow & & \swarrow \\ & R\mathbf{Hom}^\bullet(\mathbf{F}^\bullet, \mathbf{C}^\bullet) & \end{array} \quad \text{and} \quad \begin{array}{ccc} R\mathbf{Hom}^\bullet(\mathbf{A}^\bullet, \mathbf{F}^\bullet) & \longleftarrow & R\mathbf{Hom}^\bullet(\mathbf{B}^\bullet, \mathbf{F}^\bullet) \\ {\scriptstyle [1]}\searrow & & \nearrow \\ & R\mathbf{Hom}^\bullet(\mathbf{C}^\bullet, \mathbf{F}^\bullet) & \end{array}$$

1.11. *Exact Sequence of a Pair*

Let j: $Y \to X$ be the inclusion of a *closed* subspace. Denote by i: $U \to X$ the inclusion of the open complement. If $\mathbf{A}^\bullet$ is a complex of sheaves on X, there are distinguished triangles in $D^b(X)$,

$$\begin{array}{ccc} R i_! i^* \mathbf{A}^\bullet & \longrightarrow & \mathbf{A}^\bullet \\ {\scriptstyle [1]}\nwarrow & & \swarrow \\ & R j_* j^* \mathbf{A}^\bullet & \end{array} \quad \text{and} \quad \begin{array}{ccc} R j_* j^! \mathbf{A}^\bullet & \longrightarrow & \mathbf{A}^\bullet \\ {\scriptstyle [1]}\nwarrow & & \swarrow \\ & R i_* i^* \mathbf{A}^\bullet & \end{array}$$

The second triangle can be obtained from the first by Verdier duality. (see §1.12) In the case $\mathbf{A}^\bullet \cong \mathbb{Z}_X$ the hypercohomology exact sequences are simply the long exact cohomology sequences for $H^*(X, Y)$ and $H^*(X, U)$ respectively. If $\mathbf{A}^\bullet \cong \mathbb{D}^\bullet_X$ we obtain the long exact homology sequences for $H_*(X, U)$ and $H_*(X, Y)$ respectively.

1.12. *Duality*

Let X be a topological pseudomanifold.

In [6] Borel and Moore defined the dual $\mathfrak{D}(\mathbf{A}^\bullet)$ of any complex of sheaves $\mathbf{A}^\bullet$ and showed (when R is a Dedeking ring) that for any open set $U \subset X$ the cohomology groups $\mathscr{H}_c^i(U:\mathbf{A}^\cdot)$ and $\mathscr{H}^i(U;\mathfrak{D}(\mathbf{A}^\cdot))$ are dual. This means there is a natural (split) exact sequence

$$0\to \operatorname{Ext}(\mathscr{H}_c^{q+1}(U;\mathbf{A}^\cdot),R)\to \mathscr{H}^{-q}(U;\mathfrak{D}(\mathbf{A}^\cdot))$$
$$\to \operatorname{Hom}(\mathscr{H}_c^{q}(U;\mathbf{A}^\cdot),R)\to 0.$$

(Here, $\mathscr{H}_c^q$ denotes the hypercohomology with compact supports, i.e., $R^q\Gamma_c$.) In fact, this property characterizes $\mathfrak{D}(\mathbf{A}^\cdot)$ up to quasi-isomorphism. It implies, for example, that if X is compact and R is a field

$$\mathscr{H}^q(X;\mathbf{A}^\cdot)=\operatorname{Hom}(\mathscr{H}^{-q}(X;\mathfrak{D}(\mathbf{A}^\cdot)),R).$$

In [42] Verdier showed that there is a complex of sheaves $\mathbb{D}_X^\cdot$ (called the dualizing complex) such that

$$\mathfrak{D}(\mathbf{A}^\cdot)\cong R\mathbf{Hom}^\cdot(\mathbf{A}^\cdot,\mathbb{D}_X^\cdot)$$

for any bounded complex $\mathbf{A}^\cdot$. He identified $\mathbb{D}_X^\cdot=\mathfrak{D}(\mathbf{R}_X)$. If $\mathbf{B}^\cdot\cong\mathfrak{D}(\mathbf{A}^\cdot)$ then the corresponding pairing

$$\mathbf{B}^\cdot\overset{L}{\otimes}\mathbf{A}^\cdot\to\mathbb{D}_X^\cdot$$

is said to be a Verdier dual pairing.

If $\mathbf{A}^\cdot$ is a bounded topologically constructible complex of sheaves on X then there is a natural isomorphism in $D^b(X)$,

$$\mathbf{A}^\cdot\cong\mathfrak{D}(\mathfrak{D}(\mathbf{A}^\cdot)).$$

We now describe two important functors $Rf_!$: $D^b(X)\to D^b(Y)$ and $f^!$: $D^b(Y)\to D^b(X)$ which were defined by Verdier for any continuous map $f: X\to Y$ between locally compact topological spaces. $Rf_!$ is the right derived functor of the direct image functor with proper supports, $f_!$.

$\Gamma(U,f_!\mathbf{A}^\cdot)=\{\gamma\in\Gamma(f^{-1}(U),\mathbf{A}^\cdot)|\text{support of }\gamma\text{ is proper over }U\}$. The stalk $\mathbf{H}^*(Rf_!\mathbf{A}^\cdot)_y$ is the hypercohomology with compact supports of the fibre $f^{-1}(y)$, with coefficients in $\mathbf{A}^\cdot$. If f is proper than $Rf_*=Rf_!$.

If $\mathbf{I}^\cdot$ is a complex of injective sheaves on Y, $f^!(\mathbf{I}^\cdot)$ is defined to be the sheaf associated to the presheaf whose sections over an open set $U\subset X$ are $\Gamma(U;f^!\mathbf{I}^\cdot)=\operatorname{Hom}^\cdot(f_!\mathbf{K}_U^\cdot,\mathbf{I}^\cdot)$ where $\mathbf{K}_U^\cdot$ is the standard injective resolution of the constant sheaf $\mathbf{R}_U$. The *Verdier duality theorem* is a canonical isomorphism in $D^b(Y)$,

$$Rf_*R\mathbf{Hom}^\cdot(\mathbf{A}^\cdot,f^!\mathbf{B}^\cdot)\cong R\mathbf{Hom}^\cdot(Rf_!\mathbf{A}^\cdot,\mathbf{B}^\cdot)$$

for any $\mathbf{A}^\cdot\in D^b(X)$ and $\mathbf{B}^\cdot\in D^b(Y)$.

Remarks on $f^!$ and $\mathbb{D}_X^\cdot$

If $f: Z\to X$ is the inclusion of a closed subspace then $\mathscr{H}^i(Z;f^!\mathbf{A}^\cdot)$ is denoted $\mathscr{H}_Y^i(X;\mathbf{A}^\cdot)$. This group is also constructed as a derived functor in §1. There is a natural isomorphism

$$\mathbb{D}_X^\cdot\cong f^!\mathbf{R}_{pt}$$

where $f\colon X \to$point. The complex $\mathbb{D}_X^\bullet$ is quasi-isomorphic to the complex of sheaves of singular chains on X which is associated to the complex of presheaves

$$\Gamma(U;\mathbf{C}^{-p}) = C_p(X, X-U; R).$$

The associated cohomology sheaves of $\mathbb{D}_X^\bullet$ are nonzero in negative degree only, with stalks $\mathbf{H}^{-p}(\mathbb{D}_X^\bullet)_x = H_p(X, X-x; R)$. The hypercohomology groups $\mathscr{H}^*(X;\mathbb{D}_X^\bullet)$ equal the ordinary homology groups with closed support of X (i.e., Borel-Moore homology). The spectral sequence associated with the complex $\mathbb{D}_X^\bullet$ is the Grothendreck-Zeeman spectral sequence [48].

For any homology n-manifold X, $\mathbb{D}_X^\bullet[-n]$ is naturally isomorphic to the *orientation sheaf* of X. If X is a smooth oriented manifold then $\mathbb{D}_X^\bullet[-n]$ is naturally isomorphic to the complex of differential forms on X.

In terms of duality, the functors $f^!$ and $Rf_!$ may be described by

$$f^!\mathbf{B}^\bullet \cong \mathfrak{D}_X(f^*\mathfrak{D}_Y(\mathbf{B}^\bullet))$$
$$Rf_!\mathbf{A}^\bullet \cong \mathfrak{D}_Y(Rf_*\mathfrak{D}_X(\mathbf{A}^\bullet))$$

where $f\colon X \to Y$ is a continuous map between topological pseudomanifolds, $\mathbf{A}^\bullet \in D^b(X)$ and $\mathbf{B}^\bullet \in D^b(Y)$.

If $\mathbf{A}^\bullet$ is a topologically constructible complex of sheaves on X, $j\colon x \to X$ is the inclusion of a point, and N is a distinguished neighborhood of x, of the type considered in §1.1 and §1.3 then

$$H^p(j^*\mathbf{A}^\bullet) \cong \mathscr{H}^p(N;\mathbf{A}^\bullet) = \mathbf{H}^p(\mathbf{A}^\bullet)_x,$$
$$H^p(j^!\mathbf{A}^\bullet) \cong \mathscr{H}_c^p(N;\mathbf{A}^\bullet).$$

We call these groups the stalk homology and the costalk homology (respectively) of $\mathbf{A}^\bullet$ at x.

Proposition (1.12). *The dualizing complex $\mathbb{D}_X^\bullet$ is constructible with respect to any topological stratification of the topological pseudomanifold X.*

The proof follows from the local product structure of a topological pseudomanifold and the fact that

$$\mathbb{D}_{V\times W}^\bullet = \pi_1^*\mathbb{D}_V^\bullet \otimes \pi_2^*\mathbb{D}_W^\bullet$$

where π_1 and π_2 are the projections of $V\times W$ to the first and second factors respectively.

§1.13. *Standard Identities*

Suppose X and Y are topological pseudomanifolds with fixed stratifications and $f\colon X \to Y$ is a stratified map. Fix $\mathbf{A}^\bullet \in D^b(X)$ and $\mathbf{B}^\bullet$, $\mathbf{C}^\bullet$, $\mathbf{E}^\bullet \in D^b(Y)$ which are constructible with respect to these stratifications. Then there are natural isomorphisms in $D^b(X)$ and $D^b(Y)$,

(1) $\mathfrak{D}(\mathbf{A}^\bullet) = R\mathbf{Hom}^\bullet(\mathbf{A}^\bullet, \mathbb{D}_X^\bullet)$

(2) $\mathbb{D}_X^\bullet = \mathfrak{D}(\mathbf{R}_X) = f^!(\mathbb{D}_Y^\bullet)$

(3) $\mathbf{A}^\bullet \cong \mathfrak{D}(\mathfrak{D}(\mathbf{A}^\bullet))$

(4) $R\,\mathbf{Hom}^\bullet(\mathbf{B}^\bullet \overset{L}{\otimes} \mathbf{C}^\bullet, \mathbf{E}^\bullet) \cong R\,\mathbf{Hom}^\bullet(\mathbf{B}^\bullet, R\,\mathbf{Hom}^\bullet(\mathbf{C}^\bullet, \mathbf{E}^\bullet))$

(5) $f^!\mathbf{B}^\bullet \cong \mathfrak{D}_X f^* \mathfrak{D}_Y \mathbf{B}^\bullet$

(6) $Rf_!\mathbf{A}^\bullet \cong \mathfrak{D}_Y Rf_* \mathfrak{D}_X \mathbf{A}^\bullet$

(7) $f^*(\mathbf{B}^\bullet \overset{L}{\otimes} \mathbf{C}^\bullet) \cong f^*\mathbf{B}^\bullet \overset{L}{\otimes} f^*\mathbf{C}^\bullet$

(8) $f^!R\,\mathbf{Hom}^\bullet(\mathbf{B}^\bullet, \mathbf{C}^\bullet) \cong R\,\mathbf{Hom}^\bullet(f^*\mathbf{A}^\bullet, f^!\mathbf{B}^\bullet)$

(9) $Rf_* R\,\mathbf{Hom}^\bullet(f^*\mathbf{B}^\bullet, \mathbf{A}^\bullet) \cong R\,\mathbf{Hom}^\bullet(\mathbf{B}^\bullet, Rf_*\mathbf{A}^\bullet)$

(10) $Rf_* R\,\mathbf{Hom}^\bullet(\mathbf{A}^\bullet, f^!\mathbf{B}^\bullet) \cong R\,\mathbf{Hom}^\bullet(Rf_!\mathbf{A}^\bullet, \mathbf{B}^\bullet)$

(11) If $f\colon Y \times Z \to Y$ is the projection to the first factor, then

$$f^*R\,\mathbf{Hom}^\bullet(\mathbf{B}^\bullet, \mathbf{C}^\bullet) \cong R\,\mathbf{Hom}^\bullet(f^*\mathbf{B}^\bullet, f^*\mathbf{C}^\bullet)$$

(12) If X is a subset of Y with inclusion $f\colon X \to Y$ then

$$X \text{ open in } Y \Rightarrow f^!\mathbf{B}^\bullet \cong f^*\mathbf{B}^\bullet$$
$$X \text{ closed in } Y \Rightarrow Rf_!\mathbf{A}^\bullet \cong Rf_*\mathbf{A}^\bullet.$$

(13) Fibre square:
If

$$\begin{array}{ccc} Z & \xrightarrow{\bar{f}} & Z \\ {\scriptstyle \bar{\pi}}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{f} & Y \end{array}$$

is a fibre square, then

$$R\bar{f}_*\bar{\pi}^*\mathbf{A}^\bullet \cong \pi^* Rf_*\mathbf{A}^\bullet \quad \text{for any } \mathbf{A}^\bullet \in D^b(X).$$

Further identities for CLC sheaves (§1.4):

(14) If $\mathbf{B}^\bullet$ is CLC on Y then

$$R\,\mathbf{Hom}^\bullet(\mathbf{C}^\bullet, \mathbf{E}^\bullet \overset{L}{\otimes} \mathbf{B}^\bullet) \cong R\,\mathbf{Hom}^\bullet(\mathbf{C}^\bullet, \mathbf{E}^\bullet) \overset{L}{\otimes} \mathbf{B}^\bullet$$

(15) If $f\colon X^n \to Y^m$ is an inclusion of one oriented homology manifold in another one, and if $\mathbf{B}^\bullet$ is CLC on Y, then $f^!\mathbf{B}^\bullet$ is CLC on X and

$$f^!\mathbf{B}^\bullet \cong f^*\mathbf{B}^\bullet[m-n].$$

(16) If $\mathbf{B}^\bullet$ and $\mathbf{C}^\bullet$ are CLC on Y, then

(a) Each $y \in Y$ has a neighborhood U such that $\mathbf{B}^\bullet | \mathbf{U}$ is quasi-isomorphic to a complex of constant sheaves

(b) $R\,\mathbf{Hom}^{\bullet}(\mathbf{B}^{\bullet}, \mathbf{C}^{\bullet})$ is CLC

(c) $(R\,\mathbf{Hom}^{\bullet}(\mathbf{B}^{\bullet}, \mathbf{C}^{\bullet}))_x \cong R\,\mathbf{Hom}^{\bullet}(\mathbf{B}^{\bullet}_x\, \mathbf{C}^{\bullet}_x)$

(d) If $\mathbf{H}^p(\mathbf{B}^{\bullet})=0$ for all but one value of p, then $\mathbf{B}^{\bullet}$ is naturally isomorphic (in $D^b(Y)$) to a local coefficient system.

(17) If $\mathbf{B}^{\bullet}$ is CLC on $Y\times\mathbb{R}^n$ then restriction of sections induces a quasi-isomorphism

$$\pi^* R\pi_* \mathbf{B}^{\bullet} \xrightarrow{\cong} \mathbf{B}^{\bullet}$$

where $\pi\colon Y\times\mathbb{R}^n\to Y$ is the projection to the first factor. (This is because the stalk cohomology of $\pi^* R\pi_* \mathbf{B}^{\bullet}$ at any point (y, t) is equal to the hypercohomology of the restriction of $\mathbf{B}^{\bullet}$ to $\pi^{-1}(y)$. Since $\mathbf{B}^{\bullet}$ is CLC its cohomology sheaves are constant on $\pi^{-1}(y)$ and the spectral sequence for this hypercohomology group collapses, i.e., $E_2^{pq}=\mathscr{H}^p(\mathbb{R}^n;\mathbf{H}^q(\mathbf{B}^{\bullet}))=0$ unless $p=0$ and $E_2^{0q}=(\mathbf{H}^q(\mathbf{B}^{\bullet}))_{(y,t)}$. See lemma following Theorem 1.4).

