

SIMPLICIAL DIFFERENTIAL FORMS WITH POLES

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Introduction. This paper has two purposes: 1) to introduce and study a new class of differential forms on a simplicial complex and 2) to give an intuitive and elementary argument for the relation between intersection homology and L^2 cohomology. We will consider each of these purposes separately in this introduction.

Shadow forms. Consider a simplicial complex K . Whitney associated to each simplicial cochain ζ a differential form $\nu(\zeta)$ on K . Whitney's map ν from simplicial cochains to differential forms is a map of cochain complexes which induces the deRham isomorphism. We call the forms $\nu(\zeta)$ *Whitney forms*. They have been useful in several contexts, most notably integral geometry and rational homotopy theory.

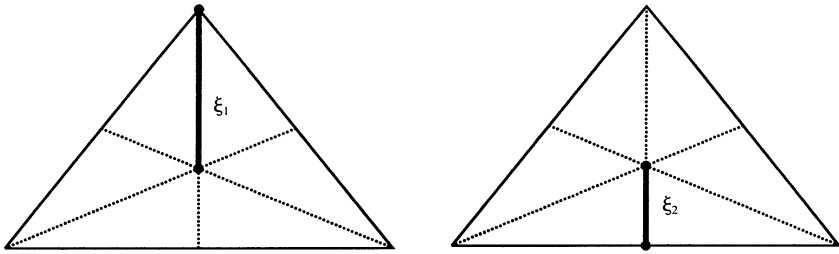
In this paper, we introduce a generalization of Whitney forms called *shadow forms*. We associate to certain simplicial chains ξ with respect to the first derived subdivision K' of K a shadow form $\omega(\xi)$ on K . Let us illustrate the construction of shadow forms when K is a single n -simplex, which we now denote by Δ . In this case, $\omega(\xi)$ is defined for k -simplices ξ of Δ' which do not lie in the boundary of Δ (or for linear combinations of these). For example, if Δ is the 2-simplex, then either of the 1-simplices ξ_1 or ξ_2 pictured below has an associated shadow form.

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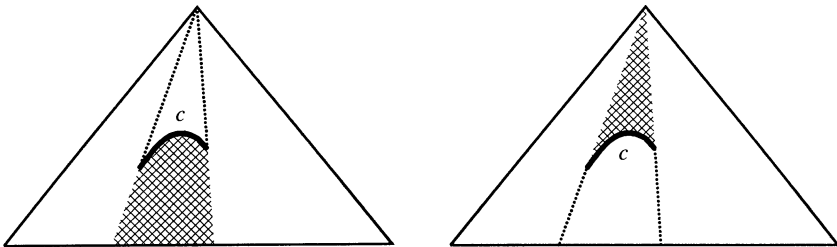
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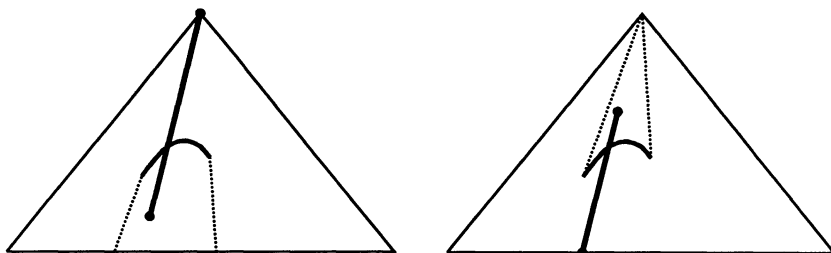
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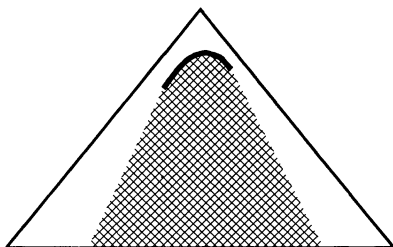
Given such a ξ , the associated shadow form will be an $(n - k)$ -form $\omega(\xi)$ which we will describe by specifying its integral over $(n - k)$ -chains c in K . The integral $\int_c \omega(\xi)$ is equal to the volume of a certain region $S_\xi(c)$ in Δ which we call the “shadow” cast by the chain c with respect to the simplex ξ . The following pictures show the shadows of a chain c with respect to ξ_1 and ξ_2 (as usual, the volume must be counted with appropriate signs and multiplicities).



In general, the shadow cast by c with respect to ξ can be defined as follows: For each point p in the interior of Δ , there is a canonical realization of the first derived subdivision Δ' such that p is the “barycenter” of Δ : the barycenter q of an $n - 1$ dimensional face F with opposite vertex v is determined by the condition that v, p , and q are colinear, and the barycenters of higher codimension faces are determined inductively. Denote by $\xi(p)$ the simplex corresponding to ξ in this subdivision. Then the shadow $S_\xi(c)$ cast by c with respect to ξ is the set of all points p for which the simplex $\xi(p)$ intersects c .



If the chain ξ is a union of dual cells of faces of Δ [Bra] [Mc], then $\omega(\xi)$ is precisely a Whitney form. For example, ξ_2 above is a dual cell so $\omega(\xi_2)$ is a Whitney form. In contrast to Whitney forms, however, general shadow forms may have poles near the faces of K of dimension $n - 2$ or less. For example, $\omega(\xi_1)$ has a pole near the top vertex of Δ . This may be seen from the fact that $\omega(\xi_1)$ integrated over a very small chain c near this vertex can produce a very big result.



In this paper, we study shadow forms on a simplex Δ by giving explicit formulas for them. Then their L^q norms (with respect to the usual Riemannian metric on the Euclidean space containing the simplex) can be written as explicit integrals. These integrals turn out to be classical: they are related to hypergeometric functions, Dirichlet integrals [WW] p. 253, and Eulerian integrals [WW] p. 258. By appropriate estimates, we determine for which combinatorial types of simplices ξ the forms $\omega(\xi)$ are L^q integrable.

Of course, we want to consider more general polyhedra than the simplex. We consider a simplicial complex K which is an n -dimensional oriented pseudomanifold. This means that K is the union of its n -simplices which are all oriented, and every $(n - 1)$ -simplex is contained in exactly two n -simplices which induce opposite orientations on it. In this

case, the associated shadow form $\omega(\xi)$ is defined whenever the chain $\xi \in C_i(K')$ satisfies a technical “transversality condition” (see Section 9). The definition of $\omega(\xi)$ amounts to the above construction in each simplex of K . The transversality condition guarantees that the resulting shadow form $\omega(\xi)$ extends continuously across the $n - 1$ dimensional faces of K . Then, ω gives a chain map from the chain complex $C_*^T(K')$ of “transverse” chains in K' to the cochain complex of shadow forms.

L^2 cohomology and intersection homology. There are a number of recent theorems which say that for certain classes of pseudomanifolds K , with an explicit Riemannian metric on the nonsingular part $K - \Sigma$, the intersection homology of K is the L^2 cohomology of K (see, for example, [B], [BC1], [BC2], [C1], [C2], [Ch1], [Ch2], [CGM], [G], [HP], [KK], [L], [N], [Sa], [SS], [Z1], [Z2], [Z3]). Intersection homology is most constructively defined by imposing an allowability condition on the simplicial chains which restricts the dimension of intersection of a chain with the singularities of K . The L^2 cohomology is defined by an integrability condition on differential forms.

Question. What do allowability conditions and integrability conditions have to do with each other?

Suppose K is an oriented pseudomanifold of dimension n and let $\Sigma = K_{(n-2)}$ denote the $n - 2$ skeleton. We may then consider the complex of transverse intersection chains in K' which satisfy the “allowability conditions” of intersection homology,

$$IC_i^T(K'; \mathbf{R}) = \{ \xi \in C_i^T(K'; \mathbf{R}) \mid \dim(|\xi| \cap F) \leq i - [\lambda/2] - 1 \text{ and}$$

$$\dim(|\partial\xi| \cap F) \leq i - [\lambda/2] - 2 \text{ for any simplex } F \subset \Sigma \}$$

where $[x]$ denotes the greatest integer less than or equal to x and λ denotes the codimension, $n - \dim(F)$, of the simplex F . The homology of this complex is the intersection homology of X (with “upper middle perversity” $p(\lambda) = [(\lambda - 1)/2]$) (see [GM2] and [GM3]).

If we linearly embed K in some Euclidean space, then this induces a metric on the manifold $K - \Sigma$, the “nonsingular part” of K . Then it makes sense to ask what the L^2 norm of a differential form on $K - \Sigma$ is. Our main result is

THEOREM 5.2. *For any chain ξ in $C_i^T(K'; \mathbf{R})$, the shadow form $\omega(\xi)$ is square integrable (i.e. $\omega(\xi) \in L^2$) if and only if ξ satisfies the allowability conditions for intersection homology (i.e. $\xi \in IC_i(K'; \mathbf{R})$).*

This theorem is proven by explicit computation of the L^2 norm of $\omega(\xi)$ as described earlier. It makes sense of the concept of the growth rate of a geometric chain and provides a direct answer to the question posed above.

Since ω is a chain map, this theorem gives a chain level proof of the following

COROLLARY 9.3. *The intersection homology of K is isomorphic to the cohomology of the complex of L^2 shadow forms on K .*

Since K with a linear embedding has conical singularities, it is a much deeper result due to Cheeger [Ch1], [Ch2] that the cohomology of the complex of all L^2 differential forms coincides with the middle intersection cohomology of X . We do not know if the methods here could be used to give a proof of Cheeger's theorem by successively refining the subdivision. However our construction provides a strong intuition behind his theorem.

