Fundamental lemma and affine Springer fibers

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Notes on two papers, jointly written with Robert Kottwitz and Robert MacPherson

Equivariant cohomology, Koszul duality, and the localization theorem (with R. Kottwitz and R. MacPherson), Inv. Math. 131 (1998), 25-83.

Homology of affine Springer fibers in the unramified case (with R. Kottwitz and R. MacPherson), Duke Math. J. 121 (2004), 509-561.

$F = \mathbb{F}_q((\epsilon)) \quad L = \overline{\mathbb{F}}_q((\epsilon))$
$\mathfrak{o}_F = \mathbb{F}_q[[\epsilon]] \quad \mathfrak{o}_L = \overline{\mathbb{F}}q[[\epsilon]].$

$\sigma = \mathrm{Frob}(\overline{\mathbb{F}}_q/\mathbb{F}_q) = \mathrm{Frob}(L/F)$.
Assume $\overline{F} \subset \overline{L}$

G reductive group over $\mathbb{F}_q$.
$T \subset G$ a maximal $\mathbb{F}_q$-torus.
$\mathfrak{t}, \mathfrak{g}$ Lie algebras

For $u \in \mathfrak{g}$ and $g \in G$ write $gug^{-1} = Ad(g)(u)$
$u, u' \in \mathfrak{g}$ are σ-conjugate $\iff$
$u' = gu\sigma(g)^{-1}$ for some $g \in G$

$u, u' \in \mathfrak{t}(\mathbb{F}_q((\epsilon)))$ are stably conjugate $\iff$
$u' = gug^{-1}$ for some $g \in G(\overline{F})$.
Then g can be chosen in $G(L) = G(\overline{\mathbb{F}}_q((\epsilon)))$.

Definition of κ-orbital integral

Let $u \in \mathfrak{t}(F)$ regular and "integral"

$\alpha(u)$ in valuation ring of $\overline{F}$ for every root α

Let $\kappa \in \widehat{T} = \text{Hom}(X_*(T), \overline{\mathbf{Q}}_\ell^\times)$
Assume κ is fixed under Gal$(\overline{F}/F)$.
Let $f : \mathfrak{g}(F) \to \overline{\mathbb{Q}}_\ell$ smooth, compact support

$$O_u^\kappa(f) = \sum_{u' \underset{st}{\sim} u} \langle inv(u,u'), \kappa \rangle \int_{G_{u'}(F)\backslash G(F)} f(g^{-1}u'g)dg$$

Sum is over representatives $u' = gug^{-1}$ of the conjugacy classes within the stable conjugacy class of u.
Then $g^{-1}\sigma(g) \in T(L)$ defines a 1-cocycle,and there is a procedure by which κ assigns a number, denoted

$$\langle inv(u,u'), \kappa \rangle.$$

Fundamental Lemma in a very special case

Assume for simplicity that G is adjoint and $S = T$ is split over $L = \overline{\mathbb{F}}_q((\epsilon))$.

Assume H is a reductive group over $\mathbb{F}_q$
$s \in \widehat{G}$ and $\widehat{H} = \mathrm{Cent}_{\widehat{G}}(s)$.
So (H, s) is endoscopic data.

After some more choices, we have a maximal $\mathbb{F}_q$-torus $T_H \subset H$ and an isomorphism

$$T_H \cong T \subset G.$$

So $u \in T$ gives $u_H \in T_H$ and $s \in Z(\widehat{H})$ gives $\kappa \in \hat{T}$.

$$\begin{array}{ccccccc} H & & G & & \widehat{H} & \subset & \widehat{G} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ T_H & \subset & T & & Z(\widehat{H}) & \subset & \hat{T} \\ u_H & & u & & s & & \kappa \end{array}$$

Conjecture (Langlands)
Let $r = \dim(X_u) - \dim(X^H_{u_H})$. Then:

$$O^\kappa_u(1_{\mathfrak{g}(\mathfrak{o}_F)}) = q^r O^\kappa_{u_H}(1_{\mathfrak{h}(\mathfrak{o}_F)})$$

Remark. The invariants $\langle inv(u_H, u'_H), \kappa \rangle$ are all 1 so $O^{\kappa}_{u_H}(1_{\mathfrak{h}(\mathfrak{o}_F)})$ is a "stable" orbital integral.

Affine Springer Fibers

There exists an *ind*-scheme $X = G/K$ (the affine Grassmannian) over $\mathbb{F}_q$ such that:

$X(\mathbb{F}_q) = G(F)/G(\mathfrak{o}_F)$ and $X(\overline{\mathbb{F}}_q) = G(L)/G(\mathfrak{o}_L)$.