1.14. *Truncation* ([13, 14, 23])

If $\mathbf{A}^{\bullet}$ is a complex of sheaves on X, define new complexes

$$(\tau_{\leqq p}\mathbf{A}^{\bullet})^n \equiv \begin{cases} \mathbf{A}^n & \text{if } n<p \\ \ker d^p & \text{if } n=p \\ 0 & \text{if } n>p, \end{cases}$$

$$(\tau^{\geqq p}\mathbf{A}^{\bullet})^n \equiv \begin{cases} 0 & \text{if } n<p \\ \operatorname{coker} d^{p-1} & \text{if } n=p \\ \mathbf{A}^n & \text{if } n>p. \end{cases}$$

These functors $\tau_{\leqq p}$ and $\tau^{\geqq p}$ determine "truncation" functors on the derived category $D^b(X)$. Notice, however, that $\tau_{\leqq p}\mathbf{A}^{\bullet}$ is naturally quasi-isomorphic to the complex

$$(\tilde{\tau}_{\leqq p}\mathbf{A}^{\bullet})^n \equiv \begin{cases} \mathbf{A}^n & \text{if } n\leqq p \\ \operatorname{Im} d^p & \text{if } n=p+1 \\ 0 & \text{if } n>p+1 \end{cases}$$

while $\tau^{\geqq p}\mathbf{A}^{\bullet}$ is quasi-isomorphic to the complex

$$(\tilde{\tau}^{\geqq p}\mathbf{A}^{\bullet})^n \equiv \begin{cases} 0 & \text{if } n<p-1 \\ \operatorname{Im} d^{p-1} & \text{if } n=p-1 \\ \mathbf{A}^n & \text{if } n\geqq p. \end{cases}$$

Theorem. *Suppose $\mathbf{A}^{\bullet}$ and $\mathbf{B}^{\bullet}$ are complexes of sheaves on X. Then,*

1. $\tau_{\leqq p}\tau_{\leqq q}\mathbf{A}^{\bullet}=\tau_{\leqq \min(p,q)}\mathbf{A}^{\bullet}$.
2. $(\tau_{\leqq p}\mathbf{A}^{\bullet})_x=\tau_{\leqq p}(\mathbf{A}^{\bullet}_x)$ *where $\mathbf{A}^{\bullet}_x$ denotes the stalk at $x\in X$.*
3. $\mathbf{H}^k(\tau_{\leqq p}\mathbf{A}^{\bullet})_x=\begin{cases}\mathbf{H}^k(\mathbf{A}^{\bullet})_x & \textit{for } k\leqq p\\ 0 & \textit{for } k>p.\end{cases}$
4. *If $\phi\colon \mathbf{A}^{\bullet}\to\mathbf{B}^{\bullet}$ is a sheaf map which induces isomorphisms on the associated cohomology sheaves,*

$$\phi_*\colon \mathbf{H}^n(\mathbf{A}^\bullet)\cong \mathbf{H}^n(\mathbf{B}^\bullet) \quad \textit{for all } n\leqq p$$

then $\tau_{\leqq p}\phi\colon \tau_{\leqq p}\mathbf{A}^\bullet\to\tau_{\leqq p}\mathbf{B}^\bullet$ *is a quasi-isomorphism.*

5. *If* $f\colon X\to Y$ *is a continuous map, and* $\mathbf{C}^\bullet$ *is a complex of sheaves on* Y, *then*

$$\tau_{\leqq p} f^*(\mathbf{C}^\bullet)\cong f^*\tau_{\leqq p}(\mathbf{C}^\bullet).$$

6. *For any* $\mathbf{A}^\bullet\in D^b(X)$ *there is a distinguished triangle*

$$\begin{array}{ccc} \tau_{\leqq p}\mathbf{A}^\bullet & \longrightarrow & \mathbf{A}^\bullet \\ {\scriptstyle [1]}\nwarrow & & \swarrow \\ & \tau^{\geqq p+1}\mathbf{A}^\bullet & \end{array}$$

7. *If* R *is a fiield and* $\mathbf{A}^\bullet$ *is a CLC complex of sheaves of* R*-modules on* X, *then there are natural quasi-isomorphisms*

$$\tau^{\geqq -p}R\,\mathbf{Hom}^\bullet(\mathbf{A}^\bullet,\mathbf{R}_X)\to\tau^{\geqq -p}R\,\mathbf{Hom}^\bullet(\tau_{\leqq p}\mathbf{A}^\bullet,\mathbf{R}_X)\leftarrow R\,\mathbf{Hom}^\bullet(\tau_{\leqq p}\mathbf{A}^\bullet,\mathbf{R}_X).$$

Deligne has also defined a "truncation over a closed subset" functor:

Definition. Let Y be a closed subset of X and let $\mathbf{S}^\bullet$ be a complex of sheaves on X. Fix an integer p. Then $\tau^Y_{\leqq p}\mathbf{S}^\bullet$ is the sheafification of the presheaf $\mathbf{T}^\bullet$, where

(a) for $i<p$, $T^i(U)=\Gamma(U;\mathbf{S}^i)$

(b) for $i=p$, $T^i(U)=\begin{cases}\Gamma(U;\mathbf{S}^i) & \text{if } U\cap Y=\phi\\ \ker d\subset\Gamma(U;\mathbf{S}^i) & \text{if } U\cap T\neq\phi\end{cases}$

(c) for $i>p$, $T^i(U)=\begin{cases}\Gamma(U;\mathbf{S}^i) & \text{if } U\cap Y=\phi\\ \mathbf{0} & \text{if } U\cap Y\neq\phi.\end{cases}$

The stalk of the associated cohomology sheaf is

$$\mathbf{H}^i(\tau^Y_{\leqq p}\mathbf{S}^\bullet)_x=\begin{cases}0 & \text{if } x\in Y \text{ and } i>p\\ \mathbf{H}^i(\mathbf{S}^\bullet)_x & \text{otherwise.}\end{cases}$$

The functor $\tau^Y_{\leqq p}$ passes to a functor on the derived category $D^b(X)$.

1.15. *Lifting Morphisms*

Proposition. *Let* $f\colon \mathbf{A}^\bullet\to\mathbf{B}^\bullet$ *be a morphism in* $D^b(X)$. *Suppose* $\mathbf{H}^i(\mathbf{A}^\bullet)=0$ *for* $i>p$ *and* $\mathbf{H}^i(\mathbf{B}^\bullet)=0$ *for* $i<p$.

Then the canonical map

$$\mathrm{Hom}_{D^b(X)}(\mathbf{A}^\bullet,\mathbf{B}^\bullet)\to\mathrm{Hom}(\mathbf{H}^p(\mathbf{A}^\bullet),\mathbf{H}^p(\mathbf{B}^\bullet))$$

is a bijection.

Proof. Up to quasi-isomorphism, $\mathbf{A}^\bullet$ and $\mathbf{B}^\bullet$ can be represented by complexes

$$\cdots \longrightarrow \mathbf{A}^{p-1} \xrightarrow{d^{p-1}} \mathbf{A}^p \longrightarrow 0 \longrightarrow \cdots$$

$$\cdots \longrightarrow 0 \longrightarrow \mathbf{I}^p \xrightarrow{d^p} \mathbf{I}^{p+1} \longrightarrow \cdots$$

where $\mathbf{I}^d$ is injective for all d. By injectivity, any morphism in $D^b(X)$, $f\colon \mathbf{A}^\bullet \to \mathbf{B}^\bullet$ corresponds to an actual map $\bar{f}$ between these complexes, i.e., a map

$$\operatorname{coker} d^{p-1} = \mathbf{H}^p(\mathbf{A}^\bullet) \to \ker d^p = \mathbf{H}^p(\mathbf{B}^\bullet) \quad \square$$

Proposition. *Suppose* $\mathbf{A}^\bullet$, $\mathbf{B}^\bullet$, *and* $\mathbf{C}^\bullet$ *are objects in* $D^b(X)$ *and that* $\mathbf{H}^n(\mathbf{A}^\bullet)=0$ *for all* $n \geqq p+1$. *Let* $\psi\colon \mathbf{B}^\bullet \to \mathbf{C}^\bullet$ *be be a morphism such that* $\psi_*\colon \mathbf{H}^n(\mathbf{B}^\bullet) \to \mathbf{H}^n(\mathbf{C}^\bullet)$ *is an isomorphism for all* $n \leqq p$. *Then the map induced by* ψ,

$$\operatorname{Hom}_{D^b(X)}(\mathbf{A}^\bullet, \mathbf{B}^\bullet) \to \operatorname{Hom}_{D^b(X)}(\mathbf{A}^\bullet, \mathbf{C}^\bullet)$$

is an isomorphism. In particular, any $\phi\colon \mathbf{A}^\bullet \to \mathbf{C}^\bullet$ *has a unique lift (in* $D^b(X)$*)* $\tilde{\phi}\colon \mathbf{A}^\bullet \to \mathbf{B}^\bullet$ *such that* $\phi = \psi \circ \tilde{\phi}$.

Proof. Let $M^\bullet$ denote the algebraic mapping cylinder of ψ. From the long exact sequence on cohomology which is associated to the triangle

$$\begin{array}{ccc} \mathbf{B}^\bullet & \xrightarrow{\psi} & \mathbf{C}^\bullet \\ {\scriptstyle [1]}\nwarrow & & \swarrow{\scriptstyle \theta} \\ & \mathbf{M}^\bullet & \end{array}$$

we see that $\mathbf{H}^n(\mathbf{M}^\bullet)=0$ for all $n \leqq p-1$. Furthermore, for $\phi\colon \mathbf{A}^\bullet \to \mathbf{C}^\bullet$ the composition $\theta \circ \phi\colon \mathbf{A}^\bullet \to \mathbf{M}^\bullet$ induces the 0 map on $\mathbf{H}^p(\mathbf{A}^\bullet) \to \mathbf{H}^p(\mathbf{M}^\bullet)$. The preceding lemma implies that the map induced by θ

$$\operatorname{Hom}_{D^b(X)}(\mathbf{A}^\bullet, \mathbf{C}^\bullet) \to \operatorname{Hom}_{D^b(X)}(\mathbf{A}^\bullet, \mathbf{M}^\bullet) \cong \operatorname{Hom}(\mathbf{H}^p(\mathbf{A}^\bullet), \mathbf{H}^p(\mathbf{M}^\bullet))$$

is the 0 map. Similarly $\operatorname{Hom}_{D^b(X)}(\mathbf{A}^\bullet, \mathbf{M}^\bullet)[-1]=0$. The conclusion now follows from the exact sequence

$$\begin{aligned} &\to \operatorname{Hom}_{D^b(X)}(\mathbf{A}^\bullet, \mathbf{M}^\bullet)[-1] \to \operatorname{Hom}_{D^b(X)}(\mathbf{A}^\bullet, \mathbf{B}^\bullet) \\ &\to \operatorname{Hom}_{D^b(X)}(\mathbf{A}^\bullet, \mathbf{C}^\bullet) \to \operatorname{Hom}_{D^b(X)}(\mathbf{A}^\bullet, \mathbf{M}^\bullet) \end{aligned}$$

§ 2. $IH_*^{\bar{p}}$ as Hypercohomology, for P.L. Pseudomanifolds

In this chapter we show how the construction of $IH_*^{\bar{p}}(X)$ from [20] actually defines a complex of sheaves on X (the complex of $(\bar{p}, *)$-allowable piecewise linear chains). We calculate the local cohomology groups of this complex and find that

$$IH_i^{\bar{p}}(X, X-x)=0 \quad \text{if } i \leqq n-p(k)-1$$

whenever x lies in a stratum of codimension k.

This vanishing condition is an essential property in the axiomatic characterization of the intersection homology sheaf.

2.0. In this chapter we will assume X is a piecewise-linear pseudomanifold with a fixed (P.L.) stratification

$$\phi = X_{-1} \subset X_0 \subset X_1 \subset \ldots \subset X_{n-2} = \Sigma \subset X$$

as in [20]. We also fix a commutative ring with unit, R.

A *perversity* is a sequence of integers $\bar{p} = (p_2, p_3, \ldots, p_n)$ with $p_2 = 0$ and $p_c \leqq p_{c+1} \leqq p_c + 1$. We shall often write $p(c)$ instead of p_c. There are several perversities of particular importance:

the zero perversity $\bar{0} = (0, 0, 0, \ldots, 0)$
the lower middle perversity $\bar{m} = (0, 0, 1, 1, 2, 2, 3, \ldots)$
the upper middle perversity $\bar{n} = (0, 1, 1, 2, 2, 3, 3, \ldots)$
the logarithmic perversity $\bar{l} = (0, 1, 2, 2, 3, 3, 4, 4, \ldots)$
the sublogarithmic perversity $\bar{s} = (0, 0, 1, 1, 2, 2, 3, \ldots)$
the top perversity $\bar{t} = (0, 1, 2, 3, 4, 5, \ldots)$.

For any perversity $\bar{p}$, the *complementary* perversity is defined to be

$$\bar{t} - \bar{p} = (0, 1 - p_1, 2 - p_2, 3 - p_3, \ldots).$$

For example, $\bar{0}$ and $\bar{t}$ are complementary, as are $\bar{m}$ and $\bar{n}$.

In this chapter we fix a perversity $\bar{p}$, and define a complex of sheaves $\mathbf{IC}^{\bullet}_{\bar{p}}$. However we shall usually omit the subscript $\bar{p}$ for notational convenience.

2.1. *The Complex of Sheaves*, $\mathbf{IC}^{\bullet}_{\bar{p}}$

The treatment in this section is parallel to that of [20] (p. 138) which may be consulted for further details. Define the complex of sheaves of P.L. chains $\mathbf{C}^{\bullet}$ on X by specifying the sections $\Gamma(U; \mathbf{C}^{\bullet})$ over any open subset $U \subset X$ as follows: If T is a locally finite triangulation of U let $C_i^T(U)$ denote the group of locally finite i-dimensional simplicial chains (with R-coefficients), with respect to this triangulation. The support of a chain $\xi \in C_i^T(U)$ is denoted $|\xi|$. Let $C_i(U)$ denote the limit of the $C_i^T(U)$ taken over all locally finite triangulations T of U. If $V \subset U$ and T is a locally finite triangulation of U, there exists a locally finite triangulation S of V such that each simplex of S is contained in a unique simplex of T. Any chain ξ in $C_i^T(U)$ thus gives rise by restriction to a chain ξ' in $C_i^S(V)$ such that $|\xi'| = |\xi| \cap V$. Taking limits over all locally finite compatible triangulations defines a *restriction homomorphism* $C_i(U) \to C_i(V)$ and thus defines a presheaf.

Definition. $\Gamma(U; \mathbf{C}^{-k}) \equiv C_k(U)$.

Remark. If $A \subset U$ is a (relatively closed) i-dimensional P.L. subset of U, and if $B \subset A$ is an $i-1$ dimensional P.L. subset then there is a one to one correspondence between those chains $\alpha \in \Gamma(U; \mathbf{C}^{-k})$ such that $|\alpha| \subset A$ and $|\partial \alpha| \subset B$, and between (Borel-Moore) homology classes with infinite supports, $\tilde{\alpha} \in H_i^{\infty}(A, B)$. The chain $\partial \alpha$ corresponds to the class $\partial_*(\tilde{\alpha}) \in H_{i-1}^{\infty}(B, \phi)$ under the connecting homomorphism.

$\mathbf{C}^\bullet$ is a complex of fine sheaves on X and is quasi-isomorphic to the dualizing complex $\mathbf{D}_X^\bullet$.

Define $\mathbf{IC}_{\bar{p}}^{-k}$ to be the subsheaf of $\mathbf{C}^{-k}$ whose sections over an open set $U \subset X$ consist of all locally finite P.L. chains $\xi \in \Gamma(U; \mathbf{C}^{-k})$ such that $|\xi|$ is $(\bar{p}, i)$-allowable and $|\partial \xi|$ is $(\bar{p}, i-1)$-allowable, with respect to the filtration $U \cap X_0 \subset U \cap X_1 \subset \ldots \subset U \cap \Sigma \subset U$. This means that for each c,

$$\dim(|\xi| \cap U \cap X_{n-c}) \leqq i - c + p(c),$$
$$\dim(|\partial \xi| \cap U \cap X_{n-c}) \leqq i - 1 - c + p(c).$$

Thus, $\mathbf{IC}_{\bar{p}}^\bullet$ is a complex of fine sheaves on X whose hypercohomology $\mathscr{H}^*(X; \mathbf{IC}_{\bar{p}}^\bullet)$ is canonically isomorphic to $IH_*^{\bar{p}}(X; R)$. The associated cohomology sheaf $\mathbf{H}^{-k}(\mathbf{IC}_{\bar{p}}^\bullet)$ is called the local intersection homology sheaf and is denoted $\mathbf{IH}_k^{\bar{p}}$. The stalk at $x \in X$ of this sheaf is denoted $IH_k^{\bar{p}}(X, X-x)$ or $IH_k^{\bar{p}}(X, X-x; R)$.

2.2. $\mathscr{L}$. *Local Coefficient Systems*

Let $\mathbf{F}$ be a local coefficient system of R modules on $X - \Sigma$. [37] (In other words, $\mathbf{F}$ is a sheaf of R modules which, viewed as an étale space over $X - \Sigma$, is a locally trivial fiber bundle. $\mathbf{F}$ is determined by the data: a base point in C and a representation of $\pi_1(C)$ for each connected component C of $X - \Sigma$.)

Consider an open subset $U \subset X$ and a locally finite triangulation T of U. Since $\mathbf{F}$ may not be defined on all of U, it is impossible to define a group $C_i^T(U, \mathbf{F})$ of i chains with coefficients in $\mathbf{F}$. Nevertheless, for any perversity $\bar{p}$, one can define $IC_i^{\bar{p}, T}(U, \mathbf{F})$ as the group of locally finite i-dimensional simplicial chains ξ with coefficients in $\mathbf{F}$ that satisfy allowability conditions: for each $c \geqq 2$,

$$\dim(|\xi| \cap U \cap X_{n-c}) \leqq i - c + \bar{p}(c),$$
$$\dim(|\partial \xi| \cap U \cap X_{n-c}) \leqq i - 1 - c + \bar{p}(c).$$

One can also define a boundary map $IC_i^{\bar{p}, T}(U, \mathbf{F}) \to IC_{i-1}^{\bar{p}, T}(U, \mathbf{F})$. This is because if Δ is any i simplex with nonzero coefficient in ξ, both the interior of Δ and the interiors of all the $i-1$ dimensional faces of Δ lie entirely in $X - \Sigma$ by the allowability conditions. This is all one needs to define by the usual methods the simplicial chain complex with local coefficients.

Definition. The complex $\mathbf{IC}_{\bar{p}}^\bullet(\mathbf{F})$ of intersection chains with coefficients in X is given by $\Gamma(U, \mathbf{IC}_{\bar{p}}^i(\mathbf{F})) = \lim IC_i^{\bar{p}, T}(U, \mathbf{F})$ where the limit is taken over locally finite compatible triangulations of U. The intersection homology groups with coefficients in $\mathbf{F}$, $IH_*^{\bar{p}}(X; \mathbf{F})$, are the hypercohomology, or global section cohomology of $\mathbf{IC}_{\bar{p}}^\bullet(\mathbf{F})$.

2.3. *Indexing Schemes*

There are four ways in the literature to index the dimension of an intersection homology group:

(a) Homology subscripts, as in [20]: a subscript k indicates chains of dimension k.

(b) Homology superscripts, as in Chap. 3 of this paper: a superscript $-k$ indicates chains of dimension k.

(c) Cohomology superscripts, as in [7, 16] a superscript l indicates chains of codimension l, i.e., dimension $n-l$.

(d) Beilinson-Bernstein-Deligne-Gabber scheme: a superscript l indicates chains of codimension $\frac{n}{2}+l$.

For an n-dimensional compact oriented pseudomanifold these schemes compare as follows: $IH_k(X)$ in scheme (a) is isomorphic to $\mathscr{H}^{-k}(X; \mathbf{IC}^\bullet)$ in scheme (b), $IH^{n-k}(X)$ in scheme (c), and $IH^{\frac{n}{2}-k}(X)$ in scheme (d).

2.4. *Calculation of the Local Intersection Homology*

Recall that a P.L. stratification of X determines a filtered P.L. isomorphism of a neighborhood U of each point $x \in X_k - X_{k-1}$ with $x * S^{k-1} * L_X$, where S^{k-1} is the (P.L.) $k-1$ sphere and L_x is a filtered P.L. space called the *link of the stratum* $X_k - X_{k-1}$ at the point x. (If x and y are points in the same connected component of $X_k - X_{k-1}$ then L_x and L_y are P.L. isomorphic.)

Proposition. *The stalk at the point* $x \in X_k - X_{k-1}$ *of the local intersection homology sheaf is*

$$IH_j(X, X-x) = IH_x^{-j} = \begin{cases} IH_{j-k-1}(L_x) & \text{if } j \geqq n-p(n-k) \\ 0 & \text{if } j \leqq n-p(n-k)-1. \end{cases}$$

Intuition. If $j < n-p(n-k)$ then any j-dimensional cycle in $IC_j^{\bar{p}}(X)$ will intersect the stratum $X_k - X_{k-1}$ in a subset of dimension less than k, and can therefore (by transversality) be moved away from the point $\{x\}$. So it represents 0 in the local homology group. If $j \geqq n-p(n-k)$ then any j-dimensional cycle which contains $\{x\}$ also contains a neighborhood of $\{x\}$ in the stratum $X_k - X_{k-1}$, so it is locally the product of $\mathbb{R}^k$ with the cone over a $j-k-1$ dimensional cycle in the link L of the stratum.