More generally, we investigate the shadow forms which have finite L^q norm and show that they are precisely those forms which are associated to chains satisfying an intersection homology restriction for a particular perversity ($p(\lambda) = \lambda/q$), which we call the L^q -perversity. It follows that the cohomology of the complex of L^q shadow forms is precisely the intersection homology with the L^q perversity: in fact the two chain complexes are isomorphic. For example, the cohomology of the complex of bounded (or L^∞) shadow forms is isomorphic to the ordinary cohomology, if K is normal. In general we are led to the following

Conjecture. If K is a pseudomanifold with conical singularities, $q \geq 2$, and the L^q cohomology of $K - \Sigma$ is finite dimensional, then it is isomorphic to the intersection homology of K with the L^q perversity.

Presumably the methods of [Ch1] could be used to prove this conjecture although they require careful L^q estimates for the homotopy operators.

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first author would like to thank Northeastern University for partial support while this paper was being prepared.

1. Differential forms on a simplex. Let $\Delta = \Delta^n$ denote the standard n -simplex

$$\Delta^n = \{(x_1, x_2, \dots, x_{n+1}) \in \mathbf{R}^{n+1} \mid 0 \leq x_i \leq 1, \sum x_i = 1\}.$$

We shall use the symbol $\Omega^k(\Delta)$ to denote the smooth differential k -forms on the interior of Δ . Recall that an (abstract) k -simplex σ in the first derived complex of Δ is a chain of incident faces $F_1 < F_2 < \dots < F_{k+1}$. For any specific choice of barycentric subdivision of Δ , (where we denote the barycenter of a face F by the symbol \hat{F}), the realization of σ is the k -simplex spanned by the vertices $\hat{F}_1, \hat{F}_2, \dots, \hat{F}_{k+1}$. A choice of barycenter $p \in \text{int}(\Delta)$ determines a barycentric subdivision of Δ : the barycenter $\hat{F}(p)$ of a face F is uniquely determined as the intersection

$$\hat{F}(p) = F \cap A$$

where A is the affine subspace spanned by the point p and the opposite face to F . This gives rise to a realization $\sigma(p)$ of σ , *i.e.* as the k -simplex which is spanned by $\hat{F}_1(p), \hat{F}_2(p), \dots, \hat{F}_{k+1}(p)$. Throughout this paper we will suppose that $\sigma(p)$ is not contained in any proper face of Δ , *i.e.* that $F_{k+1} = \Delta$.

Fix the orientation of Δ corresponding to the ordered basis $\{e_2 - e_1, e_3 - e_1, \dots, e_{n+1} - e_1\}$ of vectors in the affine space spanned by Δ (where $e_i \in \mathbf{R}^{n+1}$ is the standard unit vector with a 1 in the i^{th} position). Then Lebesgue measure on Δ is given by the Whitney form

$$\begin{aligned} \text{dvol}(\Delta) &= W(\Delta) \\ &= n! \sum_{i=1}^{n+1} (-1)^{i+1} x_i dx_1 \wedge dx_2 \wedge \dots \wedge \hat{dx}_i \wedge \dots \wedge dx_{n+1}. \end{aligned}$$

Define the incidence manifold,

$$D_\sigma = \{(p, x) \in \text{int}(\Delta) \times \text{int}(\Delta) \mid x \in \sigma(p)\}$$

with inclusion $i : D_\sigma \hookrightarrow \Delta \times \Delta$. The manifold D_σ is $n + k$ dimensional and it fibers over Δ in two ways. The projection to the first factor has fiber $\sigma(p)$ which is oriented by the canonical ordering of the vertices of σ . We orient D_σ using the orientation of Δ followed by the orientation of σ . This induces an orientation on the fibres of the projection to the second factor by requiring that the orientation of D_σ agree with the orientation of Δ followed by the orientation of the fibre.

Definition. The shadow form $\omega(\sigma) \in \Omega^{n-k}(\Delta)$ is the differential form

$$\omega(\sigma) = \int_{\pi_2} i^* \pi_1^*(W(\Delta))$$

where π_1 and π_2 denote projections to the first and second factors, and where \int_{π_2} denotes integration over the fibres of π_2 (see [GHV] p. 298 and [BT] p. 61).

This is a smooth differential form on $\text{int}(\Delta)$ since $\pi_2 : \overline{D_\sigma} \cap \pi_2^{-1}(\text{int}(\Delta)) \rightarrow \text{int}(\Delta)$ is a smooth fibre bundle whose fibre $\pi_2^{-1}(x) \cap \overline{D_\sigma}$ is a smooth compact manifold with corners, and since the differential form $i^* \pi_1^* W(\Delta)$ is the restriction to D_σ of a smooth differential form $\pi_1^* W(\Delta)$ which is defined in a neighborhood of the closure $\overline{D_\sigma}$.

2. Currents on the simplex Δ . Let Ω_o^k denote the space of smooth k -forms on $\text{int}(\Delta)$ with compact support, endowed with the C^∞ compact open topology ([F] Section 4.1.7). The space \mathcal{E}^k of currents is the dual space, $\mathcal{E}^k = \text{Hom}_c(\Omega_o^k, \mathbf{R})$ of continuous homomorphisms $\Omega_o^k \rightarrow \mathbf{R}$. There is an embedding $\Psi : \Omega^{n-k} \hookrightarrow \mathcal{E}^k$ defined by

$$\Psi(\omega)(\eta) = \int_\Delta \omega \wedge \eta.$$

We now give the dual construction of the form $\omega(\sigma)$. Fix a barycenter $p \in \text{int}(\Delta)$ and consider the induced barycentric subdivision of Δ . Let σ denote an abstract k -simplex in the first derived complex of Δ . The corresponding realization $\sigma(p)$ defines a current $\phi(\sigma(p)) \in \text{Hom}_c(\Omega_o^k, \mathbf{R})$ which associates to any smooth k -form η the number

$$\phi(\sigma(p))(\eta) = \int_{\sigma(p)} \eta.$$

We can average the current $\phi(\sigma(p))$ over all choices $p \in \Delta$ and so obtain a new current $\tilde{\phi}(\sigma)$,

$$\tilde{\phi}(\sigma)(\eta) = \int_{p \in \Delta} \left(\int_{\sigma(p)} \eta \right) \text{dvol}(\Delta)$$

where $\text{dvol}(\Delta)$ denotes the volume form on Δ . We will now show that this averaging procedure smooths the current $\sigma(p)$:

PROPOSITION 2.1. $\tilde{\phi}(\sigma) = \Psi(\omega(\sigma))$.

Proof. It suffices to show that, for any differential k -form with compact support, $\eta \in \Omega_o^k$, we have $\tilde{\phi}(\sigma)(\eta) = \Psi(\omega(\sigma))(\eta)$. But

$$\begin{aligned} \Psi(\omega(\sigma))(\eta) &= \int_{\Delta} \omega(\sigma) \wedge \eta = \int_{\Delta} \left(\int_{\pi_2} i^* \pi_1^*(W(\Delta)) \right) \wedge \eta \\ &= \int_{D_\sigma} i^*(\pi_1^*(W(\Delta)) \wedge \pi_2^*(\eta)) \\ &= \int_{\Delta} \left(\int_{\pi_1} i^* \pi_2^*(\eta) \right) \wedge W(\Delta) \\ &= \int_{p \in \Delta} \left(\int_{\sigma(p)} \eta \right) \wedge W(\Delta) = \int_{p \in \Delta} \left(\int_{\sigma(p)} \eta \right) \text{dvol}(\Delta) \\ &= \tilde{\phi}(\sigma)(\eta) \end{aligned}$$

by the projection formula (see [GHV] Proposition IX p. 303). The change of order of integration is justified since η has compact support. □

COROLLARY 2.2. *If σ is an abstract k -simplex in the first derived complex of Δ , then the differential of the associated shadow form is*

$$d\omega(\sigma) = \sum_{\tau} (-1)^{o(\tau)} \omega(\tau)$$

where the sum is taken over all codimension 1 faces τ of σ such that $\text{int}(\tau) \subset \text{int}(\Delta)$ and where $o(\tau) = 0$ if the orientation of τ which is induced

from that of σ coincides with the orientation which is determined from the ordering of the vertices of τ and $o(\tau) = 1$ otherwise. \square

3. Shadows. Imagine a piecewise linear or piecewise smooth chain c of dimension $n - k$ which is contained in the interior of Δ . Corresponding to an abstract k -simplex σ in the first derived complex of Δ (which is not contained in any proper face of Δ), we may associate the “shadow” of c which is “cast” by σ (as illustrated in the introduction),

$$S_\sigma(c) = \{p \in \Delta \mid c \cap \sigma(p) \neq \emptyset\}.$$

The differential form $\omega(\sigma)$ defined in Section 1 has the beautiful property that its integral over a chain c is precisely the volume of the shadow (counted with appropriate sign and multiplicity):

PROPOSITION 3.1. For any chain $c \subset \text{int}(\Delta)$ of dimension $n - k$ we have

$$\int_c \omega(\sigma) = \text{volume}(S_\sigma(c)).$$

Proof. The proof is a direct computation:

$$\begin{aligned} \int_c \omega(\sigma) &= \int_c \int_{\pi_2} i^* \pi_1^*(W(\Delta)) = \int_{\pi_2^{-1}(c) \cap D_\sigma} i^* \pi_1^*(W(\Delta)) \\ &= \int_{\pi_1(\pi_2^{-1}(c) \cap D_\sigma)} W(\Delta) = \text{vol}(\pi_1(\pi_2^{-1}(c) \cap D_\sigma)) \end{aligned}$$

where “vol” denotes volume counted with appropriate sign and multiplicity. \square

4. Equations of shadow forms. In this section we give explicit formulas for the forms $\omega(\sigma)$ in terms of barycentric coordinates on the n -simplex Δ . First we recall the corresponding formula for Whitney’s differential forms.