Let $u \in \mathfrak{t}(F)$ regular and "integral"

$\alpha(u)$ in valuation ring of $\overline{F}$ for every root α

The affine Springer fiber:

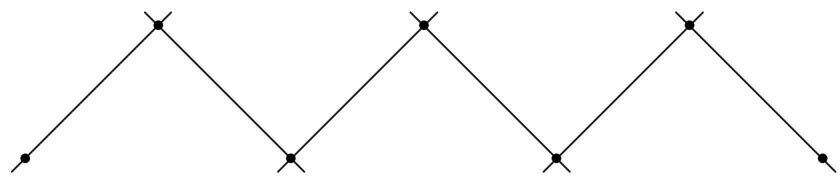
$$X_u = \left\{ xK \in X = G/K | \ Ad(x^{-1})(u) \in \mathfrak{g}(\mathfrak{o}_L) \right\}$$

Lie algebra analog to

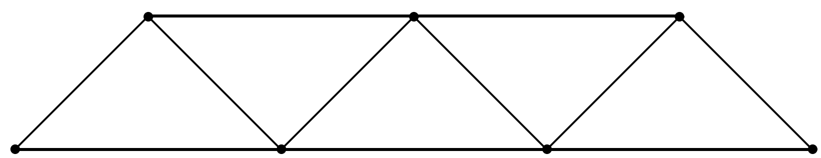
$$uxK = xK \iff x^{-1}ux \in K$$

Examples in $\mathfrak{sl}_2$

$$u = \begin{pmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{pmatrix}$$



$$u = \begin{pmatrix} \epsilon^2 & 0 \\ 0 & -\epsilon^2 \end{pmatrix}$$



$$u = \begin{pmatrix} 0 & \epsilon \\ \epsilon^3 & 0 \end{pmatrix}$$

The lattice Λ

Let $S \subset T$ be the maximal unramified subtorus. It splits over $L = \overline{\mathbb{F}}_q((\epsilon))$ so

$$X_*(S) = X_*(T)^{\mathrm{Gal}(\overline{L}/L)}$$

and this sequence is exact:

$$1 \to S(\mathfrak{o}_L) \to S(L) \overset{\mathrm{val}}{\to} X_*(S) \to 1$$

$\alpha(\mathrm{val}(s)) = \mathrm{val}(\alpha(s))$ for all $\alpha \in X^*(S)$

Splitting: $X_*(S) \to S(L)$ by $\alpha \mapsto \alpha(\epsilon)$

$\Lambda \subset S(L)$ image of this splitting.

$\Lambda \subset S \subset T$ acts on X_u.
$\Lambda \backslash X_u$ is a projective scheme of finite type/$\mathbb{F}_q$.

Let $\kappa \in \widehat{T}$, invariant under Gal$(\overline{F}/F)$. Then

$$\Lambda \cong X_*(S) \to X_*(T) \xrightarrow{\kappa^{-1}} \overline{\mathbb{Q}}_\ell^\times$$

determines line bundle $\mathcal{L}_{\kappa^{-1}}$ on $\Lambda\backslash X_u$.

$T(L)$ acts on $H^i(\Lambda\backslash X_u; \mathcal{L}_{\kappa^{-1}})$

Let $H^i(\Lambda\backslash X_u; \mathcal{L}_{\kappa^{-1}})_\kappa$ be the κ-isotypical piece.

Theorem [GKM] Assume κ has finite order. Let $u \in \mathfrak{t}(F)$ regular and integral. Then the κ-orbital integral

$$O_u^\kappa(1_{\mathfrak{g}(\mathfrak{o}_F)})$$

is equal to the trace of Frobenius,

$$\sum_{i \geq 0} (-1)^i Tr\left(\sigma^{-1} : H^i(\Lambda\backslash X_u; \mathcal{L}_{\kappa^{-1}})_\kappa\right)$$

Assuming T splits over $L = \overline{\mathbb{F}}_q((\epsilon))$, and assuming a certain purity conjecture, we give an explicit formula for this, which can be compared with the formula for the endoscopic group.

First reductions

The Frobenius acts on $H^i(\Lambda\backslash X_u; \mathcal{L}_{\kappa^{-1}})$ through an element (in a subgroup) of $\widetilde{W} = W \rtimes \mathrm{Aut}$

Aut = automorphisms of the Dynkin diagram

So to compute $O_u^\kappa(1_{\mathfrak{g}(\mathfrak{o}_F)})$ we only need to compute $H^*(\Lambda\backslash X_u; \mathcal{L}_{\kappa^{-1}})$ and the action of $\widetilde{W}$.