Proof. $S^{k-1} * L_x$ inherits a stratification from that of X. We shall find an isomorphism between the stalk at x of the sheaf $\mathbf{IC}^{-j}$ and the group $IC_{j-1}(S^{k-1} * L_x)$.

Any P.L. chain $\eta \in IC_{j-1}(S^{k-1} * L_x)$ gives rise to a germ of a chain near x, by forming the join $x * \eta$. Conversely, if ξ is a germ of a chain near x then in a sufficiently small neighborhood U it can be expressed as a sum of simplices each of which contains x as a vertex. Pseudo-radial retraction along cone lines then determines a P.L. chain $\eta \in IC_{j-1}(S^{k-1} * L_x)$ such that $(x * \eta) \cap U = |\xi| \cap U$. Thus, $\mathbf{IC}_*^{-j} \cong IC_{j-1}(S^{k-1} * L_x)$.

We now compute $IC_*(S^{k-1} * L_x)$. If $A \in C_q(S^{k-1})$ and $B \in IC_{j-q-1}(L_x)$ then $A * B \in IC_j(S^{k-1} * L_x)$ provided either (a) $q \leqq j-(n-k)+p(n-k)-1$ or (b) $q \leqq j-(n-k)+p(n-k)$ and $\partial B = 0$. Letting

$$\tau IC_j(L_x) = \begin{cases} IC_j(L_x) & \text{if } j \geqq n-k-p(n-k) \\ \ker \partial & \text{if } j = n-k-1-p(n-k) \\ 0 & \text{if } j < n-k-1-p(n-k) \end{cases}$$

we obtain a chain map

$$C_*(S^{k-1})\otimes\tau I\,C_*(L_x)\to I\,C_*(S^{k-1}*L_x)$$

which is given by the join of chains. In fact, this map induces isomorphisms on homology because it has a homotopy inverse ψ which is determined by the following (perversity-preserving) rule: If η is a j-simplex in $S^{k-1}*L_x$, let $\psi(\eta)=A*B$ where $A=\eta\cap S^{k-1}$ and B is the image of $\eta-A$ under pseudo-radial retraction along join lines, to L_x.

Applying the Kunneth formula,

$$\begin{aligned} IH_j(X,X-x)&=IH_{j-1}(S^{k-1}*L_x)\\ &=\bigoplus_{q=0}^{k-1} H_q(S^{k-1})\otimes H_{j-q-2}(\tau I\,C_*(L_x))\\ &=\begin{cases} IH^{\bar p}_{j-k-1}(L_x) & \text{if } j\geqq n-p(n-k)\\ 0 & \text{if } j<n-p(n-k).\end{cases}\end{aligned}$$

2.5. *Attaching Property of* $\mathbf{IC}^\bullet$

The following proposition will be needed in the next chapter (§3.5) where its significance will become apparent.

Define $U_k=X-X_{n-k}$. Let $i_k\colon U_k\to U_{k+1}$ and $j_k\colon (U_{k+1}-U_k)\to U_{k+1}$ be the inclusions. Set $\mathbf{IC}_k^\bullet=\mathbf{IC}^\bullet|U_k$.

Proposition. *The natural homomorphism*

$$\mathbf{IC}_{k+1}^\bullet\to R\,i_{k*}\,i_k^*\,\mathbf{IC}_{k+1}^\bullet$$

induces isomorphisms

$$\mathbf{H}^m(j_k^*\,\mathbf{IC}_{k+1}^\bullet)\to\mathbf{H}^m(j_k^*\,R\,i_{k*}\,i_k^*\,\mathbf{IC}_{k+1}^\bullet)$$

for all $m\leqq\bar p(k)-n$.

Proof. Since $\mathbf{IC}^\bullet$ is fine, we have

$$R\,i_{k*}\,i_k^*\,\mathbf{IC}^\bullet\,|U_{k+1}=i_{k*}\,i_k^*\,\mathbf{IC}^\bullet|\,U_{k+1}.$$

Note that sections of $\mathbf{IC}^\bullet|U_{k+1}$ consist of chains in U_{k+1} which can be triangulated with finitely many simplices near $x\in X_{n-k}$, and which satisfy a perversity restriction there. On the other hand, sections of $i_{k*}\,i_k^*(\mathbf{IC}^\bullet|U_{k+1})$ consist of chains *in* U_k which can be triangulated with locally (in U_k) finitely many simplices and which do not necessarily satisfy a perversity restriction near X_{n-k}. Thus $\mathbf{IC}^\bullet|U_{k+1}$ is a complex of subsheaves of $i_{k*}\,i_k^*(\mathbf{IC}^\bullet|U_{k+1})$ and we shall now show that the inclusion induces isomorphisms on the cohomology sheaves of dimensions $m\leqq p(k)-n$, thus establishing the above claim. It suffices to study the cohomology at a point $x\in X_{n-k}$. The inclusion of stalks

$$\mathbf{IC}_x^\bullet\to[i_{k*}\,i_k^*(\mathbf{IC}^\bullet|\,U_{k+1})]_x$$

is a chain map. There is a map back which can be defined as follows: For $j\geqq n-p(k)$, let $\xi\in[i_{k*}\,i_k^*(\mathbf{IC}^{-j}|\,U_{k+1})]_x$. Such a germ has a representation which is a j-dimensional P.L. chain (not necessarily compact) contained in U_{k+1}. Choose a local P.L. filtered product neighborhood of x, $U=D^{n-k}\times c(L)$ where D^{n-k} is the (P.L.) $n-k$ disc and $c(L)$ denotes the cone on a filtered space L, the *link* of the $n-k$-dimensional stratum. U may be chosen so that ξ is transverse to $D^{n-k}\times L$, i.e., so that $\dim(\xi\cap D^{n-k}\times L)\leqq j-1$ (McCrory [29]), and $\xi\cap D^{n-k}\times L$ can be triangulated with *finitely* many simplices. Let $\pi\colon U\to D^{n-k}$ denote the projection to the first factor and let $\eta=c(\xi\cap(D^{n-k}\times L))$ be the P.L. mapping cylinder of $\pi|(|\xi|\cap D^{n-k}\times L)$. η can be oriented using the product orientation from $|\xi|\times[0,1]$, so it defines a chain. We claim $\eta\in\mathbf{IC}_x^{-j}$ i.e., that $|\eta|$ is $(\bar p, j)$-allowable, since $\dim(\eta\cap X_{n-k})\leqq n-k$. Note that when $\partial\xi=0$ we get $\partial\eta=0$ and in fact η represents the same class as ξ in $[\mathbf{H}^{-j}(i_{k*}i_k^*\,\mathbf{IC}^\bullet|U_{k+1})]_x$ since $\xi-\eta$ is the boundary of the (infinite) chain $c(\xi\cap U)$. Thus the above sheaf inclusion induces a surjective on local cohomology, and a similar (relative) argument shows it is also injective on local cohomology.

§ 3. Sheaf Theoretic Construction of $\mathbf{IC}^\bullet$

In [14] Deligne suggested a new method for constructing the complex $\mathbf{IC}^\bullet$. His procedure constructs (for any perversity $\bar p$) a complex of sheaves $\mathbf{P}^\bullet$ on any topological pseudomanifold by starting with the constant sheaf on the nonsingular part and using standard sheaf theoretic operations.

In this chapter we show that $\mathbf{P}^\bullet$ is naturally isomorphic to $\mathbf{IC}^\bullet$ (provided both are constructed with respect to the same stratification - an assumption which is lifted in the next chapter). This result was suggested by Deligne.

In § 3.1 we give Deligne's construction of the complex $\mathbf{P}^\bullet$. In § 3.3 we list axioms which uniquely characterize this complex of sheaves (up to quasi-isomorphism). In § 3.4 we verify that the complex $\mathbf{IC}^\bullet$ from § 2.1 satisfies these axioms and is therefore quasi-isomorphic to $\mathbf{P}^\bullet$.

3.0. Throughout this chapter, X will be an n-dimensional topological pseudomanifold. We *fix* a topological stratification [§ 1.1]

$$\phi=X_{-1}\subset X_0\subset\ldots\subset X_{n-2}=\Sigma\subset X.$$

In § 3.5 we will also assume that X has a P.L. structure and the topological stratification is also a P.L. stratification.

In this chapter we fix a perversity $\bar p$, and a regular Noetherian ring R of finite Krull dimension. The word *sheaf* will mean a sheaf of R-modules.

The complex resulting from Deligne's construction [§ 3.1] is denoted $\mathbf{P}^\bullet$ and in § 3.5 the complex of P.L. intersection chains on X is denoted $\mathbf{IC}_{\bar p}^\bullet$. Theorem 3.5 asserts that these complexes are canonically isomorphic in $D^b(X)$ whenever they are both defined. In subsequent chapters we will use $\mathbf{IC}^\bullet$ (or $\mathbf{IC}_{\bar p}^\bullet$) to denote this isomorphism class of objects, for any topological pseudomanifold.

3.1. *Deligne's Construction*

Consider the filtration by open sets,

$$U_1=U_2\subset U_3\subset\ldots\subset U_{n+1}=X$$

where $U_k = X - X_{n-k}$ and $i_k\colon U_k \to U_{k+1}$ is the inclusion. Define complexes $\mathbb{P}_k^{\bullet} \in D^b(U_k)$ inductively as follows

$$\mathbb{P}_2^{\bullet} = \mathbb{D}_{X-\Sigma}^{\bullet} \cong \mathbf{R}[n] \quad \text{on} \quad U_2 = X - \Sigma$$
$$\mathbb{P}_{k+1}^{\bullet} = \tau_{\leqq p(k)-n} R i_{k*} \mathbb{P}_k^{\bullet} \qquad \text{for } k \geqq 2.$$

Definition. Deligne's construction is the complex $\mathbb{P}^{\bullet} = \mathbb{P}_{n+1}^{\bullet}$ which is defined by this process. In other words,

$$\mathbb{P}^{\bullet} = \tau_{\leqq p(n)-n} R i_{n*} \dots \tau_{\leqq p(3)-n} R i_{3*} \tau_{\leqq p(2)-n} R i_{2*} \mathbf{R}_{X-\Sigma}[n].$$

$\mathscr{L}$. We could equally well have started with a system of local coefficients $\mathbf{F}$ on $X - \Sigma$ in place of the constant sheaf $\mathbf{R}$, so

$$\mathbb{P}_2^{\bullet} = \mathbf{F}[n] \quad \text{on} \quad U_2 = X - \Sigma$$
$$\mathbb{P}_{k+1}^{\bullet} = \tau_{\leqq p(k)-n} R i_{k*} \mathbb{P}_k^{\bullet} \qquad \text{for } k \geqq 1$$
$$\mathbb{P}^{\bullet}(\mathbf{F}) = \mathbb{P}_{n+1}^{\bullet}.$$

The resulting complex $\mathbb{P}^{\bullet}(\mathbf{F})$ is called the "intersection homology chains with coefficients in $\mathbf{F}$".

Lemma. *$\mathbb{P}^{\bullet}$ is constructible with respect to the given stratification of X.*

Proof. Clearly $\mathbb{P}_2^{\bullet}$ is constructible on U_2. Suppose we have shown that $\mathbb{P}_k^{\bullet}$ is constructible on U_k. Fix $x \in U_{k+1} - U_k$ and let N be a distinguished neighborhood of x in U_{k+1}, of the type considered in § 1.1, i.e., there is a stratum preserving homeomorphism

$$N \cong \mathbb{R}^k \times V$$

where $V = \operatorname{cone}^{\circ}(L)$ for some stratified space L. Let $N^{\circ} = N \cap U_k \cong \mathbb{R}^k \times V^{\circ}$ where $V^{\circ} = \operatorname{cone}(L) - \text{vertex}$. Consider the fibre square

$$\begin{array}{ccc} \mathbb{R}^k \times V^{\circ} & \xrightarrow{\ i\ } & \mathbb{R}^k \times V \\ \Big\downarrow \pi & & \Big\downarrow \pi \\ V^{\circ} & \xrightarrow{\ i\ } & V \end{array}$$

By § 1.13.17, § 1.13.13, and induction,

$$R i_*(\mathbb{P}_k^{\bullet} | N^{\circ}) \cong R i_*(\pi^* R\pi_* \mathbb{P}_k^{\bullet} | N^{\circ})$$
$$\cong \pi^* R i_* R\pi_*(\mathbb{P}_k^{\bullet} | N^{\circ})$$

which shows that $R i_{k*} \mathbb{P}_k^{\bullet}$ is CLC on each stratum of U_{k+1}. It follows that $\tau_{\leqq p(k)-n} R i_{k*} \mathbb{P}_k^{\bullet}$ is also CLC on each stratum of U_{k+1}, which completes the induction.

3.2. *The Attaching Map*

If $\{X_k\}$ denotes the filtration of the space X, let $U_k = X - X_{n-k}$ denote the complementary filtration by open sets, with inclusions $i_k\colon U_k \to U_{k+1}$. Let $j_k\colon (U_{k+1} - U_k) \to U_{k+1}$ be the inclusion of the stratum of codimension k into U_{k+1}. Let $\mathbf{S}^{\bullet}$ be a complex of sheaves on X which is constructible with respect to the filtration $\{X_k\}$ (see § 1.11) and let $\mathbf{S}_k^{\bullet} = \mathbf{S}^{\bullet} | U_k$.

Definition. The attaching map of degree m associated with the complex $\mathbf{S}^{\bullet}$ over the stratum $X_{n-k}-X_{n-k-1}$ is the sheaf map

$$A_m : \mathbf{H}^m(j_k^* \mathbf{S}^{\bullet}_{k+1}) \to \mathbf{H}^m(j_k^* R i_{k*} i_k^* \mathbf{S}^{\bullet}_{k+1})$$

which is obtained by restricting the natural morphism

$$\mathbf{S}^{\bullet}_{k+1} \to R i_{k*} i_k^* \mathbf{S}^{\bullet}_{k+1}$$

to this stratum, and taking the induced map on cohomology sheaves.

We shall say the sheaf $\mathbf{S}^{\bullet}$ is r-attached across this stratum, if A_m is an isomorphism for all $m \leqq r$.

3.3. *Axioms* [AX1]

Definition. Let $\mathbf{S}^{\bullet}$ be a complex of sheaves on X, which is constructible with respect to the stratification $\{X_k\}$ and let $\mathbf{S}^{\bullet}_k = \mathbf{S}^{\bullet}|(X - X_{n-k})$. We shall say $\mathbf{S}^{\bullet}$ satisfies the axioms [AX1] (with perversity $\bar{p}$, and with respect to the stratification $\{X_k\}$) provided:

(a) Normalization: $\mathbf{S}^{\bullet}|(X-\Sigma) \cong \mathbf{F}[n]$ where F is a local coefficient system on $X - \Sigma$.

(b) Lower bound: $\mathbf{H}^i(\mathbf{S}^{\bullet}) = 0$ for all $i < -n$.

(c) Vanishing condition: $\mathbf{H}^m(\mathbf{S}^{\bullet}_{k+1}) = 0$ for all $m > p(k) - n$.

(d) Attaching: $\mathbf{S}^{\bullet}$ is $p(k)-n$ attached across each stratum of codimension k, i.e., the attaching maps

$$\mathbf{H}^m(j_k^* \mathbf{S}^{\bullet}_{k+1}) \to \mathbf{H}^m(j_k^* R i_{k*} i_k^* \mathbf{S}^{\bullet}_{k+1})$$

are isomorphisms for all $k \geqq 2$ and all $m \leqq \bar{p}(k) - n$.

Definition. We shall say $\mathbf{S}^{\bullet}$ satisfies $[\mathrm{AX1}]_R$ it if satisfies [AX1] with $\mathbf{F} = \mathbf{R}_{X-\Sigma}$ = the constant sheaf, in part (a).

3.4. *Alternate Formulations of* AX1[d]

We may replace axiom (d) with

(d') $\quad \mathbf{H}^m(j_k^! \mathbf{S}^{\bullet}_{k+1}) = 0 \quad$ for all $\quad k \geqq 2 \quad$ and all $\quad m \leqq \bar{p}(k) - n + 1$.

It is easy to see that (d')⇒(d) using the long exact cohomology sequence associated with the distinguished triangle

$$\begin{array}{ccc} j_k^* \mathbf{S}^{\bullet}_{k+1} & \longrightarrow & j_k^* R i_{k*} i_k^* \mathbf{S}^{\bullet}_{k+1} \\ & \nwarrow \quad \swarrow {\scriptstyle [1]} & \\ & j_k^! \mathbf{S}^{\bullet}_{k+1} & \end{array}$$

The same sequence also give (c) and (d)⇒(d').

Furthermore AX1[d'] is equivalent to

(d'') for all $k \geqq 2$, for all $x \in X_{n-k} - X_{n-k-1}$ and for all $m \leqq p(k) - k + 1$, we have

$$H^m(j_x^! \mathbf{S}^\bullet)=0$$

where $j_x: \{x\} \to X$ is the inclusion of a point.

To see this, factor j_x into a composition

$$x \xrightarrow[u_x]{} X_{n-k} - X_{n-k-1} \xrightarrow[j]{} X$$

Then $j_x^! \mathbf{S}^\bullet \cong u_x^! j^! \mathbf{S}^\bullet \cong u_x^* j^! \mathbf{S}^\bullet[n-k]$ by § 1.13.15. Thus the cohomology of this complex vanishes in dimensions $m \leqq p(k)-k+1$ iff the stalk cohomology of $j^! \mathbf{S}^\bullet$ vanishes in dimensions $\leqq p(k)-n+1$.

This reformulation of AX1[d] is useful because it is equivalent to AX2[d] which will appear in the next chapter.

3.5. [AX1] *Characterizes Deligne's Construction*

Theorem. *The functor* $\mathbb{P}^\bullet$ *which assigns to any locally trivial sheaf* $\mathbf{F}$ *on* $X-\Sigma$, *the complex*

$$\mathbb{P}^\bullet(\mathbf{F}) = \tau_{\leqq p(n)-n} R i_{n*} \ldots \tau_{\leqq p(2)-n} R i_{2*} \mathbf{F}[n]$$

defines an equivalence of categories between

(a) *the category of locally constant sheaves on* $X-\Sigma$ *and*

(b) *the full subcategory of* $D^b(X)$ *whose objects are all complexes of sheaves which satisfy the axioms* [AX1].

The inverse functor $\mathbf{L}$ assigns to any constructible complex of sheaves $\mathbf{S}^\bullet$ which satisfy [AX1] the locally constant sheaf $\mathbf{L}(\mathbf{S}^\bullet) = \mathbf{H}^{-n}(\mathbf{S}^\bullet|(X-\Sigma))$.