Let e_i denote the standard unit vector with a 1 in the i^{th} position. For any ordered subset $S \subset \{1, 2, \dots, n\}$ consisting of $r + 1$ numbers

$$S = \{i_1, i_2, \dots, i_{r+1}\}$$

we associate the r -dimensional oriented face F of Δ whose vertices are $\{e_{i_1}, e_{i_2}, \dots, e_{i_{r+1}}\}$ and whose orientation is given by this ordering of the vertices. Radial projection $\pi : \Delta \rightarrow F$ is then given by

$$\pi(x_1, x_2, \dots, x_{n+1}) = (x_{i_1}, x_{i_2}, \dots, x_{i_{r+1}})/\Sigma S$$

where $\Sigma S = \sum_{j=1}^{r+1} x_{i_j}$,

Definition. ([W] p. 139, [ST]) The Whitney form $W_\Delta(S)$ is the differential r -form on $\text{int}(\Delta)$ which is given by

$$W_\Delta(S) = r! \sum_{j=1}^{r+1} (-1)^{j+1} x_{i_j} dx_{i_1} \wedge \dots \wedge \hat{dx}_{i_j} \wedge \dots \wedge dx_{i_{r+1}}$$

(for $r = 0$ we interpret this as $W_\Delta(\{x_{i_j}\}) = x_{i_j}$). If $S = \{1, 2, \dots, n + 1\}$ (i.e. if $F = \Delta$) then $W_\Delta(S)$ is the volume form on Δ .

Remark. A closely related form on Δ is obtained by pulling back the volume form from F , and it is

$$\pi^*(\text{dvol}(F)) = \frac{-1}{r!} \frac{W_\Delta(S)}{(\Sigma S)^{r+1}}.$$

Let σ be an abstract k -simplex in the first derived complex of Δ , which is not contained in any proper face of Δ and which is defined by the faces $F_1 < F_2 < \dots < F_{k+1} = \Delta$. We obtain a decomposition of the set $\{1, 2, \dots, n + 1\}$ into $k + 1$ disjoint subsets S_1, S_2, \dots, S_{k+1} such that for each α , ($1 \leq \alpha \leq k + 1$), the face F_α has vertices $\{e_i | i \in S_1 \cup S_2 \cup \dots \cup S_\alpha\}$.

Let $s_i = |S_i|$ denote the cardinality of S_i , and let $W(S_i) = W_\Delta(S_i)$ denote the Whitney form on Δ which corresponds to the face whose vertices are $\{e_j | j \in S_i\}$.

For fixed positive real numbers v_1, v_2, \dots, v_k whose sum is less than one, we define the function $\Phi(v_1, v_2, \dots, v_k; s_1, s_2, \dots, s_k)$ as follows:

$$\Phi = \frac{\int_{R(v)} \phi du_k du_{k-1} \dots du_1}{\int_R \phi du_k du_{k-1} \dots du_1}$$

where

$$\phi = (u_1)^{s_1-1}(u_2)^{s_2-1} \cdots (u_k)^{s_k-1}(1 - u_1 - u_2 - \cdots - u_k)^{s_{k+1}-1},$$

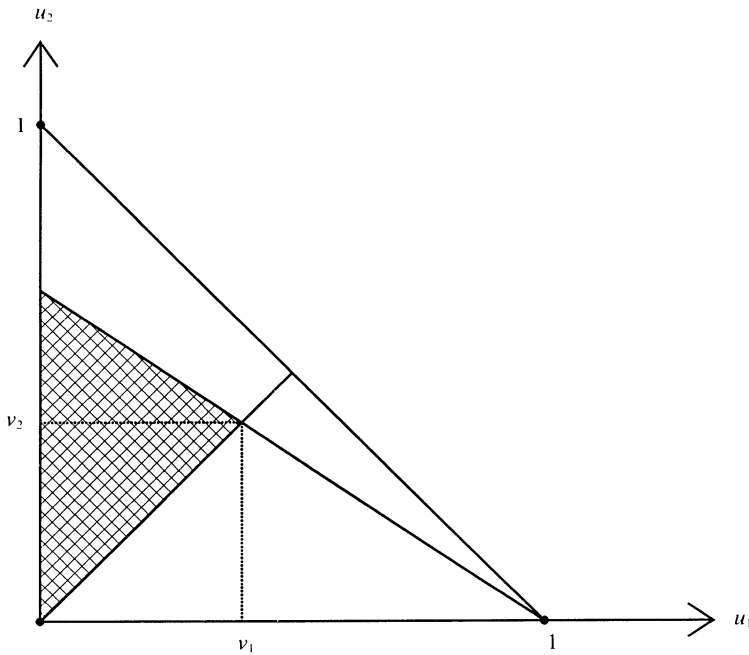
where $s_{k+1} = n + 1 - \sum_{j=1}^k s_j$, and where $R(v)$ is the subset of the simplex

$$R = \{(u_1, u_2, \dots, u_k) \in \mathbf{R}^k \mid 0 \leq u_i \leq 1 \text{ and } u_1 + u_2 + \cdots + u_k \leq 1\}$$

given by

$$R(v) = \left\{ (u_1, u_2, \dots, u_k) \mid 0 \leq \frac{u_1}{v_1} \leq \frac{u_2}{v_2} \leq \cdots \leq \frac{u_k}{v_k} \leq \frac{1 - u_1 - u_2 - \cdots - u_k}{1 - v_1 - v_2 - \cdots - v_k} \right\}.$$

The shaded region in the following diagram is the subset $R(v)$:



The numerator of Φ is a Dirichlet integral which can be explicitly computed, but we do not know a general formula for it.¹ A simple formula approximating Φ is given in Proposition 6.1. The denominator is

$$\Phi_0 = \frac{\Gamma(s_1)\Gamma(s_2) \cdots \Gamma(s_{k+1})}{\Gamma(n + 1)}.$$

THEOREM 4.1. *The differential form $\omega(\sigma)$ is equal to*

$$(-1)^{s_2+2s_3+\cdots+ks_{k+1}}\eta(\sigma)\Phi(\Sigma S_1, \Sigma S_2, \dots, \Sigma S_k; s_1, s_2, \dots, s_k)$$

where $\eta(\sigma)$ is the closed differential form of degree $n - k$,

$$\eta(\sigma) = \frac{W(S_1)}{(\Sigma S_1)^{s_1}} \wedge \frac{W(S_2)}{(\Sigma S_2)^{s_2}} \wedge \cdots \wedge \frac{W(S_{k+1})}{(\Sigma S_{k+1})^{s_{k+1}}}.$$

Proof. We will explicitly perform the integration over the fibre as indicated in the definition (Section 1) of $\omega(\sigma)$,

$$\omega(\sigma) = \int_{\pi_2} i^* \pi_1^*(W(\Delta)).$$

We shall use the following notation: a point in $\Delta \times \Delta$ will be denoted

$$(x_1, x_2, \dots, x_{n+1}, x'_1, x'_2, \dots, x'_{n+1})$$

and in general a dot will be used to indicate a variable associated with the second factor. For each of the subsets S_i we define $u_i = \Sigma S_i = \Sigma\{x_j | j \in S_i\}$, $u'_i = \Sigma S'_i$, and $f^*W(u_1, u_2, \dots, u_{k+1})$ to be the pullback of the volume form under the mapping $f : \Delta^n \rightarrow \Delta^k$ which is given by

$$f(x_1, x_2, \dots, x_{n+1}) = (\Sigma S_1, \Sigma S_2, \dots, \Sigma S_{k+1}) = (u_1, u_2, \dots, u_{k+1}).$$

Remark. With this notation, the region $R(u')$ above is precisely the projection to the plane $u'_{k+1} = 0$ of the set $f\pi_1(\pi_2^{-1}(x') \cap D)$.

We will now decompose the form $W(\Delta)$ into a product and then compute its pullback to the incidence manifold D_σ .

¹Added in proof: A. Zelevinsky has recently shown us how to evaluate this integral by calculating the associated generating function.