Suppose a lattice $\Lambda \cong \mathbb{Z}^n$ acts freely on a projective variety Y. Get

$$\begin{array}{ccc} E\Lambda \times_\Lambda Y & \xrightarrow[\pi]{} & B\Lambda \\ \downarrow p & & \\ \Lambda\backslash Y & & \end{array}$$

Let $s : \Lambda \to \mathbb{C}^\times$ be a character.
Then s determines local systems

$$\begin{array}{ccc} \mathcal{L}_s = \mathbb{C} \times_\Lambda Y & \qquad & \mathcal{L}_s^B = \mathbb{C} \times_\Lambda E\Lambda \\ \downarrow & & \downarrow \\ \Lambda\backslash Y & & B\Lambda \end{array}$$

and an isomorphism $p^*(\mathcal{L}_s) \cong \pi^*(\mathcal{L}_s^B)$.

If $H_*(Y)$ is pure and if s has finite order then the spectral sequence for

$$\begin{array}{ccc} \mathcal{L}_s & & \mathcal{L}_s^B \\ \downarrow & & \downarrow \\ E\Lambda \times_\Lambda Y & \xrightarrow{\pi} & B\Lambda \end{array}$$

collapses (use the mixed Hodge structure on $H_*(B\Lambda)$)

So:

$$\begin{aligned} H_m(\Lambda \backslash Y; \mathcal{L}_s) &\cong \bigoplus_{p+q=m} H_p(\Lambda; H_q(Y; \mathbb{C}_s)) \\ &\cong \bigoplus_{p+q=m} \mathrm{Tor}^p_{\mathbb{C}[\Lambda]}(\mathbb{C}_s, H_q(Y)) \end{aligned}$$

Assume all this works in étale (co)homology.

Symmetric algebras

Let $\mathfrak{a}$ be a finite dimensional complex vector space.

$S(\mathfrak{a})$ = polynomial functions on $\mathfrak{a}^*$.
$S(\mathfrak{a}^*) = \mathcal{D}(\mathfrak{a})$ = linear differential operators, constant coefficients, on $\mathfrak{a}^*$

If H is a module over $\mathcal{D}(\mathfrak{a})$ and if $I \subset \mathcal{D}$ is an ideal, define

$$H\{I\} = \{h \in H | \ \partial h = 0 \text{ for all } \partial \in I\}$$

Let A be an n dimensional complex torus. Identify $X_*(A) \otimes \mathbb{C} \cong \mathfrak{a}$, $X^*(A) \otimes \mathbb{C} \cong \mathfrak{a}^*$

$\alpha \in X^*(A)$ gives $L_\alpha \to BA$ with $c^1(L_\alpha) \in H^2(BA)$
$\mathcal{D}(\mathfrak{a}) = \mathbf{S}(\mathfrak{a}^*) \cong \mathbf{S}(X^*(A) \otimes \mathbb{C}) \cong H^*_A(\{\text{pt}\}; \mathbb{C})$.
$\mathbf{S}(\mathfrak{a}) \cong \mathbf{S}(X_*(A) \otimes \mathbb{C}) \cong H_*^A(\{\text{pt}\}; \mathbb{C})$.

(The equivariant homology is an algebra because BA is an H-space:

$A \times A \to A$ gives $BA \times BA \to BA$ and

$H_*^A(\{\text{pt}\}) \times H_*^A(\{\text{pt}\}) \to H_*^A(\{\text{pt}\})$.)

If A acts on Y then $Y_A = Y \times_A EA \to BA$.

$H^*_A(Y) = H^*(Y_A) = H^*(Y \times_A EA)$ (Borel) so there is a spectral sequence

$$E_2^{p,q} = H^p(BA) \otimes H^q(Y) \implies H_A^{p+q}(Y)$$

Y is *equivariantly formal* if this collapses. Then:

$$H^*(Y) \cong H^*_A(Y) \otimes_{H^*_A(\{\mathrm{pt}\})} \mathbb{C}$$
$$H_*(Y) \cong H_*^A(Y)\{I\}$$

where $I = \ker(\mathcal{D}(A) \to \mathbb{C})$ augmentation ideal.

Localization theorem [Chang and Skjelbred] Suppose Y is equivariantly formal. Then for all j the following sequence is exact

$$H^A_{j+1}(Y_1, Y_0) \to H^A_j(Y_0) \to H^A_j(Y) \to 0$$

where $Y_0 =$ fixed point set,
$Y_1 =$ union of $\leq$ 1-dimensional orbits.

If Y is a projective algebraic variety and the (co)homology of Y is pure then Y is equivariantly formal for any algebraic torus action.