Proof. In fact we will show that for each $k \geqq 2$ the functor

$$\mathbb{P}_k^\bullet = \tau_{\leqq p(k)-n} R i_{k*}$$

defines an equivalence of categories between

(a) the full subcategory C_k of $D^b(U_k)$ whose objects are complexes of sheaves which satisfy the axioms [AX1] on U_k, and

(b) the full subcategory C_{k+1} of $D^b(U_{k+1})$ whose objects are complexes of sheaves which satisfy the axioms [AX1] on U_{k+1}.

The inverse functor $\mathbf{L}_k$ is i_k^*.

This will suffice because $\mathbb{P}^\bullet = \mathbb{P}_n^\bullet \circ \ldots \circ \mathbb{P}_3^\bullet \circ \mathbb{P}_2^\bullet$ and $\mathbf{L} = \mathbf{L}_2 \circ \mathbf{L}_3 \circ \ldots \circ \mathbf{L}_n$. Using § 1.13.16(d), $\mathbf{L}_2(\mathbf{S}_3^\bullet) = \mathbf{H}^{-n}(\mathbf{S}_3^\bullet|(X-\Sigma))$ where $\mathbf{S}_3^\bullet = \mathbf{S}_3^\bullet|U_3$.

Clearly L_k is a functor from C_{k+1} to C_k. $\mathbb{P}_k^\bullet$ is a functor from C_k to C_{k+1} for the following reasons: For any $\mathbf{A}^\bullet \in C_k$, $\mathbb{P}_k^\bullet(\mathbf{A}^\bullet) = \tau_{\leqq p(k)-n} R i_{k*} \mathbf{A}^\bullet$ satisfies [AX1](a)(b)(c) by construction. [AX1](d) is also satisfied because the attaching map is the composition

$$\tau_{\leqq p(k)-n} j_k^* R i_{k*} \mathbf{A}^\bullet \cong j_k^* \mathbb{P}_k^\bullet(\mathbf{A}^\bullet) \to j_k^* R i_{k*} i_k^* \mathbb{P}_k^\bullet(\mathbf{A}^\bullet) \cong j_k^* R i_{k*} \mathbf{A}^\bullet$$

which induces isomorphisms on stalk cohomology in the dimensions $m \leqq p(k)-n$.

Clearly $\mathbf{L}_k \mathbb{P}_k^\cdot(\mathbf{S}_k^\cdot)=\mathbf{S}_k^\cdot$. For each object $\mathbf{A}^\cdot$ in C_{k+1} we must construct a quasi-isomorphism $T_{k+1}(\mathbf{A}^\cdot)$: $\mathbf{A}^\cdot \cong \mathbb{P}_k^\cdot \mathbf{L}_k(\mathbf{A}^\cdot)$ which is natural as a transformation from the identity to $\mathbb{P}_k^\cdot \circ \mathbf{L}_k$, i.e., for any morphism f: $\mathbf{A}^\cdot \to \mathbf{B}^\cdot$ in the category C_{k+1}, the following diagram commutes:

$$\begin{array}{ccc} \mathbf{A}^\cdot & \xrightarrow{\;f\;} & \mathbf{B}^\cdot \\ {\scriptstyle T_{k+1}(\mathbf{A}^\cdot)}\downarrow & & \downarrow{\scriptstyle T_{k+1}(\mathbf{B}^\cdot)} \\ \mathbb{P}_k^\cdot \mathbf{L}_k(\mathbf{A}^\cdot) & \xrightarrow[\mathbb{P}_k^\cdot L_k(f)]{} & \mathbb{P}_k^\cdot \mathbf{L}_k(\mathbf{B}) \end{array}$$

Consider the natural map $\mathbf{A}^\cdot \to R i_{k*} i_k^* \mathbf{A}^\cdot$. Define $T_{k+1}(\mathbf{A}^\cdot)$ to be the composition

$$\mathbf{A}^\cdot \cong \tau_{\leqq p(k)-n} \mathbf{A}^\cdot \to \tau_{\leqq p(k)-n} R i_{k*} i_k^* \mathbf{A}^\cdot = \mathbb{P}_k^\cdot \mathbf{L}_k(\mathbf{A}^\cdot)$$

This is a quasi isomorphism over U_k. We must check that it is also a quasi isomorphism over $U_{k+1}-U_k$. By [AX1](d) the morphism

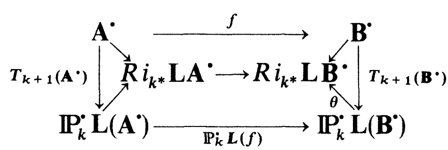
$$j_k^* \mathbf{A}^\cdot \to j_k^* R i_{k*} i_k^* \mathbf{A}^\cdot$$

induces isomorphisms on cohomology sheaves in all dimensions $m \leqq p(k)-n$. Thus

$$j_k^* \mathbf{A}^\cdot \cong j_k^* \tau_{\leqq p(k)-n} \mathbf{A}^\cdot \cong j_k^* \tau_{\leqq p(k)-n} R i_{k*} i_k^* \mathbf{A}^\cdot$$

which completes the check.

For any morphism f: $\mathbf{A}^\cdot \to \mathbf{B}^\cdot$ in C_{k+1} we have a diagram



By induction the right and left triangles and top and bottom trapezoids commute. We must show that the outside square commutes. It is clear that

$$\theta \circ \mathbb{P}_k^\cdot \mathbf{L}(f) \circ T_{k+1}(\mathbf{A}^\cdot) = \theta \circ T_{k+1}(\mathbf{B}^\cdot) \circ f.$$

However, according to § 1.15 composition with θ induces an isomorphism

$$\operatorname{Hom}_{D^b(U_{k+1})}(\mathbf{A}^\cdot, \mathbb{P}_k^\cdot \mathbf{L}\,\mathbf{B}^\cdot) \to \operatorname{Hom}_{D^b(U_{k+1})}(\mathbf{A}^\cdot, R i_{k*} \mathbf{L}(\mathbf{B}^\cdot))$$

so we can cancel the θ from the above equation. This completes the proof.

Corollary. *If a constructible complex* $\mathbf{S}^\cdot$ *satisfies* $[\mathrm{AX1}]_R$, *then* $\mathbf{S}^\cdot$ *is naturally quasi-isomorphic to Deligne's complex* $\mathbb{P}^\cdot = \mathbb{P}^\cdot(\mathbf{R}_{X-\Sigma})$ *which was defined in* § 3.1. *If in addition all the* $\mathbf{S}^i$ *are fine, the cohomology groups of the complex*

$$\to \Gamma(X; S^{i-1}) \to \Gamma(X; S^i) \to \Gamma(X; S^{i+1}) \to \dots$$

are naturally isomorphic to the intersection homology groups $IH_*^{\bar{p}}(X)$.

Example. J. Cheeger has shown [10] that if X is a Riemannian pseudomanifold with conical singularities then the complex of locally L^2 differential forms on $X - \Sigma$ is a complex of fine sheaves on X which satisfies [AX1]. This complex is defined by

$\Gamma(U; \Omega^p) \equiv$ those differential p-forms ω on $U \cap (X-\Sigma)$ such that for every point $x \in U$ there is a neighborhood V_x such that

$$\int_{V_x \cap (X-\Sigma)} \omega \wedge * \omega < \infty \quad \text{and} \quad \int_{V_x \cap (X-\Sigma)} d\omega \wedge *(d\omega) < \infty.$$

The following example explains the use of the word "attaching": the complex

$$\bigoplus_i \mathbf{H}^i(\mathbf{IC}^{\bullet})[-i]$$

satisfies $[\mathrm{AX1}]_R$ except for the attaching axiom. It clearly has the same homology sheaves as $\mathbf{IC}^{\bullet}$ but is in general not isomorphic to $\mathbf{IC}^{\bullet}$ since $\mathbf{IC}^{\bullet}$ is indecomposable (see Corollary 2 in § 4.1).

3.6. $\mathbf{IC}^{\bullet}$ *Satisfies the Axioms* [AX1]

Theorem. *If X is a P.L. pseudomanifold with a fixed P.L. stratification, then the sheaf of intersection chains* $\mathbf{IC}^{\bullet}$ *satisfies the axioms* $[\mathrm{AX1}]_R$ *with respect to that stratification.*

Proof. Axioms AX1(a)(b) are obviously satisfied. Axiom [AX1](c) and constructibility were verified in § 2.4. [AX1](d) was verified in § 2.5.

Corollary. *If X is a P.L. pseudomanifold with a fixed P.L. stratification then the sheaf of piecewise linear intersection chains as constructed in* [19] (*and* § 2.1) *is naturally quasi isomorphic to* $\mathbb{P}^{\bullet}$ *as constructed by Deligne's procedure in* § 3.1.

§ 4. Topological Invariance of $\mathbf{IC}^{\bullet}$

In this chapter we shall show that the intersection homology groups $IH_*^{\bar{p}}(X)$ are topological invariants and they do not depend on the choice of stratification of X. In fact, we shall show for any homeomorphism $f\colon X \to Y$, that the complexes $\mathbb{P}_X^{\bullet}$ and $f^* \mathbb{P}_X^{\bullet}$ are quasi-isomorphic.

A key ingredient of the proof is the construction of the canonical $\bar{p}$ filtration $X_0^{\bar{p}} \subset X_1^{\bar{p}} \subset \ldots \subset X_{n-2}^{\bar{p}} \subset X_n^{\bar{p}} = X$ of X. This depends on the choice of a perversity $\bar{p}$ but not on a previous choice of a stratification of X - it is a purely topological invariant of X. The filtration $\{X_i^{\bar{p}}\}$ is a sort of "homological stratification". For example, $X_i^{\bar{p}} - X_{i-1}^{\bar{p}}$ is a R-homology manifold of dimension i. It may be thought of as the "coarsest stratification" with respect to which Deligne's construction gives $\mathbf{IC}^{\bullet}$ - any topological stratification is a refinement of it. The role of the canonical $\bar{p}$ filtration in the proof is to compare objects of $D^b(X)$ satisfying axioms AX1 with respect to two different topological stratifications (which may not have a common refinement).

This chapter contains a set of axioms [AX2] which uniquely characterize the complex $\mathbb{P}^{\bullet}$ up to quasi-isomorphism but which do not refer to a choice of stratification of X. These axioms involve the concepts of local support and co-support of a complex of sheaves, which we now describe.

If $\mathbf{S}^{\bullet}$ is a complex of sheaves on X, and $j_x\colon \{x\} \to X$ is the inclusion of a point, there is a homomorphism

$$\mathcal{H}^m(X; \mathbf{S}^{\bullet}) \to \mathcal{H}^m(Rj_{x*} j_x^* \mathbf{S}^{\bullet}) \cong H^m(j_x^* \mathbf{S}^{\bullet}).$$

If a class $\xi \in \mathcal{H}^m(X; \mathbf{S}^{\bullet})$ does not vanish under this homomorphism then any cycle representative of ξ must contain the point x. Thus, $H^m(j_x^* \mathbf{S}^{\bullet})$ represents local classes which "cannot be pulled away from the point x", and we say

$$\{x \in X \mid H^m(j_x^* \mathbf{S}^{\bullet}) \neq 0\}$$

is the *local support* set of the complex $\mathbf{S}^{\bullet}$ (in dimension m).

Similarly, there is a homomorphism

$$H^m(j_x^! \mathbf{S}^{\bullet}) = \mathcal{H}^m(Rj_{x*} j_x^! \mathbf{S}^{\bullet}) \to \mathcal{H}^m(X; \mathbf{S}^{\bullet}).$$

A class $\eta \in \mathcal{H}^m(X; \mathbf{S}^{\bullet})$ is in the image of this homomorphism if some cycle representative of η is completely contained in a neighborhood of x. Thus $H^m(j_x^! \mathbf{S}^{\bullet})$ represents local classes which are "supported near x" and we say

$$\{x \in X \mid H^m(j_x^! \mathbf{S}^{\bullet}) \neq 0\}$$

is the local co-support set of the complex $\mathbf{S}^{\bullet}$ (in dimension m). The axioms [AX2] place restrictions on the size of the local support and co-support sets.

4.0. Throughout this chapter we shall assume X is an n-dimensional topological pseudomanifold (but we do not fix any particular stratification of X). We shall also fix a perversity $\bar{p}$ and let q denote the complementary perversity, $q(k) = k - 2 - p(k)$ for all $k \geqq 2$. By a *sheaf* on X, we shall mean a sheaf of R-modules, where R is a fixed finite dimensional regular Noetherian ring.

4.1. *Axioms* [AX2]

For the perversity $\bar{p}$, let p^{-1} denote the sub-inverse of p, i.e.,

$$p^{-1}(l) = \min\{c \mid p(c) = l\}.$$

We use the convention that $\min(\phi) = \infty$. Recall that $\bar{p}$ is a function from the set $\{2, 3, 4, \ldots\}$ to the nonnegative integers. It satisfies inequalities which imply that if it takes two integer values then it also takes each value between them.

Let $\bar{q}$ denote the complementary perversity, $q(k) = k - 2 - p(k)$ and let q^{-1} denote the sub-inverse of $\bar{q}$.

Suppose X is an n-dimensional topological pseudomanifold. For each $x \in X$, let $j_x\colon \{x\} \to X$ denote the inclusion.

Definition. A topologically constructible complex of sheaves $\mathbf{S}^{\cdot}$ on X satisfies axioms [AX2] provided:

(a) Normalization – There is a closed subset $Z \subset X$ such that $\mathbf{S}^{\cdot}|(X - Z) \cong \mathbf{R}_{X-Z}[n]$ and $\dim(Z) \leqq n-2$.

(b) Lower bound
$\mathbf{H}^m(\mathbf{S}^{\cdot}) = 0$ for all $m < -n$.

(c) Support condition
For all $m \geqq -n+1$, $\dim\{x \in X \mid H^m(j_x^* \mathbf{S}^{\cdot}) \neq 0\} \leqq n - p^{-1}(m+n)$.

(d) Cosupport condition
For all $m \leqq -1$, $\dim\{x \in X \mid H^m(j_x^! \mathbf{S}^{\cdot}) \neq 0\} \leqq n - q^{-1}(-m)$.

Where "dim" denotes the topological dimension of Hurewicz and Wallman [24]. We use the convention that a set of negative dimension (including $-\infty$) must be empty.

Uniqueness Theorem. *Up to canonical isomorphism there exists a unique complex in $D^b(X)$ which satisfies axioms* [AX2]. *It is given by* $\mathbf{IC}^{\cdot}$, *constructed* (*as in* § 2.1 *and* § 3.1) *with respect to any stratification of* X.

The proof is in § 4.3

Corollary 1. *For any topological pseudomanifold X, the groups $IH_*^{\bar{p}}(X)$ are topological invariants and exist independently of the choice of a stratification of X.*

$\mathcal{L}$. Let Z be X_{n-2} for some topological stratification $X = X_n \supset \ldots \supset X_0$ of X and let $\mathbf{F}$ be a local system on $X - Z$. Then if we replace axiom [AX2](a) by

(a′) $$\mathbf{S}^{\cdot}|(X - Z) \cong \mathbf{F}[n]$$

the uniqueness theorem still holds with $\mathbf{IC}^{\cdot}$ replaced by $\mathbf{IC}^{\cdot}(\mathbf{F})$.

Definition. A topological pseudomanifold X is *irreducible* if $X - \Sigma$ is connected for some (and hence for any) stratification of X.

Corollary 2. *If X is irreducible and* $\mathbf{F}$ *is a local coefficient system on $X - \Sigma$ which is indecomposable as a local system, then* $\mathbf{IC}^{\cdot}(\mathbf{F})$ *is indecomposable, i.e., if* $\mathbf{IC}^{\cdot} \cong \mathbf{A} \oplus \mathbf{B}^{\cdot}$ *in $D^b(X)$, then either* $\mathbf{A}^{\cdot} \cong \mathbf{zero}^{\cdot}$ *or* $\mathbf{B}^{\cdot} \cong \mathbf{zero}^{\cdot}$.

Proof. Both $\mathbf{A}^{\cdot}$ and $\mathbf{B}^{\cdot}$ satisfy the axioms [AX2] except for axiom (a).

Remark. Assume $\mathbf{S}^{\cdot}$ is a complex of sheaves which is constructible with respect to a topological stratification $X_0 \subset X_1 \subset \ldots \subset X_n = X$, and satisfies [AX2]. Then it also satisfies the axioms [AX2]′ where the sets in statements (c) and (d) are assumed to be unions of strata and where "$X - Z$" is replaced by "$X - X_{n-2}$". This is because $\mathbf{S}^{\cdot}|(X - X_{n-2})$ is a local coefficient system and $\pi_1(X - X_{n-2} - Z) \to \pi_1(X - X_{n-2})$ is surjective since Z has topological codimension 2

4.2. *Construction of the Canonical $\bar{p}$-Filtration*

Let $U^{\bar{p}}$ be the largest open set on which $\mathbb{D}_X^{\cdot}$ is CLC (§ 1.4). Let $\Sigma^{\bar{p}} = X - U^{\bar{p}}$, and let $n - m$ be the topological dimension of $\Sigma^{\bar{p}}$. Then $m \geqq 2$ because for any topological stratification of X as a topological pseudomanifold, $\Sigma^{\bar{p}}$ will be a union of connected components of interiors of strata (by Prop. 1.12), and the stratum of dimension n is contained in $U^{\bar{p}}$.

Define $X^{\bar{p}}_{n-2}=\Sigma^{\bar{p}}=X-U^{\bar{p}}$.

Suppose by induction that $X^{\bar{p}}_{n-k}$, $X^{p}_{n-k+1},\dots,X^{\bar{p}}_{n-2}\subset X$ has been defined. Let $U^{\bar{p}}_k=X-X^{\bar{p}}_{n-k}$ and let $\mathbb{P}^{\cdot}_k\in D^b(U^{\bar{p}}_k)$ be the complex obtained from Deligne's construction using the filtration by $\{X^{\bar{p}}_i\}$. Let $h_k\colon U^{\bar{p}}_k\to X$ be the inclusion. Let V' be the largest open subset of $X^{\bar{p}}_{n-k}$ on which $\mathbb{D}^{\cdot}_{X^{\bar{p}}_{n-k}}$ and $(Rh_{k*}\mathbb{P}^{\cdot}_k)|X^{\bar{p}}_{n-k}$ are both CLC. Let V be the union of the connected components of V' which have topological dimension $n-k$.

Define $X^{\bar{p}}_{n-k-1}=X^{\bar{p}}_{n-k}-V$.

This completes the inductive step in the definition.

Proposition. 1. *This procedure terminates after finitely many steps. In fact,* $\dim(X^{\bar{p}}_{n-k-1})\leqq n-k-1$.