LEMMA 4.2. *The restriction to $D_\sigma = \{(x, x') \in \text{int } \Delta \times \text{int } \Delta \mid x' \in \sigma(x)\}$ of the form $\pi_1^*W(\Delta)$ is equal to the restriction to D_σ of the form*

$$\eta(\sigma) = a_1 \pi_2^* \left[\frac{W(S'_1) \wedge \cdots \wedge W(S'_{k+1})}{(u'_1)^{s_1} \cdots (u'_{k+1})^{s_{k+1}}} \right] \\ \wedge \pi_1^* f^* W(u_{k+1}, \dots, u_2, u_1) u_1^{s_1-1} u_2^{s_2-1} \cdots u_{k+1}^{s_{k+1}-1}$$

where

$$a_1 = (-1)^{s_2+2s_3+\cdots+ks_{k+1}} \frac{\Gamma(n+1)}{\Gamma(s_1)\Gamma(s_2) \cdots \Gamma(s_{k+1})}.$$

This lemma will complete the proof of Theorem 4.1 because

$$\omega(\sigma) = \int_{\pi_2} i^* \pi_1^* W(\Delta) = \int_{\pi_2} \eta(\sigma) \\ = a_1 \int_{\pi_2} \pi_2^*(\eta(\sigma)) \wedge \pi_1^* f^* W(u_{k+1}, \dots, u_2, u_1) u_1^{s_1-1} u_2^{s_2-1} \cdots u_{k+1}^{s_{k+1}-1}$$

and the value of this integral, at a point $x' \in \Delta$ is

$$a_1 \eta(x') \int_{\pi_2^{-1}(x') \cap D_\sigma} \pi_1^* f^* W(u_{k+1}, \dots, u_2, u_1) u_1^{s_1-1} u_2^{s_2-1} \cdots u_{k+1}^{s_{k+1}-1} \\ = a_1 \eta(x') \int_{\pi_1(\pi_2^{-1}(x') \cap D_\sigma)} f^* W(u_{k+1}, \dots, u_2, u_1) u_1^{s_1-1} u_2^{s_2-1} \cdots u_{k+1}^{s_{k+1}-1} \\ = a_1 \eta(x') \int_{\pi_1(\pi_2^{-1}(x') \cap D_\sigma)} f^*(\text{dvol}(\Delta^k)) u_1^{s_1-1} u_2^{s_2-1} \cdots u_{k+1}^{s_{k+1}-1} \\ = (-1)^{s_2+s_3+\cdots+s_{k+1}} \eta(x') \Phi(u'; s_1, \dots, s_k)$$

by the above remark. □

Proof of Lemma 4.2. The lemma follows immediately from two calculations:

Calculation 1:

$$W(\Delta) = a_1 W(S_1) \wedge \cdots \wedge W(S_{k+1}) \wedge \frac{f^* W(\Sigma S_{k+1}, \dots, \Sigma S_2, \Sigma S_1)}{(\Sigma S_1)(\Sigma S_2) \cdots (\Sigma S_{k+1})}.$$

Calculation 2:

$$i^*((\Sigma S_i)^{s_i} \pi_2^* W(S'_i)) = i^*((\Sigma S'_i)^{s_i} \pi_1^* W(S_i)).$$

Calculation 1 is proven by induction on k , using the following formulas which can be verified directly:

(a)
$$W(S_1 \cup S_2) = (-1)^{s_2} \frac{\Gamma(s_1 + s_2)}{\Gamma(s_1)\Gamma(s_2)} W(S_1) \wedge W(S_2) \wedge \frac{f^* W(\Sigma S_2, \Sigma S_1)}{(\Sigma S_1)(\Sigma S_2)}$$

(b)
$$W(\Sigma S_2, \Sigma S_1) = W(\Sigma S_2 + \Sigma S_1, \Sigma S_1).$$

Calculation 2 is proven by induction on $s = |S_i|$. By relabelling the coordinate axes, we can assume that $S = S_i = \{1, 2, \dots, s\}$. Writing $u = \Sigma S = x_1 + x_2 + \cdots + x_s$ it suffices to show that

$$u^s \pi_2^* W(u', x'_2, \dots, x'_s) = (u')^s \pi_1^* W(u, x_2, \dots, x_s)$$

on the incidence manifold D_σ , since $W(x_1, x_2, \dots, x_s) = W(u, x_2, \dots, x_s)$. The manifold D_σ is contained in the plane

$$\frac{x_1}{x'_1} = \frac{x_2}{x'_2} = \cdots = \frac{x_s}{x'_s} = \frac{u}{u'}.$$

We conclude

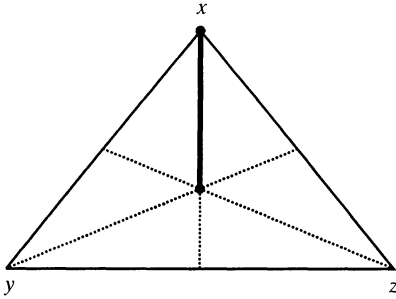
$$u dx'_s - x_s du' = u' dx_s - x'_s du.$$

Multiplying by uu' we have

$$u^2 \pi_2^* W(u', x'_s) = (u')^2 \pi_1^* W(u, x_s)$$

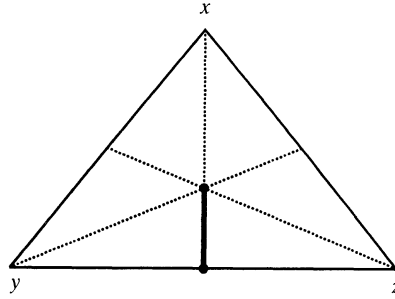
when restricted to D_σ . This is the case $s = 2$ of calculation 2. The general case is easily checked by induction, using part (a) above. □

Examples. If an abstract k -simplex σ is defined by the sequence of groups of indices S_1, S_2, \dots, S_{k+1} then we will write either $\sigma = (S_1)(S_2) \cdots (S_{k+1})$ or $\sigma = (\{x_{ij}\}_{i \in S_1})(\{x_{ij}\}_{i \in S_2}) \cdots (\{x_{ij}\}_{i \in S_{k+1}})$.



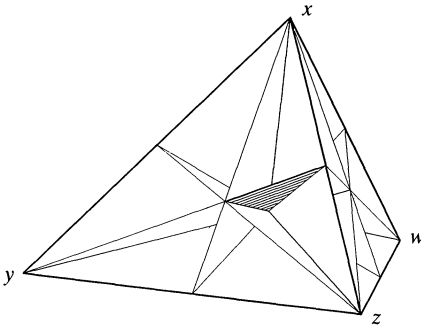
$$\sigma = (x)(yz)$$

$$\omega(\sigma) = \frac{x(2-x)}{(1-x)^2} (ydz - zdy)$$



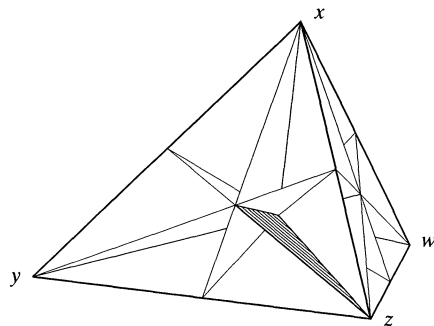
$$\sigma = (yz)(x)$$

$$\omega(\sigma) = ydz - zdy$$



$$\sigma = (xz)(y)(w)$$

$$\omega(\sigma) = \frac{y}{(y+w)} (xdz - zdx)$$



$$\sigma = (z)(xy)(w)$$

$$\omega(\sigma) = \frac{z(2-z)}{(1-z)^2} (xdy - ydx)$$

5. Growth rates of shadow forms. In this section we study the relationship between the growth rate of a shadow form $\omega(\sigma)$ and the allowability properties of σ (in the sense of intersection homology). We

fix the standard orientation of \mathbf{R}^n and orient Δ by requiring that the projection $\Delta \rightarrow \mathbf{R}^n$ to the first n coordinates is orientation preserving. The standard inner product on \mathbf{R}^{n+1} induces an inner product on the tangent spaces of Δ and therefore also on the differential forms $\Omega^k(\Delta)$ which are defined on the interior of Δ . Let $*$ denote the associated Hodge star operator. Thus we may define the q -norm $\|\omega\|_q$ of a differential form ω , where $1 \leq q \leq \infty$ by

$$\|\omega\|_q = \left(\int_{\Delta} f^{q/2} \, \text{dvol}(\Delta) \right)^{1/q}$$

where $\omega \wedge * \omega = f \, \text{dvol}(\Delta)$. A differential form ω is said to be of type L^q if $\|\omega\|_q < \infty$.

Recall [GM1] that a perversity is an integer valued function \bar{p} defined on the positive integers such that $p(0) = p(1) = p(2) = 0$ and $p(\lambda) \leq p(\lambda + 1) \leq p(\lambda) + 1$. A k -dimensional simplex σ in the first barycentric subdivision of Δ is (\bar{p}, k) -allowable iff for each proper face F of Δ we have

$$\dim(\sigma \cap F) \leq k - \lambda + p(\lambda)$$

where $\lambda = n - \dim(F)$.

Definition. If σ is a k -simplex in the first barycentric subdivision of Δ whose vertices are the barycentres of the faces $F_1 < F_2 < \dots < F_{k+1} = \Delta$, we define the profile p_σ of σ to be the following integer valued function:

$$p_\sigma(0) = 0$$

$$p_\sigma(\lambda) = \lambda - (k - i + 1) \text{ if } \lambda \text{ is one of the special values,}$$

$$\lambda = s_{k+1} + s_k + \dots + s_{i+1}.$$

If λ is between two special values, say $\lambda = \lambda_0 + t$ where $\lambda_0 = s_{k+1} + s_k + \dots + s_{i+1}$ and $1 \leq t < s_i$, define $p_\sigma(\lambda) = p_\sigma(\lambda_0) + (t - 1)$.