Application to Springer fibers over $\mathbb{C}$

Now let $F = \mathbb{C}((\epsilon)) = L$.
Assume T defined over $\mathbb{C}$, $u \in \mathfrak{t}(L)$ is regular
$Y = X_u \subset X = G(L)/G(\mathfrak{o}_L)$.
Then $T(\mathbb{C})$ acts on X, preserving X_u.

Conjecture: The cohomology of X_u is pure.

Let $x_0 = 1.G(\mathfrak{o}_L) \subset X$ be the base point.
The fixed point set of $T(\mathbb{C})$ is

$$X_0 = (X_u)_0 = \Lambda x_0$$

$$H_*^T(X_0) \cong \mathbb{C}[\Lambda] \otimes \mathbf{S}$$

where $\mathbf{S} = \mathbf{S}(X_*(T) \otimes \mathbb{C}) = \mathbf{S}(\mathfrak{t})$.

If $\alpha \in \Phi(G,T)$ is a root it determines
$\partial_\alpha \in \mathcal{D}(\mathfrak{t})$ a degree 1 differential operator
$x_\alpha \in \mathbf{S}(\mathfrak{t})$ a degree 1 monomial
$\alpha^\vee \in \Lambda$ a co-root in Λ.

Define the following submodule:

$$L_{\alpha,u} = \sum_{d=1}^{\mathrm{val}(\alpha(u))} (1-\alpha^\vee)^d \mathbb{C}[\Lambda] \otimes \mathbf{S}(\mathfrak{t})\{\partial_\alpha^d\}$$

in $\mathbb{C}[\Lambda] \otimes \mathbf{S}(\mathfrak{t})$.

Theorem [GKM] Suppose X_u is pure.
Then the following sequence is exact,

$$0 \longrightarrow \sum_{\alpha \in \Phi^+} L_{\alpha,u} \longrightarrow \mathbb{C}[\Lambda] \otimes \mathbf{S}(\mathfrak{t}) \longrightarrow H_*^T(X_u) \longrightarrow 0$$

Moreover the group $\Lambda \rtimes W \rtimes \mathrm{Aut}$ acts in an obvious way on $\mathbb{C}[\Lambda] \otimes \mathbf{S}$, preserving the $\oplus L_{\alpha,u}$ so we obtain an action on $H_*^T(X_u)$ and also on $H_*(X_u)$.

Proof

It turns out that

$$X_1 = \bigcup_{\alpha \in \Phi^+} X_u^\alpha$$

where X_u^α is the Springer fiber for the group H^α of semisimple rank 1 determined by α.

In the semisimple rank 1 case there is a further 1-dimensional torus that acts on X_u. The 1-dimensional orbits of this extended torus action are isolated and can be explicitly described.

Fundamental lemma

Let (H, s) be endoscopic data with $\widehat{H} = \hat{G}_s$ and $T_H \cong T$. Then

$$\Phi^\vee(H, T_H) = \left\{ \alpha^\vee \in \Phi^\vee(G, T) | \ s(\alpha^\vee) = 1 \right\}$$

Set

$$\Delta = \prod \partial_\alpha^{\mathsf{val}(\alpha(u))} \in \mathcal{D} = H_T^*(\{\mathsf{pt}\}).$$

product over those $\alpha \in \Phi^\vee(G) - \Phi^\vee(H)$. Set

$$r = \deg(\Delta) = \sum_{\text{same } \alpha} \mathsf{val}(\alpha(u))$$

Theorem [GKM] The mapping

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \sum_{\alpha\in\Phi^+(G)} L_{\alpha,u} & \longrightarrow & \mathbb{C}[\Lambda]\otimes\mathbf{S} & \longrightarrow & H_*^T(X_u) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle 1\otimes\Delta} & & \downarrow & & \\
0 & \longrightarrow & \sum_{\alpha\in\Phi^+(H)} L_{\alpha,u} & \longrightarrow & \mathbb{C}[\Lambda]\otimes\mathbf{S} & \longrightarrow & H_*^T(X_{u_H}^H) & \longrightarrow & 0
\end{array}
$$

is surjective and induces a homomorphism

$$H_*^T(X_u) \to H_*^T(X_{u_H}^H)[-2r]$$

which becomes an isomorphism after localizing with respect to the multiplicative subset

$$J = \langle\{1-\alpha^\vee\}\rangle$$

as $\alpha^\vee$ varies over $\Phi^\vee(G) - \Phi^\vee(H)$.
This induces an isomorphism

$$H_m(\Lambda\backslash X_u; \mathcal{L}_s) \cong H_{m-2r}(\Lambda\backslash X_{u_H}^H; \mathcal{M}_s).$$

equivariant with respect to $W \rtimes \mathrm{Aut}$.