2. *For any topological stratification of X, each $X^{\bar{p}}_{j-k-1}$ is a closed union of connected components of strata.*

3. *Each $Z_{n-k}=X^{\bar{p}}_{n-k}-X^{\bar{p}}_{n-k-1}$ is either empty or else it is an $n-k$ dimensional R-homology manifold.*

4. *Let $\mathbf{Q}^{\cdot}$ be the complex of sheaves obtained by applying Deligne's construction to the filtration $\{X^{\bar{p}}_i\}$. Then for any k, $\mathbf{Q}^{\cdot}|Z_{n-k}$ is CLC.*

Proof. We prove all these propositions simultaneously by induction. Suppose they are true for all integers $<k$. Fix a topological stratification $X_0\subset X_1\subset\dots\subset X_n=X$ of X.

Lemma. *$Rh_{k*}\mathbf{Q}_k|X^{\bar{p}}_{n-k}$ is constructible with respect to this stratification.*

Proof of Lemma. We must show this complex is CLC on each stratum of $X^{\bar{p}}_{n-k}$. Choose any point $x\in X^{\bar{p}}_{n-k}$ and let S_r be the stratum which contains x. Choose a conical filtered space $V=V_n\supset V_{n-1}\supset\dots\supset V_r=$a point, and a continuous homeomorphism of $V\times\mathbb{R}^r$ to a neighborhood N of x in X (see the definition of a stratification in § 1.1). Let $\pi\colon N\to V$ denote the resulting projection to the first factor. V inherits a filtration

$$V_0=V^{\bar{p}}_r\subset V^{\bar{p}}_{n-k}\subset V^{\bar{p}}_{n-k+1}\subset\dots\subset V^{\bar{p}}_n$$

such that $X^{\bar{p}}_{n-j}=V^{\bar{p}}_{n-j}\times\mathbb{R}^r$.

We now use a tilda to denote the intersection of a subset with N, and we use a bar to denote the projection of such a subset to V, as follows:

Let $\tilde{U}^p=U^p\cap N$, $\tilde{U}^{\bar{p}}_j=U^{\bar{p}}_j\cap N$.

Let $\tilde{i}_j\colon\tilde{U}^{\bar{p}}_j\to\tilde{U}^{\bar{p}}_{j+1}$ and $\tilde{h}_j\colon\tilde{U}^{\bar{p}}_j\to N$ denote the inclusions.

Let $\bar{U}^p=\pi(\tilde{U}^{\bar{p}})$, $\bar{U}^{\bar{p}}_j=\pi(\tilde{U}^{\bar{p}}_j)$.

Let $\bar{i}_j\colon\bar{U}^{\bar{p}}_j\to\bar{U}^{\bar{p}}_{j+1}$ and $h_j\colon\bar{U}^{\bar{p}}_j\to V$ denote the inclusions.

It follows from the inductive hypothesis (2) that $\tilde{U}^{\bar{p}}_j=\pi^{-1}(\bar{U}^{\bar{p}}_j)$.

Therefore,

$$\begin{aligned}(Rh_{k*}\mathbf{Q}^{\cdot}_k)|N&\cong Rh_{k*}(\mathbf{Q}^{\cdot}_k|N)\\&\cong R\tilde{h}_{k*}\tau_{\leqq p(k-1)-n}R\tilde{i}_{k-1*}\dots\tau_{\leqq p(2)-n}R\tilde{i}_{2*}\mathbf{R}_{\tilde{U}^{\bar{p}}}[n]\\&\cong R\tilde{h}_{k*}\tau_{\leqq p(k-1)-n}R\tilde{i}_{k-1*}\dots\tau_{\leqq p(2)-n}R\tilde{i}_{2*}\pi^*\mathbf{R}_{\bar{U}^{\bar{p}}}[n].\end{aligned}$$

Now the π^* moves to the left changing tildas to bars, giving

$$\cong R\tilde{h}_{k*}\pi^*\tau_{\leqq p(k-1)-n}\, R\bar{i}_{k-1*}\ldots\tau_{\leqq p(2)-n}R\bar{i}_{2*}\mathbf{R}_{\bar{U}^{\bar{p}}}[n]$$
$$\cong \pi^* R\bar{h}_{k*}\tau_{\leqq p(k)-n}R\bar{i}_{k-1*}\ldots\tau_{\leqq p(2)-n}R\bar{i}_{2*}\mathbf{R}_{\bar{U}^{\bar{p}}}[n]$$

which is CLC when restricted to $\pi^{-1}(V_0)$.

This lemma implies that the set V' above is a union of connected components of strata and it contains the $n-k$ dimensional strata in X. Thus V is also a union of connected components of strata which contains all the $n-k$ dimensional strata in V'. This proves (1) and (2). Property (3) is guaranteed by the condition that $\mathbb{D}^{\bullet}_{X^{\bar{p}}_{n-k}}$ be CLC on $V' \supset V = Z_{n-k}$. Finally, $Rh_{k*}\mathbf{Q}^{\bullet}_k|Z_{n-k}$ is CLC so $\tau_{\leqq p(k)-n}Rh_{k*}\mathbf{Q}^{\bullet}_k|Z_{n-k} = \mathbf{Q}^{\bullet}|Z_{n-k}$ is also CLC, which proves (4).

4.3. *Proof of Topological Invariance* (Theorem 4.1)

For any topological stratification $\{X_j\}$ of X, define $S\{X_j\}$ to be the full subcategory of $D^b(X)$ consisting of complexes which are construtible with respect to $\{X_j\}$. Define $\mathbf{Q}^{\bullet}$ to be the object (in $D^b(X)$) obtained by Deligne's construction with respect to the canonical $\bar{p}$ filtration.

The proof of Theorem 4.1 will follow from three statements:

1. An object $\mathbf{A}^{\bullet}$ in $S\{X_j\}$ satisfies [AX2] if and only if it satisfies $[\mathrm{AX1}]_R$ with respect to the stratification $\{X_j\}$ (see Lemma 1 below).
2. $\mathbf{Q}^{\bullet}$ satisfies [AX2] (Lemma 2 below).
3. $\mathbf{Q}^{\bullet}$ is an object in $S\{X_j\}$ for any stratification $\{X_j\}$ of X (Proposition 4.2).

Statement (2) clearly guarantees the existence part of Theorem 4.1.

Uniqueness. Let $\mathbf{S}^{\bullet}$ be a complex of sheaves which satisfies [AX2]. By assumption $\mathbf{S}^{\bullet}$ is constructible with respect to some topological stratification $\{X_j\}$ of X. By statement (3), $\mathbf{Q}^{\bullet}$ is also an object in $S\{X_j\}$. By statement (1), both $\mathbf{S}^{\bullet}$ and $\mathbf{Q}^{\bullet}$ satisfy $[\mathrm{AX1}]_R$ with respect to this stratification, so by Theorem 3.5 and its corollary, $\mathbf{S}^{\bullet}$ and $\mathbf{Q}^{\bullet}$ are canonically isomorphic in $D^b(X)$.

Now let $\mathbf{A}^{\bullet}$ be the object obtained from Deligne's construction with respect to any other topological stratification of X. By Theorem 3.5, it satisfies $[\mathrm{AX1}]_R$ with respect to that stratification and by statement (1) it also satisfies axioms [AX2]. Thus it is canonically isomorphic to $\mathbf{Q}^{\bullet}$. Q.E.D.

Lemma 1. *Let $\{X_j\}$ be a topological stratification of X and suppose $\mathbf{S}^{\bullet}$ is an object in $S\{X_j\}$. Then $\mathbf{S}^{\bullet}$ satisfies $[\mathrm{AX1}]_R$ if and only if it satisfies* [AX2].

Proof. $[\mathrm{AX1}]_R$(a)(b) is equivalent to [AX2](a)(b) using the remark in §4.1. [AX1](c)$\Rightarrow$[AX2](c) as follows:

By constructibility the set $\{x \in X \mid H^m(j_x^* \mathbf{S}^{\bullet}) \neq 0\}$ is a union of strata. AX1(c) states that these strata may not include $X - X_{n-k}$ if $p(k)-n < m$. Thus, the only allowable strata are contained in $X - X_{n-k}$ for $p(k)-n \geqq m$, or $k \geqq p^{-1}p(k) \geqq p^{-1}(m+n)$. This set has dimension less than or equal to $n-k \leqq n - p^{-1}(m+n)$ which verifies AX2(c). The same calculation gives AX2(c) $\Rightarrow$AX1(c).

By § 3.4, [AX1](c)(d)$\Leftrightarrow$[AX1](d″) which is equivalent to [AX2](d) by a counting argument analogous to the one in the preceding paragraph.

Lemma 2. *Let* $\mathbf{Q}^\bullet$ *be the object obtained by applying Deligne's construction* (§ 2.1) *to the canonical* $\bar{p}$*-filtration* $\{X_i^{\bar{p}}\}$ *of* X. *Then* $\mathbf{Q}^\bullet$ *satisfies* [AX2].

Proof. $\mathbf{Q}^\bullet$ satisfies [AX2](a)(b)(c) by construction. It remains to verify axiom [AX2](d).

We will verify that

$$x \in X_l^{\bar{p}} - X_{l-1}^{\bar{p}} \Rightarrow H^m(j_x^! \mathbf{Q}^\bullet) = 0 \qquad \text{for all } m \leqq -q(n-l)-1.$$

This will suffice because if

$$(X_l^{\bar{p}} - X_{l-1}^{\bar{p}}) \cap \{x \mid H^m(j_x^! \mathbf{Q}^\bullet) \neq 0\} \neq \phi$$

then

$$m > -q(n-l)-1$$

so

$$q^{-1}(-m) \leqq q^{-1} q(n-l) \leqq n-l$$

or

$$l \leqq n - q^{-1}(-m).$$

Verification. Let j: $X_l^{\bar{p}} - X_{l-1}^{\bar{p}} \to U$ and i: $U - X_{l-1}^{\bar{p}} \to U$ be the inclusions, where $U = X - X_{l-1}^{\bar{p}}$.

Consider the long exact sequence on the stalk cohomology at x, which is associated to the distinguished triangle

$$\begin{array}{ccc} j_* j^! \mathbf{Q}^\bullet | U & \longrightarrow & \mathbf{Q}^\bullet | U \\ {}_{[1]}\nwarrow & & \swarrow \\ & R i_* i^* \mathbf{Q}^\bullet | U & \end{array}$$

since $\mathbf{Q}^\bullet | U \cong \tau_{\leqq p(n-l)-n} R i_* i^* \mathbf{Q}^\bullet | U$ we have

$$\mathbf{H}^m(j_* j^! \mathbf{Q}^\bullet)_x = 0 \qquad \text{for } m \leqq -n + p(n-l) + 1.$$

Now factor j_x into a composition

$$x \xrightarrow[u_x]{} X_l^{\bar{p}} - X_{l-1}^{\bar{p}} \xrightarrow[j]{} U = X - X_{l-1}^{\bar{p}}.$$

Then $j_x^! \mathbf{Q}^\bullet = u_x^! j^! \mathbf{Q}^\bullet = u_x^* j^! \mathbf{Q}^\bullet[l]$ since $X_l^{\bar{p}} - X_{l-1}^{\bar{p}}$ is a homology manifold. Thus the cohomology of this complex vanishes in dimensions $m \leqq l - n + p(n-l) + 1 = -q(n-l)+1$, as desired.

§ 5. Basic Properties of $IH^{\bar{p}}(X)$

In this chapter we prove the basic results of intersection homology without assuming X has a P.L. structure, using the methods of sheaf theory.

5.0. Throughout this chapter, X will denote an n-dimensional topological pseudomanifold, but we do not necessarily fix a stratification of X. Since we will be considering several perversities at once, we will denote the complex of intersection chains with perversity $\bar{p}$ by $\mathbf{IC}_{\bar{p}}^\bullet$.

Fix a regular Noetherian ring R of finite dimension. By *sheaf* we shall mean a sheaf of R-modules.

In some parts of this chapter we will assume that X has an R-orientation.

Definition. An R-orientation for X is a chosen quasi-isomorphism

$$\mathbb{D}^{\bullet}_{X-\Sigma} \to \mathbf{R}_{X-\Sigma}[n].$$

If $\mathrm{char}(R) \neq 2$ then an R-orientation of X is equivalent to an orientation of $X-\Sigma$ in the usual topological sense.

5.1. *The Maps from Cohomology and to Homology*

Choose an orientation on X.

Let $j\colon \Sigma \to X$ be the inclusion of the singularity set of X (for some topological stratification of X) and let $i\colon X-\Sigma = U \to X$ be the inclusion of the non-singular part.

Definition. The "cap product with the orientation class" is the morphism $\phi\colon \mathbf{R}_X[n] \to \mathbb{D}^{\bullet}_X$ which is obtained as the canonical lift (in $D^b X$) of the orientation:

$$\mathbf{R}_X[n] \to Ri_* \mathbf{R}_{X-\Sigma}[n] \overset{\cong}{\dashrightarrow} Ri_* \mathbb{D}^{\bullet}_{X-\Sigma}.$$

The lift exists and is unique in $D^b(X)$ because in the distinguished triangle,

$$\begin{array}{ccc} Rj_* \mathbb{D}^{\bullet}_{\Sigma}[1] & \xrightarrow{\quad [1] \quad} & \mathbb{D}^{\bullet}_X \\ & \nwarrow \qquad \swarrow & \end{array}$$

$$\mathbf{R}_X[n] \to Ri_* \mathbf{R}_{X-\Sigma} \to Ri_* \mathbf{R}_U[n]$$

the cohomology sheaves associated to $Rj_* \mathbb{D}^{\bullet}_{\Sigma}[1]$ vanish in dimensions $t \leqq -n$ (see § 1.15).

Proposition. *Suppose X is oriented. Let $\mathbf{IC}^{\bullet}$ denote the complex of intersection chains on X with respect to the perversity $\bar{p}$. Let $i\colon X-\Sigma \to X$ denote the inclusion. There are unique morphism in $D^b(X)$,*

$$\mathbf{R}_X[n] \to \mathbf{IC}^{\bullet} \to \mathbb{D}^{\bullet}_X$$

such that the induced morphism

$$i^* \mathbf{R}_X[n] \to i^* \mathbf{IC}^{\bullet}$$

is the evident one and the induced morphism

$$i^* \mathbf{IC}^{\bullet} \to i^* \mathbb{D}^{\bullet}_X$$

is given by the orientation. These morphism factor the cap product with the orientation.

Proof. Choose a stratification $\{X_k\}$ of X. With notation as in §3.1, suppose by induction that $\mathbb{P}_k^\bullet \to \mathbb{D}_{U_k}^\bullet$ has been constructed. We obtain a morphism

$$\mathbb{P}_{k+1}^\bullet \to Ri_{k*}\mathbb{P}_k^\bullet \to Ri_{k*}\mathbb{D}_{U_k}^\bullet$$

which has a unique lift to $\mathbb{D}_{U_{k+1}}^\bullet$ by §1.15 since the local cohomology sheaves associated to $Rj_{k*}\mathbb{D}_{X_{n-k}-X_{n-k-1}}^\bullet$ vanish in dimensions $t \leqq k-n-1$. Similarly a morphism $\mathbf{R}_{U_k} \to \mathbb{P}_k^\bullet$ (defined by induction) gives rise to a morphism

$$\mathbf{R}_{U_{k+1}} \to Ri_{k*}i_k^*\mathbf{R}_{U_k} \to Ri_{k*}\mathbb{P}_k^\bullet$$

which has a unique lift to $\tau_{\leqq p(k)-n}Ri_{k*}\mathbb{P}_k^\bullet$ by §1.15.

$\mathscr{L}$. For all stratified pseudomanifolds X, oriented or not, there is an orientation local system $\mathcal{O}$ on the nonsingular part $X-\Sigma$ of X

$$\mathcal{O} \cong \mathbf{H}^{-n}(\mathbb{D}^\bullet)\,|\,X-\Sigma.$$

(If X is oriented then $\mathcal{O} \cong \mathbf{R}_{(X-\Sigma)}$.) In general there are canonical morphisms

$$\mathbf{R}_X[n] \to \mathbf{IC}^\bullet \quad \text{and} \quad \mathbf{IC}^\bullet(\mathcal{O}) \to \mathbb{D}^\bullet.$$

5.2. *Construction of the Intersection Pairings*

Suppose $\bar{l}+\bar{m} \leqq \bar{p}$ are perversities. We shall define a product morphism

$$\mathbf{IC}_{\bar{l}}^\bullet \overset{L}{\otimes} \mathbf{IC}_{\bar{m}}^\bullet \to \mathbf{IC}_{\bar{p}}^\bullet[n].$$

The product is defined using a stratification, but turns out to be independent of the stratification. (In fact, one will obtain the same product morphism by following this construction, using the common refinement of the canonical $\bar{l}$, $\bar{m}$, $\bar{p}$ filtrations, in place of the stratification.)

For a stratification $\{X_k\}$ of X, let $\mathbf{L}^\bullet$, $\mathbf{M}^\bullet$, and $\mathbb{P}^\bullet$ denote the complex from §3.1 associated to the perversities $\bar{l}$, $\bar{m}$, and $\bar{p}$ respectively. Using notation as in §3.1, we shall define morphisms $\mathbf{L}_k^\bullet \overset{L}{\otimes} \mathbf{M}_k^\bullet \to \mathbb{P}_k^\bullet$ inductively over $U_k = X - X_{n-k}$ as follows:

On $U_2 = X - \Sigma$ the morphism is multiplication,

$$\mathbf{R}_{X-\Sigma}[n] \overset{L}{\otimes} \mathbf{R}_{X-\Sigma}[n] \to \mathbf{R}_{X-\Sigma}[n][n].$$

Now suppose μ_k: $\mathbf{L}_k^\bullet \otimes \mathbf{M}_k^\bullet \to \mathbb{P}_k^\bullet[n]$ has been constructed. We must define a morphism

$$(\tau_{\leqq l(k)-n}Ri_{k*}\mathbf{L}_k^\bullet) \overset{L}{\otimes} (\tau_{\leqq m(k)-n}Ri_{k*}\mathbf{M}_k^\bullet) \to \tau_{\leqq p(k)-n}Ri_{k*}\mathbb{P}_k^\bullet[n].$$

The pairing μ_k induces morphisms

$$\begin{aligned}(\tau_{\leqq l(k)-n}Ri_{k*}\mathbf{L}_k^\bullet) \overset{L}{\otimes} (\tau_{\leqq m(k)-n}Ri_{k*}\mathbf{M}_k^\bullet) &\to Ri_{k*}\mathbf{L}_k^\bullet \overset{L}{\otimes} Ri_{k*}\mathbf{M}_k^\bullet \\ &\to Ri_{k*}(\mathbf{L}_k^\bullet \overset{L}{\otimes} \mathbf{M}_k^\bullet) \\ &\to Ri_{k*}(\mathbb{P}_k^\bullet[n]).\end{aligned}$$

By §1.15, this composition has a canonical lift (in $D^b(U_{k+1})$) to $\tau_{\leq p(k)-n} Ri_{k*}\mathbb{P}_k^{\bullet}[n]$ since the cohomology sheaves associated to

$$(\tau_{\leq l(k)-n} Ri_{k*}\mathbf{L}_k^{\bullet}) \overset{L}{\otimes} (\tau_{\leq m(k)-n} Ri_{k*}\mathbf{M}_k^{\bullet})$$

vanish in dimensions $j \geq l(k)+m(k)-2n+1$.