LEMMA 5.1. *The profile p_σ is the smallest perversity \bar{p} for which σ is (\bar{p}, k) -allowable.*

Proof. This is a simple counting argument. □

THEOREM 5.2. *Let σ be an abstract k -simplex in the first derived complex of Δ , let p_σ denote the profile of σ and let $\omega(\sigma)$ denote the shadow form corresponding to σ . Fix a real number q with $1 \leq q \leq \infty$. Then $\omega(\sigma)$ is in L^q iff $p_\sigma(\lambda) < \lambda/q$ for all integers λ with $1 \leq \lambda \leq n + 1 - s_1$.*

The proof will occupy Section 6, Section 7, and Section 8. In order to estimate $\|\omega(\sigma)\|_q = \|\Phi\eta(\sigma)\|_q$ we will estimate separately $|\Phi|^q$ and $\|\eta(\sigma)\|_q$.

6. Estimates on $|\Phi|$. Fix nonnegative integers $s_1, s_2, \dots, s_k, s_{k+1}$ such that $s_{k+1} = n + 1 - s_1 - s_2 - \dots - s_k$. Fix positive real numbers $v_i, (1 \leq i \leq k)$ such that $0 < v_i < 1$ and $\sum_{i=1}^k v_i < 1$.

PROPOSITION 6.1. *The function $\Phi(v_1, \dots, v_k; s_1, \dots, s_k)$ is bounded as follows:*

$$J \leq \Phi \leq a_2 J$$

where

$$a_2 = \frac{\Gamma(n + 1)}{\Gamma(s_1)\Gamma(s_2) \cdots \Gamma(s_{k+1})} \frac{1}{(s_1)(s_1 + s_2) \cdots (s_1 + s_2 + \cdots + s_{k+1})}$$

and where

$$J = v_1^{s_1} \left(\frac{v_2}{1 - v_1} \right)^{s_2} \cdots \left(\frac{v_k}{1 - v_1 - v_2 - \cdots - v_{k-1}} \right)^{s_k}.$$

Proof. To compute the numerator of Φ , use the change of variables given by $t_1 = u_1/v_1$ and $t_i = (1 - v_1 - v_2 - \cdots - v_{i-1})((u_i/v_i) - (u_{i-1}/v_{i-1}))$ for $2 \leq i \leq k$. The inequalities in the definition of the domain of integration imply that $t_i \geq 0$ and $\sum t_i \leq 1$, and this defines a new domain of integration. Solving for u_i ,

$$u_i = v_i \left(\frac{t_i}{1 - v_1 - \cdots - v_{i-1}} + \frac{t_{i-1}}{1 - v_1 - \cdots - v_{i-2}} + \cdots + \frac{t_2}{1 - v_1} + t_1 \right)$$

we obtain the bound

$$\frac{v_i}{1 - v_1 - v_2 - \cdots - v_{i-1}} \leq u_i \leq \frac{v_i(t_1 + \cdots + t_i)}{1 - v_1 - v_2 - \cdots - v_{i-1}}$$

and

$$1 - t_1 - t_2 - \cdots - t_k \leq 1 - u_1 - u_2 - \cdots - u_k \leq 1.$$

The volume form becomes

$$\begin{aligned} du_k du_{k-1} \cdots du_1 \\ = v_1 \left(\frac{v_2}{1 - v_1} \right) \cdots \left(\frac{v_k}{1 - v_1 - v_2 - \cdots - v_{k-1}} \right) dt_k dt_{k-1} \cdots dt_1 \end{aligned}$$

so we conclude

$$\begin{aligned} J \int t_1^{s_1-1} \cdots t_k^{s_k-1} (1 - t_1 - \cdots - t_k)^{s_{k+1}-1} dt &\leq \Phi \Phi_0 \\ &\leq J \int t_1^{s_1-1} (t_1 + t_2)^{s_2-1} \cdots (t_1 + \cdots + t_k)^{s_k-1} dt \end{aligned}$$

where the integrals are taken over the region $t_i \geq 0$ and $\sum t_i \leq 1$. The integral in the lower limit was previously defined as the ‘‘Dirichlet integral’’ and has the value Φ_0 , while the integral in the upper limit can be evaluated explicitly as $1/(s_1)(s_1 + s_2) \cdots (s_1 + s_2 + \cdots + s_k)$. \square

7. Estimates on $\|\eta(\sigma)\|$. The Whitney form $W_\Delta(S)$ is the pullback to $\Delta = \Delta^n$ of a differential form on \mathbf{R}^{n+1} . We wish to compute $*W_\Delta(S)$ by working with differential forms on the ambient Euclidean space. For this purpose we will use the following two lemmas.

LEMMA 7.1. *Let $i : \Delta \hookrightarrow \mathbf{R}^{n+1}$ denote the inclusion of the standard n simplex into Euclidean space (see Section 1) and let $\theta \in \Omega^r(\mathbf{R}^{n+1})$ be a differential form. Let*

$$dL = \frac{1}{\sqrt{n+1}} \sum_{i=1}^{n+1} dx_i$$

denote the differential form of unit norm, whose kernel is parallel to the simplex Δ , then

$$*i^*(\theta) = \frac{(-1)^{n-r}}{\sqrt{n+1}} i^* * (\theta \wedge dL).$$

Proof. We have an orientation preserving orthogonal decomposition as Riemannian manifolds, $\mathbf{R}^{n+1} \cong H \times L$, where L is the line spanned by the vector $(1, 1, 1, \dots, 1)$ and H is the affine subspace containing Δ . Let π_H and π_L denote projections to H and L respectively. Then

$$\Omega^r(\mathbf{R}^{n+1}) \cong \pi_H^* \Omega^r(H) \otimes \pi_L^* \Omega^0(L) \oplus \pi_H^* \Omega^{r-1}(H) \otimes \pi_L^* \Omega^1(L).$$

With respect to this decomposition, the operations $*$ and i^* are given by

$$i^*(\pi_H^* \alpha_1 \wedge \pi_L^* \beta_1 + \pi_H^* \alpha_2 \wedge \pi_L^* \beta_2) = \beta_1(1, 1, \dots, 1)\alpha_1$$

and

$$\begin{aligned} &*(\pi_H^* \alpha_1 \wedge \pi_L^* \beta_1 + \pi_H^* \alpha_2 \wedge \pi_L^* \beta_2) \\ &= \pi_H^*(*\alpha_1) \wedge \pi_L^*(*\beta_1) + (-1)^{n-r+1} \pi_H^*(*\alpha_2) \wedge \pi_L^*(*\beta_2). \end{aligned}$$

The result follows immediately. □

LEMMA 7.2. *Suppose F is a r dimensional face of the n -simplex, Δ^n . By relabelling coordinates if necessary, let us assume that F is spanned by the vertices $\{e_1, e_2, \dots, e_{r+1}\}$. We denote by $W(F)$ the Whitney form on F : it is the pullback to F of the r -form on \mathbf{R}^{n+1} which is given by*

$$\tilde{W}(F) = r! \sum_{j=1}^{r+1} (-1)^{j+1} x_j dx_1 \wedge dx_2 \wedge \dots \wedge \hat{dx}_j \wedge \dots \wedge dx_{r+1}.$$

Then

$$(1) \quad *h^* \tilde{W}(F) = (-1)^r r! \sum_{j=1}^{r+1} x_j dx_j$$

where $h : \mathbf{R}^{r+1} \rightarrow \mathbf{R}^{n+1}$ is the inclusion of the first $r + 1$ coordinates.

$$(2) \quad \tilde{W}(F) \wedge \sum_{j=1}^{r+1} dx_j = (-1)^r r! \binom{r+1}{\sum_{j=1}^{r+1} x_j} dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{r+1}$$

$$(3) \quad \tilde{W}(F) \wedge \sum_{j=1}^{r+1} x_j dx_j = (-1)^r r! \binom{r+1}{\sum_{j=1}^{r+1} x_j^2} dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{r+1}.$$

Proof. The proof is straightforward. □

PROPOSITION 7.3. *Suppose S_1, S_2, \dots, S_{k+1} forms a partition of the set $\{1, 2, \dots, n + 1\}$ as in Section 4, and let*

$$\theta = W(S_1) \wedge W(S_2) \wedge \cdots \wedge W(S_{k+1})$$

be the corresponding differential $(n - k)$ -form on the n -simplex Δ . Then

$$\theta \wedge * \theta = f \, \text{dvol}(\Delta)$$

where f is bounded above and below, $a_3 F \leq f \leq a_4 F$, with

$$F = (\Sigma S_1)^2 (\Sigma S_2)^2 \cdots (\Sigma S_{k+1})^2$$

$$a_3 = \frac{(k + 1)}{(s_1 s_2 \cdots s_{k+1})} \frac{(\Gamma(s_1) \Gamma(s_2) \cdots \Gamma(s_{k+1}))^2}{(n + 1)}$$

$$a_4 = (k + 1) \frac{(\Gamma(s_1) \Gamma(s_2) \cdots \Gamma(s_{k+1}))^2}{(n + 1)}.$$

Proof. We shall show that

$$\theta \wedge * \theta = a_5 \sum_{i=1}^{k+1} (\Sigma S_1^2) (\Sigma S_2^2) \cdots (\Sigma S_i)^2 \cdots (\Sigma S_{k+1}^2) \, \text{dvol}(\Delta)$$

where $\Sigma S_i^2 = \Sigma \{x_j^2 | j \in S_i\}$ and where $a_5 = (\Gamma(s_1) \Gamma(s_2) \cdots \Gamma(s_{k+1}))^2 / (n + 1)$. The conclusion will then follow from the inequalities of Hölder and Minkowski:

$$\frac{1}{s} (\Sigma S)^2 \leq (\Sigma S^2) \leq (\Sigma S)^2.$$

Clearly θ is the pullback (to Δ) of a differential form $\tilde{\theta} \in \Omega^*(\mathbf{R}^{n+1})$ which is given by the same formula. By Lemma 7.1 we have,

$$\theta \wedge * \theta = i^*(\tilde{\theta}) \wedge *i^*(\tilde{\theta}) = \frac{(-1)^k}{\sqrt{n+1}} i^*(\tilde{\theta} \wedge *(\tilde{\theta} \wedge dL)).$$