Remark. The compatibility between the intersection pairings defined for different choices of $\bar{l}, \bar{m}$, and $\bar{p}$ is easily checked. In particular these products are compatible with the cup product $(\mathbf{R}_X \overset{L}{\otimes} \mathbf{R}_X \to \mathbf{R}_X)$ and the cap product $(\mathbf{R}_X \overset{L}{\otimes} \mathbb{D}_X^{\bullet} \to \mathbb{D}_X^{\bullet})$.

Corollary. *Let X be a topological pseudomanifold. If $\bar{l}+\bar{m} \leq \bar{p}$ there exist canonical "intersection" pairings*

$$IH_i^{\bar{l}}(X) \overset{L}{\otimes} IH_j^{\bar{m}}(X) \to IH_{i+j-n}^{\bar{p}}(X).$$

These pairings are compatible with the cup and cap products.

$\mathscr{L}$. *Remark.* It was not necessary to have an orientation of X in the preceding construction.

$\mathscr{L}$. We could also have started with local coefficient systems $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ on $X-\Sigma$ and a product $\mathbf{F}_1 \otimes \mathbf{F}_2 \to \mathbf{F}_3$. This gives rise to "intersection pairings"

$$IH_i^{\bar{l}}(X; \mathbf{F}_1) \overset{L}{\otimes} IH_j^{\bar{m}}(X; \mathbf{F}_2) \to IH_{i+j-n}^{\bar{p}}(X; \mathbf{F}_3).$$

5.3. $\mathbf{IC}^{\bullet}$ *and Verdier Duality*

In this paragraph we shall assume the coefficient ring R is a field, which we now denote by k. In this section (except for the last paragraph) we shall assume X is k-orientable and that a k orientation has been chosen.

One of the most important properties of $\mathbf{IC}^{\bullet}$ is the duality between $\mathbf{IC}_{\bar{p}}^{\bullet}$ and $\mathbf{IC}_{\bar{q}}^{\bullet}$ when $\bar{p}+\bar{q}=\bar{t}$. In particular if X has even codimension strata, $\mathbf{IC}_{\bar{m}}^{\bullet}$ is self dual (for example, if X is a complex analytic variety). For the reader who is only interested in the statement that the sheaf $\mathbb{P}^{\bullet}$ as constructed by Deligne satisfies this duality, the following rather simple proof can be extracted from Chap. 3: $\mathbb{P}^{\bullet}$ is characterized by [AX1] (with $\mathbf{F}=\mathbf{k}_X$) as shown in §3.5. We may replace axiom [AX1](d) with [AX1](d″) as shown in §3.4. This set of axioms is "self dual": [AX1](c) for a complex $\mathbf{A}^{\bullet}$ and a perversity $\bar{p}$ is equivalent to [AX1](d″) for the dual complex $\mathfrak{D}(\mathbf{A}^{\bullet})$ and perversity $\bar{q}=\bar{t}-\bar{p}$, and vice versa.

In the following detailed argument we will use the axioms [AX2] (which we believe are more natural) instead of [AX1].

Definition. A pairing $\mathbf{A}^{\bullet} \overset{L}{\otimes} \mathbf{B}^{\bullet} \to \mathbb{D}_X^{\bullet}[n]$ of objects in $D^b(X)$ (where $n=\dim(X)$) is called a *Verdier dual pairing* if it induces an isomorphism in $D^b X$,

$$\text{(§1.12)} \qquad \mathbf{A}^{\bullet} \xrightarrow{\cong} R\mathbf{Hom}^{\bullet}(\mathbf{B}^{\bullet}, \mathbb{D}_X^{\bullet})[n]$$

Theorem. *Suppose $\bar{p}+\bar{q}=\bar{t}$ are perversities. Then the intersection pairing of* § 5.2, *followed by the map to homology*

$$\mathbf{IC}^{\bullet}_{\bar{p}} \overset{L}{\otimes} \mathbf{IC}^{\bullet}_{\bar{q}} \to \mathbf{IC}^{\bullet}_{\bar{t}}[n] \to \mathbb{D}^{\bullet}_X[n]$$

is a Verdier dual pairing.

Corollary. *If X is compact, the pairings*

$$IH^{\bar{p}}_{*}(X;k) \otimes IH^{\bar{q}}_{*}(X;k) \to H_{*}(X;k) \to k$$

induce isomorphisms

$$IH^{\bar{p}}_i(X;k) \cong \mathrm{Hom}(IH^{\bar{q}}_{n-i}(X;k),k) \qquad ([6]).$$

Remark. Field coefficients are used in an essential way during the following proof of Theorem 5.3. The dualizing complex of a point $\{x\}$ with coefficients in k is

$$\mathbb{D}^m_{\{x\}} = \begin{cases} k & \text{for } m=0 \\ 0 & \text{for } m \neq 0 \end{cases}$$

and this complex is injective as a complex of k-modules.

Proof of Theorem 5.3. Let $\mathbf{S}^{\bullet} = R\mathbf{Hom}^{\bullet}(\mathbf{IC}^{\bullet}_{\bar{q}}, \mathbb{D}^{\bullet}_X)[n]$. The intersection pairing induces an isomorphism

$$\mathbf{IC}^{\bullet}_{\bar{p}}|U = \mathbf{k}_U[n] \xrightarrow{\cong} \mathbf{S}^{\bullet}|U$$

where $U = X - \Sigma$ is the nonsingular set. We must check that this isomorphism extends to a quasi-isomorphism over the rest of X. It suffices to check that $\mathbf{S}^{\bullet}$ satisfies the axioms AX2(c) and (d).

Let $j_x\colon \{x\} \to X$ denote the inclusion of a point. Then

$$\begin{aligned} j^*_x \mathbf{S}^{\bullet} &= j^*_x \mathfrak{D}(\mathbf{IC}^{\bullet}_{\bar{q}})[n] \\ &= j^!_x(\mathfrak{D}(\mathfrak{D}\mathbf{IC}^{\bullet}_{\bar{q}})[n] \\ &\cong \mathrm{Hom}(j^!_x \mathbf{IC}^{\bullet}_{\bar{q}}, k)[n]. \end{aligned}$$

Therefore $H^m(j^*_x \mathbf{S}^{\bullet}) = \mathrm{Hom}(H^{-m-n}(j^!_x(\mathbf{IC}^{\bullet}_{\bar{q}}), k)$.

Since $\mathbf{IC}^{\bullet}_{\bar{q}}$ satisfies AX2(d), the set of points $x \in X$ for which this group is nonzero, has dimension $\leq n - p^{-1}(m+n)$, which verifies AX2(c).

Similarly, $H^m(j^!_x \mathbf{S}^{\bullet}) \cong \mathrm{Hom}(H^{-m-n} j^*_x \mathbf{IC}^{\bullet}_{\bar{q}}, k)$. The set of points $x \in X$ for which this group is nonzero has dimension $\leq n - q^{-1}(m)$ which verifies AX2(d).

$\mathscr{L}$. We now drop the assumption that X is oriented and we let $\mathcal{O}$ be the orientation local system of k modules on $X - \Sigma$ considered in § 5.1 $\mathscr{L}$. A pairing $\mathbf{F}_1 \otimes \mathbf{F}_2 \to \mathcal{O}$ of local systems on $X - \Sigma$ is called perfect if the induced mapping $\mathbf{F}_1 \to \mathrm{Hom}(\mathbf{F}_2, \mathcal{O})$ is an isomorphism. The above proof gives the

Theorem. *Suppose $\bar{p}+\bar{q}=\bar{t}$ are perversities and the pairing $\mathbf{F}_1 \otimes \mathbf{F}_2 \to \mathcal{O}$ is perfect. Then the intersection pairing followed by the map to homology*

$$\mathbf{IC}^{\bullet}_{\bar{p}}(\mathbf{F}_1) \overset{L}{\otimes} \mathbf{IC}_{\bar{q}}(\mathbf{F}_2) \to \mathbf{IC}^{\bullet}_{\bar{t}}(\mathcal{O})[u] \to \mathbb{D}^{\bullet}_X[u]$$

is a Verdier dual pairing.

5.4. *Functoriality for Normally Nonsingular Maps*

5.4.1. *Normally Nonsingular Inclusions* [15, 18].

Definition. An inclusion of oriented topological pseudomanifolds α: $Y \to X$ is said to be *normally nonsingular* with codimension c, if Y has a c-dimensional tubular neighborhood in X, i.e., an open neighborhood $N \subset X$ and a retraction π: $N \to Y$ such that (π, N, Y) is homeomorphic to an $\mathbb{R}^c$-vectorbundle over Y (where Y is identified with the 0-section).

For example, suppose X is a Whitney stratified subset of some manifold M, and $Y = H \cap X$ where H is a smooth submanifold of M which is transverse to each stratum of X. Then the inclusion $Y \to X$ is normally nonsingular with codimension $c = \dim(M) - \dim(H)$.

Theorem. *Suppose* α: $Y \to X$ *is a normally nonsingular inclusion with codimension* c. *Fix a perversity* $\bar{p}$ *and let* $\mathbf{IC}^{\bullet}_X$ *and* $\mathbf{IC}^{\bullet}_Y$ *denote the intersection homology complex on* X *and* Y *respectively. Then there are canonical isomorphisms* $\alpha^* \mathbf{IC}^{\bullet}_X \cong \mathbf{IC}^{\bullet}_Y[c]$ *and* $\alpha^! \mathbf{IC}^{\bullet}_X \cong \mathbf{IC}^{\bullet}_Y$.

Proof. Let π: $N \to Y$ denote the tubular neighborhood of Y in X and suppose $\dim(X) = n$. From the topological invariance of $\mathbf{IC}^{\bullet}_X$ we have a quasi-isomorphism $\mathbf{IC}^{\bullet}_X | N \cong \pi^* \alpha^* \mathbf{IC}^{\bullet}_X$.

We shall now check the axioms [AX2] for the complex $\alpha^* \mathbf{IC}^{\bullet}_X[-c]$ on Y. To verify AX2(c) we must find

$$\begin{aligned}\beta &= \dim\{y \in Y \mid H^m(j_y^* \alpha^* \mathbf{IC}^{\bullet}_X[-c]) \neq 0\}\\ &= \dim\{y \in Y \mid H^{m-c}(j_y^* \alpha^* \mathbf{IC}^{\bullet}_X) \neq 0\}\end{aligned}$$

where j_y: $\{y\} \to Y$ is the inclusion of a point. Suppose $x \in N$ and $\pi(x) = y$. Then

$$j_x^* \mathbf{IC}^{\bullet}_X \cong j_x^* \pi^* (\alpha^* \mathbf{IC}^{\bullet}_X) \cong (\pi j_x)^* (\alpha^* \mathbf{IC}^{\bullet}_X) \cong j_y^* \alpha^* \mathbf{IC}^{\bullet}_X.$$

Consequently,

$$\begin{aligned}n - p^{-1}(m - c + n) &\geqq \dim\{x \in X \mid H^{m-c} j_x^* \mathbf{IC}^{\bullet}_X \neq 0\}\\ &\geqq \dim\{y \in Y \mid H^{m-c} j_y^* \alpha^* \mathbf{IC}^{\bullet}_X \neq 0\} + c\end{aligned}$$

which shows $\beta \leqq n - c - p^{-1}(m - c + n)$ as desired.

To verify AX2(d) we must calculate

$$\begin{aligned}\gamma &= \dim\{y \in Y \mid H^m(j_y^! \alpha^* \mathbf{IC}^{\bullet}_X[-c]) \neq 0\}\\ &= \dim\{y \in Y \mid H^{m-c}(j_y^! \alpha^* \mathbf{IC}^{\bullet}_X[-c]) \neq 0\}.\end{aligned}$$

If $\pi(x) = y$, then

$$j_x^! \mathbf{IC}^{\bullet}_X \cong j_x^! \pi^* \alpha^* \mathbf{IC}^{\bullet}_X \cong j_x^! \pi^! \alpha^* \mathbf{IC}^{\bullet}_X[-c] \cong j_y^! \alpha^* \mathbf{IC}^{\bullet}_X[-c].$$

Therefore $H^m(j_x^! \mathbf{IC}^{\bullet}_X) \cong H^{m-c}(j_y^! \alpha^* \mathbf{IC}^{\bullet}_X)$ and

$$\begin{aligned}n - q^{-1}(-m) &\geqq \dim\{x \in X \mid H^m(j_x^! \mathbf{IC}^{\bullet}_X) \neq 0\}\\ &\geqq \dim\{y \in Y \mid H^{m-c}(j_y^! \alpha^* \mathbf{IC}^{\bullet}_X) \neq 0\} + c\end{aligned}$$

which shows $\gamma \leqq n - c - q^{-1}(-m)$ as desired.

The remaining axioms in [AX2] may be easily verified.

Part (2) of the proposition follows because

$$\begin{aligned}\alpha^!\mathbf{IC}_X^\bullet &\cong \mathfrak{D}_Y(\alpha^*\mathfrak{D}_X(\mathbf{IC}_X^\bullet))\\ &\cong \mathfrak{D}_Y(\alpha^*\mathfrak{D}_N(\mathbf{IC}_X^\bullet|N))\\ &\cong \mathfrak{D}_Y(\alpha^* R\mathbf{Hom}^\bullet(\pi^*\alpha^*\mathbf{IC}_X^\bullet, \pi^*\alpha^*\mathbb{D}_X^\bullet))\\ &\cong \mathfrak{D}_Y(\alpha^*\pi^* R\mathbf{Hom}^\bullet(\alpha^*\mathbf{IC}_X^\bullet, \alpha^*\mathbb{D}_X^\bullet))\\ &\cong \mathfrak{D}_Y(R\mathbf{Hom}^\bullet(\alpha^*\mathbf{IC}_X^\bullet, \alpha^!\mathbb{D}_X^\bullet)[c])\\ &\cong \mathfrak{D}_Y(\mathfrak{D}_Y(\alpha^*\mathbf{IC}_X^\bullet))[-c]\\ &\cong \alpha^*\mathbf{IC}_X^\bullet[-c] = \mathbf{IC}_Y^\bullet\end{aligned}$$

5.4.2. *Normally Nonsingular Projections*

Definition. An oriented topological fibre bundle $\pi\colon Y \to X$ is normally nonsingular with codimension $(-c)$ if the fibre $\pi^{-1}(x)$ is a topological manifold of dimension c.

Theorem. *Let $\pi\colon Y \to X$ be a normally nonsingular fibration with codimension $-c$. Fix a perversity $\bar{p}$ and let $\mathbf{IC}_X^\bullet$ and $\mathbf{IC}_Y^\bullet$ denote the intersection homology complexes on X and Y. Then $\pi^*\mathbf{IC}_X^\bullet \cong \mathbf{IC}_Y^\bullet[-c]$ and $\pi^!\mathbf{IC}_X^\bullet = \mathbf{IC}_Y^\bullet$.*

The proofs are similar to those in § 5.4.1.

5.4.3. *Normally Nonsingular Maps*

Definition. A normally nonsingular map $f\colon Y \to X$ between oriented topological pseudomanifolds, is one which can be factored as a composition of a normally nonsingular inclusion, followed by a normally nonsingular fibration. The relative dimension of f is defined to be the sum of the codimensions of the two factors. Topological pseudomanifolds and normally nonsingular maps form a category (see [15]).

Definition. Let $f\colon Y \to X$ be a proper normally nonsingular map of relative dimension c. Then the induced homomorphisms

$$f_*\colon IH_k^{\bar{p}}(Y) \to IH_k^{\bar{p}}(X)$$

and

$$f^*\colon IH_k^{\bar{p}}(X) \to IH_{k-c}^{\bar{p}}(Y)$$

are constructed as follows.

Consider the canonical "adjunction morphisms"

$$Rf_! f^! \mathbf{IC}_X^\bullet \to \mathbf{IC}_X^\bullet \quad \text{and} \quad \mathbf{IC}_X^\bullet \to Rf_* f^* \mathbf{IC}_X^\bullet.$$

By Theorems 5.4.1 and 5.4.2 these become morphisms

$$Rf_! \mathbf{IC}_Y^\bullet \to \mathbf{IC}_X^\bullet \quad \text{and} \quad \mathbf{IC}_X^\bullet \to Rf_* \mathbf{IC}_Y^\bullet[c]$$

Since f is proper, $Rf_! = Rf_*$. Taking hypercohomology gives homomorphisms

$$IH_k^{\bar{p}}(Y) = \mathscr{H}^{-k}(X; Rf_* \mathbf{IC}_Y^{\bullet}) \to \mathscr{H}^{-k}(X; \mathbf{IC}_X^{\bullet}) = IH_k^{\bar{p}}(X)$$

and

$$IH_k^{\bar{p}}(X) = \mathscr{H}^{-k}(X; \mathbf{IC}_X^{\bullet}) \to \mathscr{H}^{-k}(X; Rf_* \mathbf{IC}_Y^{\bullet}[c]) = IH_{k-c}^{\bar{p}}(Y)$$

Proposition. *$IH_k^{\bar{p}}$ is both a covariant functor (via f_*) and a contravariant functor (via f^*) on the category of topological pseudomanifolds and normally nonsingular maps.*

$\mathscr{L}$. If f: $Y \to X$ is a normally nonsingular map of topological pseudomanifolds, then X and Y can be stratified so that the inverse image of the largest stratum $X - \Sigma$ of X is the largest stratum of Y. If c is the relative dimension of f, then for any local system $\mathbf{F}$ on $X - \Sigma$,

$$\pi^* \mathbf{IC}_X^{\bullet}(\mathbf{F}) \cong \mathbf{IC}_Y^{\bullet}(\pi^* \mathbf{F})[-c]$$

and

$$\pi^! \mathbf{IC}_X^{\bullet}(\mathbf{F}) \cong \mathbf{IC}_Y^{\bullet}(\pi^* \mathbf{F} \otimes \mathbf{Hom}(\pi^* \mathcal{O}_X, \mathcal{O}_Y)).$$

5.5. *The Obstruction Sequence for Comparing two Perversities*

It is clear from Deligne's construction that whenever $\bar{p} \leqq \bar{q}$ are perversities, there is a canonical morphism $\mathbf{IC}_{\bar{p}}^{\bullet} \to \mathbf{IC}_{\bar{q}}^{\bullet}$. Thus we obtain a distinguished triangle

$$\begin{array}{ccc} \mathbf{IC}_{\bar{p}}^{\bullet} & \longrightarrow & \mathbf{IC}_{\bar{q}}^{\bullet} \\ {\scriptstyle [1]}\nwarrow & & \swarrow \\ & \mathbf{S}^{\bullet} & \end{array}$$

and a long exact sequence on hypercohomology,

$$\to IH_i^{\bar{p}}(X) \to IH_i^{\bar{q}}(X) \to \mathscr{H}^{-i}(X; \mathbf{S}^{\bullet}) \to IH_{i-1}^{\bar{p}}(X) \to \ldots$$

which is called the obstruction sequence because $\mathscr{H}^{-i}(X; \mathbf{S}^{\bullet})$ is the obstruction to lifting a class from $IH_i^{\bar{q}}(X)$ to $IH_i^{\bar{p}}(X)$.