By abuse of notation we write

$$\tilde{\theta} = W(S_1) \wedge W(S_2) \wedge \dots \wedge W(S_{k+1})$$

and

$$dL = (d\Sigma S_1 + d\Sigma S_2 + \dots + d\Sigma S_{k+1})/\sqrt{n+1}$$

from which we see

$$\tilde{\theta} \wedge dL = \frac{1}{\sqrt{n+1}} \left(\sum_{i=1}^{k+1} (-1)^{t_i} W(S_i) \wedge W(S_2) \wedge \dots \wedge [W(S_i) \wedge d\Sigma S_i] \wedge \dots \wedge W(S_{k+1}) \right)$$

where $t_i = n - k + i + s_{i+1} + s_{i+2} + \dots + s_{k+1}$. Since this differential form on \mathbf{R}^{n+1} has been written as a product of forms on orthogonal subspaces of dimensions s_1, s_2, \dots, s_{k+1} , the Hodge star operator acts on each factor separately according to equation (1) of Lemma 7.2. The result now follows directly from equations (2) and (3) in Lemma 7.2. □

COROLLARY 7.4. *The pointwise norm of*

$$\eta(\sigma) = \frac{\theta}{(\Sigma S_1)^{s_1} \dots (\Sigma S_{k+1})^{s_{k+1}}}$$

is bounded above and below by a constant times the following quantity

$$(\Sigma S_1)^{1-s_1} (\Sigma S_2)^{1-s_2} \dots (\Sigma S_{k+1})^{1-s_{k+1}}. \quad \square$$

8. Integration of ω . By Proposition 6.1 and Corollary 7.4 we see that $\|\omega(\sigma)\|^q$ is bounded above and below by a constant times the integral of the q^{th} power of the following quantity:

$$\frac{(\Sigma S_1)(\Sigma S_2) \cdots (\Sigma S_{k+1})}{(1 - \Sigma S_1)^{s_2}(1 - \Sigma S_1 - \Sigma S_2)^{s_3} \cdots (1 - \Sigma S_1 - \Sigma S_2 - \cdots - \Sigma S_k)^{s_{k+1}}}.$$

We will integrate this quantity over the simplex Δ .

Let $S = \{i_j\} \subset \{1, 2, \dots, n + 1\}$ be a subset of s elements which corresponds to a face F of Δ as in Section 4. For simplicity of notation we will write y_1, y_2, \dots, y_s in place of $x_{i_1}, x_{i_2}, \dots, x_{i_s}$. Let

$$\Sigma S = \sum_{i=1}^s y_i.$$

We will use the symbol $\int_{\Sigma S=0}^A f dV(S)$ to denote the following integral

$$\int_{y_1=0}^A \int_{y_2=0}^{A-y_1} \cdots \int_{y_s=0}^{A-y_1-y_2-\cdots-y_{s-1}} f dy_s dy_{s-1} \cdots dy_1.$$

LEMMA 8.1. *Assuming $q \geq 1$ as above, the function*

$$J(A) = \int_{\Sigma=0}^A (\Sigma S)^q (A - \Sigma S)^{r-1} dV(S)$$

converges iff $r > 0$, and in this case there are constants a_6 , and a_7 such that

$$a_6 A^{q+r+s-1} \leq J(A) \leq a_7 A^{q+r+s-1}.$$

Proof. Use the change of variables $y_s = At - y_1 - y_2 - \cdots - y_{s-1}$ (so $\Sigma S = At$) in the first integral (let us call it J_1), and set $\Sigma S' = y_1 + y_2 + \cdots + y_{s-1}$ to obtain

$$J_1 = \int_{y_s=0}^{A-\Sigma S'} (\Sigma S)^q (A - \Sigma S)^{r-1} dy_s = A^{q+r} \int_{t=(\Sigma S')/A}^1 t^q (1 - t)^{r-1} dt$$

which diverges if $r \leq 0$. Now suppose $r > 0$. Since t^q is monotonically increasing on $[0, 1]$, the integral J_1 is bounded above and below by

$$\frac{1}{r} (\Sigma S')^q (A - \Sigma S')^r \leq J_1 \leq A^{q+r} \int_0^1 t^q (1 - t)^{r-1} dt.$$

This last integral is an ‘‘Eulerian integral’’ and has the value $\Gamma(q + 1)\Gamma(r)/(q + r + 1)$ ([WW]). Therefore $J(A)$ is bounded as follows:

$$\begin{aligned} \frac{1}{r} \int_{\Sigma'=0}^A (\Sigma S')^q (A - \Sigma S')^r dV(S') &\leq J(A) \\ &\leq A^{q+r} \frac{\Gamma(q + 1)\Gamma(r)}{\Gamma(q + r + 1)} \int_{\Sigma'=0}^A 1 dV(S') \end{aligned}$$

from which we conclude by induction that

$$\begin{aligned} \frac{A^{q+r+s-1}}{\Gamma(r + s - 1)} \int_0^1 t^q (1 - t)^{r+s-2} dt &\leq J(A) \\ &\leq A^{q+r} \frac{\Gamma(q + 1)\Gamma(r)}{\Gamma(q + r + 1)} A^{s-1} \frac{1}{\Gamma(s)}. \end{aligned}$$

This completes the proof and in fact we see that we may take

$$a_6 = \frac{\Gamma(q + 1)}{\Gamma(q + r + s)} \quad \text{and} \quad a_7 = \frac{\Gamma(q + 1)\Gamma(r)}{\Gamma(q + r + 1)\Gamma(s)}. \quad \square$$

Proof of Theorem 5.2. We must decide whether the following integral converges:

$$\begin{aligned} &\int_{\Sigma_1=0}^1 (\Sigma_1)^q (1 - \Sigma_1)^{-qs_2} \int_{\Sigma_2=0}^{1-\Sigma_1} (\Sigma_2)^q (1 - \Sigma_1 - \Sigma_2)^{-qs_3} \cdots \\ &\int_{\Sigma_{k-1}=0}^{1-\Sigma_1-\cdots-\Sigma_{k-2}} (\Sigma_{k-1})^q (1 - \Sigma_1 - \cdots - \Sigma_{k-1})^{-qs_k} \\ &\int_{\Sigma_k=0}^{1-\Sigma_1-\cdots-\Sigma_{k-1}} (\Sigma_k)^q (1 - \Sigma_1 - \cdots - \Sigma_k)^{q-qs_{k+1}} \int_{\Sigma'_{k+1}=0}^{1-\Sigma_1-\cdots-\Sigma_k} 1 d\Delta' \end{aligned}$$

where Σ_k denotes $\Sigma S_k = \{x_j | j \in S_k\}$, where $\Sigma'_{k+1} = \Sigma_{k+1} - x_{n+1}$, and where $d\Delta' = d\Sigma'_{k+1} \Sigma_k \cdots d\Sigma_1$. (In other words, no integration is per-

formed over the last coordinate y_{n+1} because the sum of all the coordinates is 1).

The value of the last integral is

$$\frac{1}{\Gamma(s_{k+1})} (1 - \Sigma S_1 - \Sigma S_2 - \dots - \Sigma S_k)^{s_{k+1}-1}.$$

We now perform the remaining integrals successively, using Lemma 8.1. Thus the ΣS_k integral converges iff $q - qs_{k+1} + s_{k+1} > 0$, and in this case it is bounded above and below by a constant times

$$(1 - \Sigma S_1 - \dots - \Sigma S_{k-1})^{q(2-s_{k+1})+s_{k+1}+s_k-1}.$$

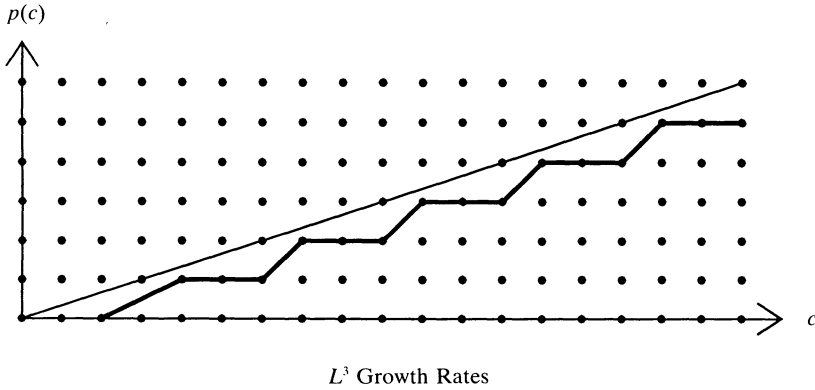
Similarly the ΣS_{k-1} integral converges iff $-qs_k + q(2 - s_{k+1}) + s_{k+1} + s_k > 0$, and so on. We conclude by induction that $\omega(\sigma)$ is in L^q iff the following conditions hold:

$$\begin{aligned} & s_{k+1} - 1 < (s_{k+1})/q \\ & s_k + s_{k+1} - 2 < (s_k + s_{k+1})/q \\ & \dots \\ (*) \quad & s_{i+1} + \dots + s_{k+1} - (k - i + 1) < (s_{i+1} + \dots + s_{k+1})/q \\ & \dots \\ & n + 1 - s_1 - k < (n + 1 - s_1)/q. \end{aligned}$$