Now fix a stratification $\{X_k\}$ of X.

Proposition. *Suppose $p(c) = q(c)$ for all $c \neq k$, and $q(k) = p(k) + 1$. Then:*

(1) *$spt\, \mathbf{H}^*(\mathbf{S}^{\bullet}) = closure(spt\, \mathbf{H}^*(\mathbf{S}^{\bullet}) \cap (X_{n-k} - X_{n-k-1}))$ where spt denotes the support of a sheaf.*

(2) *If $x \in X_{n-k} - X_{n-k-1}$ then*

$$\begin{array}{lll} \mathbf{H}^i(\mathbf{S}^{\bullet})_x = 0 & \text{for all} & i \neq q(k) - n, \\ \mathbf{H}^i(\mathbf{S}^{\bullet})_x \cong \mathbf{H}^i(\mathbf{IC}_{\bar{q}}^{\bullet})_x & \text{if} & i = q(k) - n. \end{array}$$

(3) *If $\mathbf{H}^{q(k)-n}(\mathbf{IC}_{\bar{q}}^{\bullet})_x = 0$ for all $x \in X_{n-k} - X_{n-k-1}$ then $\mathbf{IC}_{\bar{p}}^{\bullet} \to \mathbf{IC}_{\bar{q}}^{\bullet}$ is a quasi isomorphism.*

The proof follows directly from Deligne's construction.

$\mathscr{L}$. The results of this section hold equally well when $\mathbf{IC}^{\bullet}_{\bar p}$ is replaced by $\mathbf{IC}^{\bullet}_{\bar p}(F)$ and $\mathbf{IC}^{\bullet}_{\bar q}$ is replaced by $\mathbf{IC}^{\bullet}_{\bar q}(F)$.

5.6. *Normal Varieties, Local Complete Intersection, and Witt Spaces*

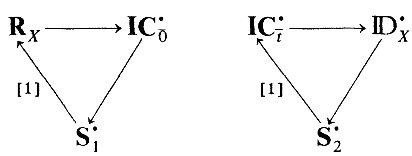
5.6.1. *Witt Spaces.* Let k be a field and let X be a n-dimensional stratified piecewise linear pseudomanifold [20].

Definition. X is a k-Witt space if $IH^{\bar m}_l(L_x;k)=0$ for all $x\in X_{n-2l-1}-X_{n-2l-2}$. Here, L_x is the link of the stratum containing x, and $\bar m=(0,0,1,1,2,2,\ldots)$.

Proposition (P. Siegel [35]). *X is a k-Witt space if and only if the canonical morphism $\mathbf{IC}^{\bullet}_{\bar m}\to\mathbf{IC}^{\bullet}_{\bar n}$ is a quasi isomorphism. Thus, if X is a Witt space $IH^{\bar m}_n(X;k)$ is self dual.*

The proof follows from the identification (§ 2.2) of the stalk $H^{\bar n(2l+1)-n}(IC^{\bullet}_{\bar n})_x$ with $IH_l(L_x)$ for any $x\in X_{n-2l-1}-X_{n-2l-2}$. Then apply Proposition 5.5(2).

5.6.2. We also have obstruction groups relating perversities $\bar 0$ and $\bar t$ to cohomology and homology (if X is oriented),



The cohomology sheaf $\mathbf{H}^i(\mathbf{S}^{\bullet}_1)$ vanishes except for $i=0$ and $\mathbf{H}^i(\mathbf{S}_2)$ vanishes except for $i=-n$. For a point $p\in X$ the stalks of $\mathbf{H}^0(\mathbf{S}^{\bullet}_1)$ and $\mathbf{H}^{-n}(\mathbf{S}^{\bullet}_2)$ at p are both free R modules of rank $r-1$ where r is the local Betti number, rank $H_n(X,X-p;R)$, at p. The number r has two other interpretations: If $X=X_n\supset X_{n-1}\supset\ldots$ is any stratification of X and N is a distinguished neighborhood of p (see § 1.1), then r is the number of connected components of $(X-X_{n-2})\cap N$. If X is a complex analytic variety, then r is the number of analytic branches at p.

Definition. A *normal* topological pseudomanifold X of dimension n is one such that rank $H_n(X,X-p;R)=1$ for all $p\in X$.

By Zariski's main theorem, a normal complex algebraic variety is normal as a topological pseudomanifold. By the above remarks, we have the

Proposition. *For a normal oriented n-dimensional topological pseudomanifold X, we have*

$$\mathbf{R}_X\cong\mathbf{IC}^{\bullet}_{\bar 0}\quad\text{and}\quad\mathbf{IC}^{\bullet}_{\bar t}\cong\mathbb{D}^{\bullet}_X$$

so

$$H^i(X)\cong IH^{\bar 0}_{n-i}(X)\quad\textit{and}\quad IH^{\bar t}_i(X)\cong H_i(X).$$

5.6.3. *Local Complete Intersections*

Proposition. *Let Y be a compact complex algebraic variety which is normal and is a local complete intersection. Let $\bar{p}$ be a perversity such that $p(k) \geqq \frac{k}{2}$ for each $k \geqq 4$. Then for all i we have $IH_i^{\bar{p}}(Y) \cong H_i(Y)$.*

Proof. Let us say that an n-dimensional triangulable space X is a space of type Q if it is a normal pseudomanifold, has a stratification by even codimension and orientable strata, and if for each $x \in X_{n-c} - X_{n-c-1}$ the local homology groups $H_i(X, X-x)$ vanish for all $i \leqq n-1-\frac{c}{2}$. Hamm [22] shows that a normal local complete intersection is a space of type Q, and we shall now show that the conclusion of the proposition holds for any n-dimensional space X of type Q.

For each $x \in X_{n-c} - X_{n-c-1}$ the link L_x is an $n-1$ dimensional space of type Q so by induction, the proposition applies to L_x. Thus,

$$IH_i^{\bar{p}}(X, X-x) \cong IH_{i-1}^{\bar{p}}(L_x) \cong H_{i-1}(L_x) \cong H_i(X, X-x) = 0$$

provided $i \leqq n-1-\frac{c}{2}$ and $p(k) \geqq \frac{k}{2}$ for all k.

Applying Proposition 5.5 to any string of perversities between $\bar{p}$ and $\bar{t}$ (where $t(c) = (c-2)$) we conclude that for all i,

$$IH_i^{\bar{p}}(X) \cong IH_i^{\bar{t}}(X) \cong H_i(X)$$

since X is normal.

§ 6. The Middle Group

6.0. In this chapter, X will denote an *oriented* topological pseudomanifold, except in the paragraph marked $\mathscr{L}$ but we do not fix any particular stratification of X. Except in § 6.2 we will assume the ring R is a field, which we now denote by k. We shall consider the middle perversities $\bar{m} = (0, 0, 1, 1, 2, 2, \ldots)$ and $\bar{n} = (0, 1, 1, 2, 2, 3, \ldots)$ with their corresponding complexes $\mathbf{IC}^{\cdot}_{\bar{m}}$ and $\mathbf{IC}^{\cdot}_{\bar{n}}$ in the derived category of the category of sheaves of k-vectorspaces.

In § 6.2 and § 6.3, X will be a k-Witt space, so $\mathbf{IC}^{\cdot}_{\bar{m}} = \mathbf{IC}^{\cdot}_{\bar{n}}$ which will be denoted simply $\mathbf{IC}^{\cdot}$ or $\mathbf{IC}^{\cdot}_X$.

6.1. *Axioms* [AX3]

Definition. Let $\mathbf{S}^{\cdot}$ be a topologically constructible complex of sheaves (of k-modules) on X. We shall say $\mathbf{S}^{\cdot}$ satisfies axioms [AX3] provided:

(a) Normalization
There is a closed subspace $Z \subset X$ such that $\mathbf{S}^{\cdot}|(X-Z) \cong \mathbf{k}_{X-Z}[n]$ and $\dim(Z) \leqq n-2$.

(b) Lower bound

$\mathbf{H}^m(\mathbf{S}^\cdot)=0 \quad$ for all $m<-n$.

(c) Support condition

$\dim\{x\in X \mid \mathbf{H}^i(\mathbf{S}^\cdot)_x \neq 0\} \leqq -2i-n-2 \quad$ for $i\geqq -n+1$.

(d) Duality

There is an isomorphism $\mathbf{S}^\cdot \cong \mathfrak{D}(\mathbf{S}^\cdot)[n]$.

Theorem. *If the complex* $\mathbf{S}^\cdot$ *satisfies axioms* [AX3] *then there is a natural isomorphism in* $D^b(X)$, $\mathbf{S}^\cdot \cong \mathbf{IC}^\cdot_{\bar{n}}$. *It follows that* $\mathbf{IC}^\cdot_{\bar{m}} \cong \mathbf{IC}^\cdot_{\bar{n}}$ *so* X *is a k-Witt space.*

Proof. We shall show $\mathbf{S}^\cdot \cong \mathbf{IC}^\cdot_{\bar{m}}$ by verifying the axioms [AX2]. Note that $m^{-1}(c)=2c+2$.

Axioms AX2(a)(b)(c) are obviously satisfied. For any $x\in X$ we have

$$\begin{aligned} j_x^! \mathbf{S}^\cdot &\cong R\,\mathbf{Hom}^\cdot(j_x^* R\,\mathbf{Hom}^\cdot(\mathbf{S}^\cdot, \mathbb{D}^\cdot_X), \mathbb{D}^\cdot_{\{x\}}) \\ &\cong \mathrm{Hom}(j_x^* \mathbf{S}^\cdot[-n], k) \\ &\cong \mathrm{Hom}(j_x^* \mathbf{S}^\cdot, k)[n] \end{aligned}$$

so $H^i(j_x^! \mathbf{S}^\cdot) \cong \mathrm{Hom}(H^{-n-i}(j_x^* \mathbf{S}^\cdot), k)$.

The set of points for which this does not vanish has (by AX3(c)) dimension $\leqq n+2i-2 \leqq n+2i-1 = n-p^{-1}(-i)$ where $\bar{p}$ is the perversity complementary to $\bar{m}$. This verifies axiom AX2(d), so $\mathbf{S}^\cdot = \mathbf{IC}^\cdot_{\bar{m}}$. By duality we also obtain $\mathbf{S}^\cdot = \mathbf{IC}^\cdot_{\bar{n}} \cong \mathbf{IC}^\cdot_{\bar{m}}$.

$\mathscr{L}$. **Definition.** A complex of sheaves $\mathbf{S}^\cdot$ is a *middle intersection homology sheaf* if for some stratification of X and for some local coefficient system $\mathbf{F}$ on $X-\Sigma$,

$$\mathbf{S}^\cdot = \mathbf{IC}^\cdot_{\bar{m}}(\mathbf{F}) = \mathbf{IC}^\cdot_{\bar{n}}(\mathbf{F}).$$

Theorem. *A complex of sheaves* $\mathbf{S}^\cdot$ *is a middle intersection homology sheaf if and only if both* $\mathbf{S}^\cdot$ *and* $\mathfrak{D}(\mathbf{S}^\cdot)$ *satisfy* [AX3](b) (*lower bound*) *and* [AX3](c) (*support condition*).

In this case, $\mathfrak{D}(\mathbf{S}^\cdot) = \mathbf{IC}^\cdot_{\bar{m}}(\mathbf{Hom}(\mathbf{F}, \mathcal{O}))$.

6.2. *Small Maps and Resolutions*

In contrast to the rest of this chapter, the results in this section are valid over an arbitrary finite dimensional regular Noetherian ring R of coefficients.

Definition. A proper surjective algebraic map $f\colon Y\to X$ between irreducible complex n-dimensional algebraic varieties is *homologically small* if for all $q>-2n$.

$$\mathrm{cod}_{\mathbb{C}}\{x\in X \mid \mathbf{H}^q(Rf_* \mathbf{IC}^\cdot_Y)_x \neq 0\} > q+2n.$$

It follows that there is a Zariski open set $U\subset X$ such that $f|f^{-1}(U)$ is a finite covering projection. The above criterion is satisfied, for example, if Y is the normalization of X or if f is a *small map*, i.e., if Y is nonsingular and for all $r>0$,

$$\operatorname{cod}_{\mathbb{C}}\{x\in X \mid \dim_{\mathbb{C}} f^{-1}(x)\geqq r\} > 2r.$$

Examples of Small Maps. If X is one or two dimensional than a small map $f\colon Y\to X$ must be a finite map. If X is a threefold then the fibres of a small map f must be zero dimensional except possibly over a set of isolated points in X where the fibres may be at most curves.

Theorem. *Let $f\colon Y\to X$ be a homologically small map of degree 1. Then $Rf_*\mathbf{IC}_Y^{\bullet}\cong\mathbf{IC}_X^{\bullet}$ and in particular $\mathbf{IH}_*^{\bar m}(Y)\cong\mathbf{IH}_*^{\bar m}(X)$.*

Proof. The complex $Rf_*\mathbf{IC}_Y^{\bullet}$ satisfies the criteria (AX2) of Theorem 4.1.

Definition. A *small resolution* $f\colon Y\to X$ is a resolution of singularities which is a small map.

Corollary. *If $f\colon Y\to X$ is a small resolution then the intersection homology groups of X equal the (ordinary) homology groups of Y, and $Rf_*\mathbf{R}_Y\cong\mathbf{IC}_X^{\bullet}$.*

Remark. Small resolutions do not always exist, and are not necessarily unique when they do exist. However, if X has several small resolutions, their cohomologies are isomorphic as groups, but not as rings. Two small resolutions must have the same signature and Euler characteristic.

$\mathscr{L}$. **Theorem.** *If $f\colon Y\to X$ is homologically small, then*

$$Rf_*\mathbf{IC}_Y^{\bullet}=\mathbf{IC}_X^{\bullet}(R^0(f|U)_*\mathbf{R}_{f^{-1}(U)}),$$

where U is the Zariski open set over which f is a covering projection.

6.3. *Kunneth Formula*

For $k=\mathbb{R}$ the following proposition was first proved by Cheeger using L^2 cohomology [10].

Proposition. *Suppose X and Y are Witt spaces. Then*

$$IH_l^{\bar m}(X\times Y)=\bigoplus_{a+b=l} IH_a^{\bar m}(X)\otimes IH_b^{\bar m}(Y).$$

In particular, the signature of $X\times Y$ is the product of the signatures of X and Y.

Proof. Let $\pi_1\colon X\times Y\to X$ and $\pi_2\colon X\times Y\to Y$ be the projections. It suffices to prove that $\mathbf{IC}_{X\times Y}^{\bullet}\cong\pi_1^*\mathbf{IC}_X^{\bullet}\overset{L}{\otimes}\pi_2^*\mathbf{IC}_Y^{\bullet}$, which is done by verifying the axioms. The support condition is easy to verify, and duality holds because

$$\begin{aligned}
&R\,\mathbf{Hom}^{\bullet}(\pi_1^*\mathbf{IC}_X^{\bullet}\overset{L}{\otimes}\pi_2^*\mathbf{IC}_Y^{\bullet},\mathbb{D}_{X\times Y}^{\bullet})\\
&\quad\cong R\,\mathbf{Hom}^{\bullet}(\pi_1^*\mathbf{IC}_X^{\bullet}\overset{L}{\otimes}\pi_2^*\mathbf{IC}_Y^{\bullet},\pi_1^*\mathbb{D}_X^{\bullet}\overset{L}{\otimes}\pi_2^*\mathbb{D}_Y^{\bullet})\\
&\quad\cong R\,\mathbf{Hom}^{\bullet}(\pi_1^*\mathbf{IC}_X^{\bullet},\pi_1^*\mathbb{D}_X^{\bullet})\overset{L}{\otimes}R\,\mathbf{Hom}^{\bullet}(\pi_2^*\mathbf{IC}_Y^{\bullet},\pi_2^*\mathbb{D}_Y^{\bullet})\\
&\quad\cong\pi_1^*\mathbf{IC}_X^{\bullet}\overset{L}{\otimes}\pi_2^*\mathbf{IC}_Y^{\bullet}[-n-m]
\end{aligned}$$

where $n=\dim(X)$ and $m=\dim(Y)$.

$\mathscr{L}$. Similarly, the characterization of middle intersection homology sheaves in § 6.1 $\mathscr{L}$ can be used to show

$$\mathbf{IC}^{\bullet}_{X\times Y}(\pi_1^*\mathbf{F}_1 \otimes \pi_2^*\mathbf{F}_2) \cong \pi_1^*\mathbf{IC}^{\bullet}_X(\mathbf{F}_1) \overset{L}{\otimes} \pi_2^*\mathbf{IC}^{\bullet}_Y(\mathbf{F}_2)$$

so

$$IH_l^{\bar{m}}(X\times Y; \pi_1^*\mathbf{F}_1 \otimes \pi_1^*\mathbf{F}_2) \cong \bigoplus_{a+b=l} IH_a^{\bar{m}}(X; \mathbf{F}_1) \otimes IH_b^{\bar{m}}(Y; \mathbf{F}_2).$$

§ 7. The Lefschetz Theorem on Hyperplane Sections

The purpose of this chapter is to show that the classical theorem of Lefschetz (on the homology of a hyperplane section of a nonsingular projective variety) continues to hold in the singular case provided we replace homology by intersection homology. This holds for a range of perversities which include the middle perversity $\bar{m}$ and the logarithmic perversity $\bar{l}$. In § 7.4 we deduce as corollaries of this theorem some results about ordinary homology and cohomology.

Our original proof of this theorem proceeded by replacing Thom's Morse-theoretic argument in the nonsingular case ([2, 31]) with a stratified Morse theoretic argument in the singular case (using the techniques of [21]). The proof we present here was pointed out to us by Deligne (who had also observed that the theorem is true). It is essentially the same as the proof in the theorem of Artin [1]. We have also made use of some ideas of K. Vilonen [46].

7.0 Throughout this chapter, X will denote a complex projective n dimensional variety with its canonical orientation, and all homology groups will be understood to take coefficients in a finite dimensional regular Noetherian ring R.

7.1. **Theorem.** *Suppose X is a complex n-dimensional algebraic variety embedded in complex projective space. Fix a perversity $\bar{p}$ such that $p(c) \leqq c/2$. Let $Y = H \cap X$ where H is a hyperplane which is transverse to each stratum of a Whitney stratification of X. Then the normally nonsingular inclusion $\alpha\colon Y \to X$ induces isomorphisms $\alpha_*\colon IH_k^{\bar{p}}(Y) \xrightarrow{\cong} IH_k^{\bar{p}}(X)$ for all $k < n-1$ and a surjection $\alpha_*\colon IH_{n-1}^{\bar{p}}(Y) \to IH_{n-1}^{\bar{p}}(X)$.*

7.2. *Intersection Homology of Affine Varieties*

This section contains the technical tools needed in the proof of the Lefschetz theorem.

Lemma. *Suppose* **A** *is an algebraically constructible sheaf on C. Then $H^2(\mathbb{C}, \mathbf{A}) = 0$.*

Proof. **A** is locally trivial except on a finite set of points ([8] Exp. 7, 8). Let K be the union of the line segments joining the origin to each of these points. Then $H^*(K, \mathbf{A}|K) \cong H^*(\mathbb{C}, \mathbf{A})$. But K is one dimensional.

Proposition. *Let* $\mathbf{S}^\bullet$ *be a complex of algebraically constructible sheaves on* $\mathbb{C}^n$, *which satisfies a support condition*

$$\dim_{\mathbb{C}}\{x\in\mathbb{C}^n \mid \mathbf{H}^m(\mathbf{S}^\bullet)_x \neq 0\} \leqq f(m)$$

where $f\colon \mathbb{Z}\to\mathbb{Z}$ *is a nonincreasing integer valued function. Then* $\mathscr{H}^j(\mathbb{C}^n;\mathbf{S}^\bullet)=0$ *for all* $j>\max\{m+f(m)\mid f(m)\geqq 0\}$. *The same holds if we replace* $\mathbb{C}^n$ *by an arbitrary affine algebraic variety.*

Proof. If $n=1$ the proposition follows from studying the spectral sequence associated to $\mathbf{S}^\bullet$ with E^2 term

$$E^2_{rs}=H^r(\mathbb{C};\mathbf{H}^s(\mathbf{S}^\bullet)) \Rightarrow \mathscr{H}^{r+s}(\mathbb{C};\mathbf{S}^\bullet).$$

For $r\geqq 2$ we have $E^2_{rs}=0$ by the previous lemma. Furthermore,

$$E^2_{0s}=H^0(\mathbb{C};\mathbf{H}^s(\mathbf{S}^\bullet))=0 \quad \text{if } s>f^{-1}(0),$$
$$E^2_{1s}=H^1(\mathbb{C};\mathbf{H}^s(\mathbf{S}^\bullet))=0 \quad \text{if } s>f^{-1}(1)$$

where $f^{-1}(r)=\max\{m\mid f(m)\geqq r\}$ is the "super-inverse" of f. Consequently

$$\bigoplus_{r+s=j} E^2_{rs}=0 \quad \text{if } j>\max_{r=0,1}[r+f^{-1}(r)]=\max\{f(m)+m\mid 0\leqq f(m)\leqq 1\}.$$

which implies $\mathscr{H}^j(\mathbb{C};\mathbf{S}^\bullet)=\mathbf{0}$ in this range also.

Remark. The super-inverse, f^{-1}, may be interpreted as follows: Suppose $\mathbf{S}^\bullet$ is constructible with respect to a stratification $X_0\subset X_1\subset\ldots\subset X_n=\mathbb{C}^n$ by complex k dimensional algebraic varieties X_k. The support condition means that for any point $x\in X_r-X_{r-1}$ the stalk cohomology $\mathbf{H}^k(\mathbf{S}^\bullet)_x$ vanishes for all $k>f^{-1}(r)$.

We now proceed by induction on n. Suppose $\mathbf{S}^\bullet$ is a complex of sheaves on $\mathbb{C}^n$ which satisfies the hypotheses of the proposition. Choose a stratification of $\mathbb{C}^n$,

$$X_0\subset X_1\subset\ldots\subset X_{n-1}\subset X_n=\mathbb{C}^n$$

by complex i-dimensional algebraic varieties X_i such that $\mathbf{S}^\bullet$ is constructible with respect to this stratification (§1.4).

We can find a linear projection $\pi\colon \mathbb{C}^n\to\mathbb{C}^{n-1}$ with the property that the stalk homology of the complex of sheaves $\mathbf{B}^\bullet=R\pi_*\mathbf{S}^\bullet$ at any point $y\in\mathbb{C}^{n-1}$ may be identified with the hypercohomology of the fibre $\pi^{-1}(Y)$, i.e.,

$$\mathbf{H}^i(R\pi_*(\mathbf{S}^\bullet))_y=\mathscr{H}^i(\pi^{-1}(y);\mathbf{S}^\bullet).$$

Such a $\pi\colon \mathbb{C}^n\to\mathbb{C}^{n-1}$ may be constructed as follows: Embed $\mathbb{C}^n$ in $\mathbb{CP}^n$ by adding a $\mathbb{CP}^{n-1}$ at infinity. Complete the stratification $X_0\subset X_1\subset\ldots\subset X_n$ of $\mathbb{C}^n$ to a stratification of $\mathbb{CP}^n$ by stratifying the $\mathbb{CP}^{n-1}$ at infinity (i.e. without refining the stratification of $\mathbb{C}^n$). Let p be a point in the largest stratum of the $\mathbb{CP}^{n-1}$ at infinity, and define π by projection along the parallel lines which pass through p. This fibration of $\mathbb{C}^n$ by lines contains a subbundle D of compact discs such that X_{n-1} is contained in the interior of D, Now, $R\pi_*\mathbf{S}^\bullet = R\pi_*\mathbf{S}^\bullet|D$ and $\pi|D$ is proper. By [17] §4.17.1 we have,

$$\mathbf{H}^i(R\pi_*(\mathbf{S}^\bullet))_y=\mathscr{H}^i(\pi^{-1}(y)\cap D;\mathbf{S}^\bullet)=\mathscr{H}^i(\pi^{-1}(y);\mathbf{S}^\bullet).$$

We will now apply the case $n=1$ to the computation of these stalk cohomology groups.

It is possible to find a stratification $Y_0 \subset \ldots \subset Y_{n-1} = \mathbb{C}^{n-1}$ by complex i-dimensional algebraic varieties Y_i, and a refinement $X_0^1 \subset X_1^1 \subset \ldots \subset X_n^1 = \mathbb{C}^n$ of the stratification $\{X_i\}$ of $\mathbb{C}^n$ such that $\pi\colon \mathbb{C}^n \to C^{n-1}$ takes strata to strata. (In other words, stratify the map π).

Suppose $y \in Y_r - Y_{r-1}$. We claim that $H^k(R\pi_* \mathbf{S}^\cdot)_y = 0$ for all $k > \max[f^{-1}(r), 1+f^{-1}(r+1)]$. To see this, consider the stratification of $\pi^{-1}(y)$ obtained from intersecting with the $\{X_i'\}$. Clearly $\mathbf{S}^\cdot|\pi(y)$ is constructible with respect to this stratification. A point $x \in \pi^{-1}(y)$ which lies in a zero dimensional stratum of $\pi^{-1}(y)$ may be an element of any stratum of X except those strata in X_{r-1}. Therefore $\mathbf{H}^k(\mathbf{S}^\cdot)_x = 0$ for all

$$k > \max[f^{-1}(r), f^{-1}(r+1), \ldots, f^{-1}(n)] = f^{-1}(r).$$

A point $x' \in \pi^{-1}(y)$ which lies in the one-dimensional stratum of $\pi^{-1}(y)$ may be an element of any stratum in $X - X_r$. Therefore, $\mathbf{H}^k(\mathbf{S}^\cdot)_{x'} = 0$ for all

$$k > \max[f^{-1}(r+1), f^{-1}(r+2), \ldots, f^{-1}(n)] = f^{-1}(r+1).$$

Applying the case $n=1$ to the fibre $\pi^{-1}(y)$ we obtain $\mathscr{H}^k(\pi^{-1}(y); \mathbf{S}^\cdot) = 0$ for all $k > \max[1+f^{-1}(r+1), f^{-1}(r)]$.

We will now apply the induction hypothesis to the complex of sheaves $R\pi_* \mathbf{S}^\cdot$ on $\mathbb{C}^{n-1}$. This complex satisfies a support condition

$$\dim\{y \in \mathbb{C}^{n-1} \mid \mathbf{H}^m(R\pi_* \mathbf{S}^\cdot)_y \neq 0\} \leqq g(m)$$

where, for every $r \geqq 0$

$$g^{-1}(r) \leqq \max[1+f^{-1}(1+r), f^{-1}(r)].$$

Therefore $H^k(\mathbb{C}^n; \mathbf{S}^\cdot) = H^k(\mathbb{C}^{n-1}; R\pi_* \mathbf{S}^\cdot) = 0$ whenever

$$k > \max\{m + g(m) \mid 0 \leqq g(m) \leqq n-1\} = \max\{r + g^{-1}(r) \mid 0 \leqq r \leqq n-1\}.$$

This condition will be satisfied if

$$k > \max\{r + f^{-1}(r) \mid 0 \leqq r \leqq n\} = \max\{m + f(m) \mid 0 \leqq f(m) \leqq n\}$$

as claimed.

7.3. *Proof of the Lefschetz Theorem*

Fix a perversity $\bar{p}$ with $p(c) \leqq c/2$ and let $\mathbf{IC}_X^\cdot$ denote the associated intersection homology complex on X. By § 5.4.1 the triangle

$$\begin{array}{ccc} R\alpha_* \alpha^! \mathbf{IC}_X^\cdot & \longrightarrow & \mathbf{IC}_X^\cdot \\ {\scriptstyle [1]}\nwarrow & & \swarrow \\ & Ri_* i^* \mathbf{IC}_X^\cdot & \end{array} \quad \text{becomes} \quad \begin{array}{ccc} R\alpha_* \mathbf{IC}_Y^\cdot & \longrightarrow & \mathbf{IC}_X^\cdot \\ {\scriptstyle [1]}\nwarrow & & \swarrow \\ & Ri_* i^* \mathbf{IC}_X^\cdot & \end{array}$$

where $i\colon (X-Y)\to X$ is the inclusion of the complement of the hyperplane section. The long exact sequence on hypercohomology associated with this triangle is

$$\to IH_i^{\bar p}(Y)\to IH_i^{\bar p}(X)\to \mathscr{H}^{-i}(Ri_* i^*\mathbf{IC}_X^\bullet)\to IH_{i-1}^{\bar p}(Y)\to$$

so we must show $\mathscr{H}^{-i}(i^*\mathbf{IC}_X^\bullet)=\mathscr{H}^{-i}(Ri_* i^*\mathbf{IC}_X^\bullet)=0$ for all $i\leqq n-1$.

Since X can be stratified by algebraic subvarieties, the complex $\mathbf{IC}_X^\bullet$ is algebraically constructible. Therefore the complex $i^*\mathbf{IC}_X^\bullet$ on the affine variety $X-X\cap H$ satisfies the hypotheses of Prop. 7.2 with the support condition given by axioms (AX2):

$$\dim_{\mathbb{C}}\{x\,|\,\mathbf{H}^{-2n}(i^*\mathbf{IC}_X^\bullet)_x\neq 0\}\leqq n$$

$$\dim_{\mathbb{C}}\{x\,|\,\mathbf{H}^{m}(i^*\mathbf{IC}_X^\bullet)_x\neq 0\}\leqq n-\frac{p^{-1}(m+2n)}{2}\qquad \text{for } m\geqq -2n+1.$$

Since $p(c)\leqq \frac{c}{2}$ we have $\frac{p^{-1}(m+2n)}{2}\geqq m+2n$ so

$$\dim_{\mathbb{C}}\{x\,|\,\mathbf{H}^{m}(i^*\mathbf{IC}_X^\bullet)_x\neq 0\}\leqq -m-n\qquad \text{for } m\geqq -2n.$$

The conclusion of Prop. 7.2 now reads

$$\mathscr{H}^{j}(X-Y;i^*\mathbf{IC}_X^\bullet)=0\qquad \text{for all } j>-n \text{ as desired.}$$

Remark. Decreasing the perversity does not give better bounds on j because the dimension of the support of $\mathbf{H}^{-2n}(i^*\mathbf{IC}_X^\bullet)$ is always n. However, if we increase the perversity past $p(c)=c/2$ the Lefschetz theorem continues to hold, although only for a smaller range of dimensions j. For a general perversity $\bar p\geqq c/2$ we have the following theorem:

$$IH_i(Y)\to IH_i(X) \text{ is an isomorphism for } i<j^*$$

and is a surjection for $i=j^*$, where

$$j^*=\max_m\left[m+n-\frac{p^{-1}(m+2n)}{2}\right]-1.$$

7.4. *Consequences in Ordinary Homology*

7.4.1. **Corollary.** *Let X be a normal n-dimensional projective variety and let $Y=X\cap H$ be a generic hyperplane section of X. Then the Gysin homomorphism (in ordinary cohomology)*

$$H^k(Y)\to H^{k+2}(X)$$

is an isomorphism for $k>n-1$ and is a surjection for $k=n-1$.

Proof. Take $p=0$ in Theorem 7.1. (This corollary can also be proved using the result of [27] and [32] on the vanishing of cohomology of Stein spaces.)

The following corollary was proved by Kato [26], Oka [34] and Ogus [33], and Kaup [27], [28].

7.4.2. **Corollary.** *Let X be a local complete intersection which is normal and let $Y=H\cap X$ be a generic hyperplane section of X. Then the homomorphism induced by inclusion*

$$H_i(Y)\to H_i(X)$$

is an isomorphism for $i<n-1$ and a surjection for $i=n-1$.

Proof. Let $p(c)=c/2$. Then by Prop. 2.3.3 we have $IH_i^{\bar p}(X)\cong H_i(X)$ for all i. The same holds for Y. Therefore the Lefschetz theorem (7.1) for intersection homology implies the same result in ordinary homology. The following corollary was pointed out to us by Horrocks [11].

7.4.3. **Corollary.** *Let X be a normal projective algebraic variety and let $\beta_1=\operatorname{rank}(H_1(X))$. Then β_1 is even.*

Proof. For a normal variety, $IH_1^{\bar p}(X)\cong H_1(X)$ for any perversity. Apply the Lefschetz theorem to successive hyperplane sections of X until we arrive at a two-dimensional variety Z with isolated singularities. Then $\beta_1(X)=\beta_1(Z)$ which is even (one verifies this directly).

7.4.4. *Remark.* The Lefschetz theorem for the middle perversity $\bar m$ is discussed in [11] as evidence that $IH_*^{\bar m}(X)$ has a pure Hodge structure.

§ 9. Generalized Deligne's Construction and Duality

We have already proved that if $\bar p(c)+\bar q(c)=c-2$ for all c, then the sheaves $\mathbf{P}^\bullet$ and $\mathbf{Q}^\bullet$ resulting from Deligne's construction with perversities $\bar p$ and $\bar q$ respectively have a canonical Verdier dual pairing. The proof was dispersed throughout §3, §4, and §5. Here we use the techniques of those chapters to study directly the relation between a single step in Deligne's construction and Verdier duality.

9.0. In this chapter R is a finite dimensional regular Noetherian ring (however in Sect. 9.2 we will assume R is a field). The space X is a stratified n-dimensional topological pseudomanifold and $U\subset X$ is an open union of connected components of strata. The closed subspace $Y=X-U$ is also a union of connected components of strata. Let i: $U\to X$ and j: $Y\to X$ denote the inclusions. Let $D_c^b(X)$ denote the derived category of sheaves of R-modules which are constructible with respect to this stratification

9.1. Fix an integer p. Let $\mathscr{C}(p)$ be the full subcategory of $D_c^b(X)$ whose objects $\mathbf{B}^\bullet$ satisfy the following conditions:

(a) $\mathbf{H}^i(j^*\mathbf{B}^\bullet)=0$ for all $i\geqq p$,

(b) $\mathbf{H}^i(j^!\mathbf{B}^\bullet)=0$ for all $i\leqq p$.

Theorem. *The functor $\tau^Y_{\leqq p-1}Ri_*$: $D_c^b(U)\to D_c^b(X)$ (see §1.4 and §1.14) takes its values in $\mathscr{C}(p)$. This functor determines an equivalence of categories $D_c^b(U)\cong\mathscr{C}(p)$ whose inverse is i^*.*

Proof. $\tau^Y_{\leqq p-1}Ri_*\mathbf{A}^\bullet$ is constructible by an argument similar to that in Lemma 3.1. The equivalence of categories argument is similar to that in the proof of Theorem 3.5.

9.2. Now suppose R is a field and Y is a k-dimensional R-homology manifold (for some $k<n$). Define $\mathscr{C}'_1$ to be the full subcategory of $D^b_c(U)$ whose objects $\mathbf{A}^\bullet$ satisfy

(c) $j^* R i_* \mathbf{A}^\bullet$ is CLC (§1.4),

(d) $j^! R i_* \mathbf{A}^\bullet$ is CLC.

Let $\mathscr{C}'_2(p)$ be the full subcategory of $D^b_c(X)$ whose objects $\mathbf{B}^\bullet$ satisfy conditions (a) and (b) of §9.1 and satisfy $\mathbf{B}^\bullet|U$ is in $\mathscr{C}'_1$.

Theorem. *The functor $\tau^Y_{\leqq p-1} R i_*\colon \mathscr{C}'_1 \to \mathscr{C}'_2(p)$ is an equivalence of categories.*

(The proof is similar to the proof of Theorem 9.1.)

Corollary. *If $p+q=-k$ then there is a one to one correspondence between pairings on U*

$$\sigma\colon\ \mathbf{A}^\bullet \otimes \mathbf{B}^\bullet \to \mathbb{D}^\bullet_U$$

and pairings on X,

$$\sigma'\colon\ \tau^Y_{\leqq p-1} R i_* \mathbf{A}^\bullet \otimes \tau^Y_{\leqq q-1} R i_* \mathbf{B}^\bullet \to \mathbb{D}^\bullet_X$$

which is given by the rule:

$$\sigma \text{ corresponds to } \sigma' \text{ if } \sigma'|U=\sigma.$$

Furthermore, σ is a Verdier dual pairing on U if and only if σ' is a Verdier dual pairing on X.

Proof of Corollary. Since $\tau^Y_{\leqq q-1} R i_* \mathbf{B}^\bullet$ is an object in $\mathscr{C}'_2(q)$, $R\mathbf{Hom}^\bullet(\tau^Y_{\leqq q-1} R i_* \mathbf{B}^\bullet, \mathbb{D}^\bullet_X)$ is an object in $\mathscr{C}'_2(p)$. This is a calculation as in §5.3. Therefore a morphism

$$\mathbf{A}^\bullet \to R\mathbf{Hom}^\bullet(\mathbf{B}^\bullet, \mathbb{D}^\bullet_U)$$

corresponds to a morphism

$$\tau^Y_{\leqq p-1} R i_* \mathbf{A}^\bullet \to R\mathbf{Hom}^\bullet(\tau^Y_{\leqq q-1} R i_* \mathbf{B}^\bullet, \mathbb{D}^\bullet_X).$$

Furthermore, isomorphisms correspond to isomorphisms.

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