Finally we claim the conditions (*) hold iff the profile p_σ of σ satisfies

$$(**) \quad p_\sigma(\lambda) < \lambda/q$$

for all λ with $1 \leq \lambda \leq n + 1 - s_1$. First note that condition (*) is equivalent to the conditions $p_\sigma(\lambda) < \lambda/q$ for every ‘‘special value’’ $\lambda = s_{i+1} + s_{i+2} + \dots + s_{k+1}$ because for these values the profile satisfies $p_\sigma(\lambda) = \lambda - (k - i + 1)$. But it is easy to see that $p_\sigma(\lambda) < \lambda/q$ for the special values of λ iff $p_\sigma(\lambda) < \lambda/q$ for all values of λ because the special values are the only points of concavity on the graph of $p_\sigma(\lambda)$. \square



9. Globalization. In this section we consider a triangulated pseudomanifold K and we show how the L^2 forms $\omega(\sigma)$ of the preceding sections may be patched together to form a chain complex whose homology is the intersection homology of K . Our goal is to associate to any (\bar{m}, i) -allowable chain $c \in IC_i(K')$ a collection of L^2 differential forms on the n -simplices of K which agree on the $n - 1$ dimensional simplices. Unfortunately there is a technical complication introduced by the possibility that c might be contained in the $n - 1$ skeleton of K , for in this case we have no way of constructing corresponding differential forms on the n -simplices of K . We will rule out this possibility by assuming the chain c is “transverse” to the $n - 1$ skeleton of K .

Throughout this section, K will denote a purely n dimensional simplicial complex linearly embedded in \mathbf{R}^n , which is also an oriented pseudomanifold, stratified by the simplices of K . Let K' denote the barycentric subdivision of K , with respect to some fixed choice of barycenters. Let $K_{(j)}$ denote the j -skeleton of K and $\Sigma = K_{(n-2)}$ denote the “singularity set” of K .

Definition. A chain $c \in C_i(K')$ is *transverse* if $\dim(|c| \cap K_{(n-1)}) \leq i - 1$ and if, for each $n - 1$ dimensional simplex F in K , we have $\dim(|\partial c| \cap F) \leq i - 2$. The transverse chains (with \mathbf{R} coefficients) form a complex, which we denote by $C_*^T(K'; \mathbf{R})$.

Remark. If $c = \sum a_j \sigma_j$ is a transverse i -chain then each σ_j is contained in some n -dimension simplex Δ of K , and is not contained in any proper face of Δ . Consequently σ_j is the kind of simplex which was

considered in Section 1 and for which we were able to associate a shadow form.

Now fix a perversity \bar{p} . Define $I^{\bar{p}}C_i(K'; \mathbf{R})$ to be the group of all chains $\xi \in C_i(K'; \mathbf{R})$ in the first barycentric subdivision of K , such that ξ is (\bar{p}, i) -allowable, and $\partial\xi$ is $(\bar{p}, i - 1)$ -allowable with respect to the stratification of K which is given by the simplices of K . Define the complex of allowable transverse chains $I^{\bar{p}}C_*^T(K'; \mathbf{R})$ to be the intersection $I^{\bar{p}}C_*(K'; \mathbf{R}) \cap C_*^T(K'; \mathbf{R})$.

PROPOSITION 9.1. *The homology of the complex $I^{\bar{p}}C_*^T(K'; \mathbf{R})$ is the intersection homology $IH_*^{\bar{p}}(K; \mathbf{R})$.*

Proof. The proof is surprisingly delicate since the chains are restricted to those which come from the first barycentric subdivision. See [GM3] for a proof without the transversality assumption. The addition of the transversality condition is handled as follows: Any chain $\xi \in IC_i(K')$ is a sum of i -dimensional simplices of two types: those which are contained in an $(n - 1)$ -dimensional simplex of K , and those which are not. This gives rise to a canonical direct sum decomposition of $IC_i(K')$ into two subgroups, which we denote as follows:

$$IC_i(K') \cong IC_i(K' - K_{(n-1)}) \oplus IC_i(K'_{(n-1)}).$$

The boundary homomorphism preserves the second subgroup but not the first, so we have

$$\partial(\xi_1, \xi_2) = (\partial_1\xi_1, \partial_2\xi_1 + \partial\xi_2)$$

where $\partial_1\xi_1$ (resp. $\partial_2\xi_1$) is the component of $\partial\xi_1$ which lies in the first (resp. second) subgroup. Clearly,

$$IC_i^T(K') = IC_i(K' - K_{(n-1)}) \cap \ker(\partial_2).$$

However, $\partial_2|_{IC_i(K' - K_{(n-1)})}$ is surjective since each allowable $(i - 1)$ -simplex in $K'_{(n-1)}$ is the face of an allowable i -simplex in K' . It is a simple exercise in homological algebra to verify that the surjectivity of ∂_2 implies that the subcomplex IC_*^T has the same homology as the full complex. \square

Definition. A differential form on $K - \Sigma$ is a collection of smooth forms ω_Δ defined on each n dimensional simplex Δ of K , which agree on the interior of each $n - 1$ dimensional simplex, i.e. whenever $F = \Delta_1 \cap \Delta_2$ is an $n - 1$ dimensional face of two n -simplices, then

$$\omega_{\Delta_1}|_{F^\circ} = \omega_{\Delta_2}|_{F^\circ}.$$

Definition. Let $\tilde{\Omega}^i(K)$ denote the vectorspace of all linear combinations of shadow forms $\omega(\sigma)$ such that σ is an $n - i$ dimensional simplex in K' . Let Ω^i denote the subspace of $\tilde{\Omega}^i$ consisting of those linear combinations which agree on each $n - 1$ dimensional simplex of K . Let $\Omega_{(q)}^i$ denote the subspace of Ω^i consisting of forms ω such that $\omega \in L^q$ and $d\omega \in L^q$.

THEOREM 9.2. *The operation ω which associates to each transverse chain $c = \sum a_j \sigma_j \in C_i^T(K')$ the differential form $\omega(c)$*

$$\omega(c)_\Delta = \sum a_j \omega_\Delta(\sigma_j) \cdot \chi_\Delta(\sigma_j)$$

(where $\omega_\Delta(\sigma)$ is the shadow form defined in Section 1, and where χ_Δ is the characteristic function of Δ , i.e. it takes the value 1 on σ iff $\sigma \subset \Delta$) induces a chain isomorphism

$$\omega : C_i^T(K'; \mathbf{R}) \rightarrow \Omega^{n-i}(K).$$

Furthermore if $1 \leq q \leq \infty$ and if $\bar{p} = \bar{p}(q)$ is the largest perversity function such that $p(\lambda) < \lambda/q$ for all λ , then ω restricts to a chain isomorphism

$$I^{\bar{p}}C_i^T(K'; \mathbf{R}) \rightarrow \Omega_{(q)}^{n-i}(K).$$

Proof. The mapping ω is an isomorphism of groups, by definition. It commutes with the boundary because of Stokes' theorem (see [BT] p. 62). □

COROLLARY 9.3. *If \bar{p} and q are related as above, then the cohomology of the complex $\Omega_{(q)}^*(K)$ is naturally isomorphic to the intersection homology $IH_{*}^{\bar{p}}(K; \mathbf{R})$.*

10. Products of shadow forms. For differentiable manifolds, the wedge of differential forms corresponds, by deRham-Poincaré to the

intersection of cycles. In this section we relate the wedge of shadow forms $\omega(\sigma) \wedge \omega(\sigma')$ to the shadow form of the intersection $\omega(\sigma \cap \sigma')$. For this, we need the following notations and definitions:

Notation. Let σ be an abstract k -simplex in the first derived complex of Δ . For $p \in \text{int}(\Delta)$ we denote by $[\sigma(p)]$ the affine plane spanned by $\sigma(p)$. If σ' is another abstract simplex, of dimension k' in the first derived complex of Δ , we denote the intersection dimensions by:

$$q = \dim([\sigma(p)] \cap [\sigma'(p)]) \quad \text{and} \quad r = \dim(\sigma(p) \cap \sigma'(p)).$$

These dimensions do not depend of the choice of $p \in \text{int}(\Delta)$ and we have $q \geq r$ and $q \geq k + k' - n$.

Definition. The simplices σ and σ' are affine transverse iff $q = k + k' - n$, i.e.:

$$\text{codim}([\sigma(p)] \cap [\sigma'(p)]) = \text{codim}[\sigma(p)] + \text{codim}[\sigma'(p)].$$

Definition. The simplices σ and σ' are topologically transverse iff $r = k + k' - n$, i.e.:

$$\text{codim}(\sigma \cap \sigma') = \text{codim}(\sigma) + \text{codim}(\sigma').$$

Denote by S_1, S_2, \dots, S_{k+1} and $S'_1, S'_2, \dots, S'_{k'+1}$ respectively the groups of indices corresponding to σ and σ' as in Section 4.

THEOREM 10.1. 1) *The simplices σ and σ' are affine transverse iff $\omega(\sigma) \wedge \omega(\sigma') \neq 0$.*

2) *If the simplices σ and σ' are affine transverse and topologically transverse (i.e. $q = r = k + k' - n$) then:*

$$\omega(\sigma) \wedge \omega(\sigma') = \pm \frac{\Psi_\sigma \Psi_{\sigma'}}{\Psi_{\sigma \cap \sigma'}} \left(\prod_{i=1}^{n+1} x_i \right) \omega(\sigma \cap \sigma')$$

where, for $\sigma = (S_1)(S_2) \cdots (S_{k+1})$,

$$\Psi_\sigma = \frac{\Phi(\Sigma S_1, \Sigma S_2, \dots, \Sigma S_k; s_1, s_2, \dots, s_k)}{(\Sigma S_1)^{s_1} (\Sigma S_2)^{s_2} \cdots (\Sigma S_{k+1})^{s_{k+1}}}.$$

Note that $\sigma \cap \sigma' = (T_1)(T_2) \cdots (T_{r+1})$ where the groups of indices T_j are described as follows: Set $r_0 = r'_0 = 0$ and define by induction r_j and r'_j as the minimal numbers such that:

$$\bigcup_{\alpha=r_j+1}^{r_{j+1}} S_\alpha = \bigcup_{\beta=r'_j+1}^{r'_{j+1}} S'_\beta.$$

Then this union is precisely T_j .

The rest of this section is devoted to the proof of Theorem 10.1. Define a bipartite graph $G(\sigma, \sigma')$, as follows:

The vertices are labelled $S_1, S_2, \dots, S_{k+1}, S'_1, S'_2, \dots, S'_{k'+1}$. The edges connecting S_α and S'_β are labelled by the indices they have in common.

LEMMA 10.2. *The graph $G(\sigma, \sigma')$ has $q + 1$ connected components.*

Proof. Suppose the vertices $S_{\alpha_1}, S_{\alpha_2}, \dots, S_{\alpha_r}, S'_{\beta_1}, S'_{\beta_2}, \dots, S'_{\beta_r}$, form a connected component of the graph. Then we have a minimal expression

$$(*) \quad S_{\alpha_1} \cup S_{\alpha_2} \cup \cdots \cup S_{\alpha_r} = S'_{\beta_1} \cup S'_{\beta_2} \cup \cdots \cup S'_{\beta_r}.$$

We denote this union by L , and we associate to it a vertex in the intersection $[\sigma(p)] \cap [\sigma'(p)]$, namely the barycenter of the face $\langle x_i \rangle_{i \in L}$.

The vertices obtained in this way are clearly linearly independent. They span the plane $[\sigma(p)] \cap (\sigma'(p))$ because every vertex in this plane is described by an expression (*) but not necessarily a minimal such expression. In summary, there are $q + 1$ such vertices, each of which is associated to a connected component of $G(\sigma, \sigma')$. □

LEMMA 10.3. (a) *If $S_1 \cap S_2 = \{x_i\}$ then:*

$$W(S_1) \wedge W(S_2) = \pm \frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1 + s_2 - 1)} x_i W(S_1 \cup S_2).$$

(b) *If $\text{card}(S_1 \cap S_2) \geq 2$ then:*

$$W(S_1) \wedge W(S_2) = 0.$$

Proof. The proof is straightforward (the case $S_1 \cap S_2 = \emptyset$ was the formula (a) in the proof of Lemma 4.2). □

To prove Theorem 10.1 we will calculate the product $W(S_1) \wedge \cdots \wedge W(S_{k+1}) \wedge W(S'_1) \wedge \cdots \wedge W(S'_{k'+1})$ by repeated application of Lemma 10.3 within each connected component of the graph $G(\sigma, \sigma')$.

Proof of the Theorem 10.1. (1) The simplices σ and σ' are affine transverse iff the graph $G(\sigma, \sigma')$ does not contain a cycle, since both statements are equivalent to the equality:

$$\#(\text{edges}) = \#(\text{vertices}) - \#(\text{connected components}).$$

However, if some connected component contains a cycle, then Lemma 10.3(b) implies that the corresponding wedge of differential forms vanishes.

(2) If σ and σ' are affine transverse and topologically transverse, the differential part of $\omega(\sigma) \wedge \omega(\sigma')$ is the product of Whitney forms:

$$\begin{aligned} &W(S_1) \wedge W(S_2) \wedge \cdots \wedge W(S_{k+1}) \wedge W(S'_1) \wedge W(S'_2) \wedge \cdots \wedge W(S'_{k'+1}) \\ &= \pm \frac{\Gamma(s_1) \cdots \Gamma(s_{k+1}) \cdot \Gamma(s'_1) \cdots \Gamma(s'_{k'+1})}{\Gamma(t_1) \cdots \Gamma(t_{r+1})} \left(\prod_{i=1}^{n+1} x_i \right) W(T_1) \\ &\qquad \qquad \qquad \wedge W(T_2) \wedge \cdots \wedge W(T_{r+1}) \end{aligned}$$

where $s_i = |S_i|$, $s'_i = |S'_i|$ and $t_j = |T_j|$. This gives the differential part of $\omega(\sigma \cap \sigma')$ since in this case, the groups of indices L_j and T_j coincide up to reordering. The formula is then a consequence of Theorem 4.1. □

Remarks. 1) Even if σ and σ' are affine transverse, it is not true that $\omega(\sigma) \wedge \omega(\sigma')$ and $\omega(\sigma \cap \sigma')$ are proportional (by a function), but there is a differential $q - r$ -form η such that $\omega(\sigma \cap \sigma') = \omega(\sigma) \wedge \omega(\sigma') \wedge \eta$.

2) The differential $n - r$ -form $\omega(\sigma \cap \sigma')$ is never 0.

Examples (in Δ^3). 1) Let $\sigma = (x_1 x_2)(x_3)(x_4)$ and $\sigma' = (x_4)(x_1 x_3)(x_2)$. Then $\sigma \cap \sigma' = \{p\}$ and $[\sigma] \cap [\sigma']$ is the line spanned by $(\hat{x}_4)(\widehat{x_1 x_2 x_3})$. In this case, σ and σ' are affine transverse, but not topologically transverse. We have: $r = 0$ and $q = k + k' - n = 1$; $L_1 = (1, 2, 3)$, $L_2 = (4)$. The differential forms are $\omega(\sigma \cap \sigma') = W(\Delta)$ and

$$\omega(\sigma) \wedge \omega(\sigma') = \frac{x_1 x_3 x_4 (2 - x_4)}{(x_3 + x_4)(1 - x_4)^2} W(x_1, x_2, x_3).$$

2) Let $\sigma = (x_1, x_2)(x_3)(x_4)$ and $\sigma' = (x_4)(x_1, x_2, x_3)$. Then $\sigma \cap \sigma' = \{p\}$ and $[\sigma] \cap [\sigma']$ is the line spanned by $(\hat{x}_4)(\widehat{x_1 x_2 x_3})$. In this case, σ and σ' are topologically transverse, but not affine transverse. We have: $r = k + k' - n = 0$ and $q = 1$; $L_1 = (1, 2, 3)$, $L_2 = (4)$. The differential forms are $\omega(\sigma \cap \sigma') = W(\Delta)$ and $\omega(\sigma) \wedge \omega(\sigma') = 0$.

3) Let $\sigma = (x_1, x_2)(x_3)(x_4)$ and $\sigma' = (x_4)(x_1, x_2)(x_3)$. Then $\sigma \cap \sigma' = \{p\}$ and $[\sigma] \cap [\sigma']$ is the plane spanned by $(\widehat{x_1 x_2})(\hat{x}_3)(\hat{x}_4)$. In this case, σ and σ' are neither topologically transverse nor affine transverse. We have: $r = 0$, $k + k' - n = 1$ and $q = 2$; $L_1 = (1, 2)$, $L_2 = (3)$ and $L_3 = (4)$. The differential forms are $\omega(\sigma \cap \sigma') = W(\Delta)$ and $\omega(\sigma) \wedge \omega(\sigma') = 0$.

4) Let $\sigma = (x_1)(x_2, x_3)(x_4)$ and $\sigma' = (x_1, x_2)(x_3)(x_4)$. Then $\sigma \cap \sigma' = (x_1, x_2, x_3)(x_4)$ and $[\sigma] \cap [\sigma']$ is the line spanned by $(\widehat{x_1 x_2 x_3})(\hat{x}_4)$. In this case, σ and σ' are topologically transverse and affine transverse. We have: $r = k + k' - n = q = 1$; $L_1 = (1, 2, 3)$ and $L_2 = (4)$. The differential forms are $\omega(\sigma \cap \sigma') = W(x_1, x_2, x_3)$ and

$$\omega(\sigma) \wedge \omega(\sigma') = \frac{x_1 x_2 x_3 (2 - x_1)}{(1 - (x_1 + x_2))(1 - x_1)^2} W(x_1, x_2, x_3).$$

11. Whitney forms. Whitney's form $W(S)$ may be recovered as sums of shadow forms corresponding to the dual cone decomposition of K . Fix an n dimensional simplex Δ and let F denote an r dimensional face of Δ . The dual cell $D(F)$ consists of the union of all abstract simplices σ in the first derived complex of Δ which are spanned by vertices $\hat{F}, \hat{F}_1, \dots, \hat{F}_r$ with $F < F_1 < \dots < F_r$.

PROPOSITION 11.1. *The Whitney form $W(F)$ is equal to the sum of the shadow forms*

$$W(F) = \Sigma\{\omega(\sigma) \mid \sigma \text{ is an } n - r \text{ dimensional simplex in } D(F)\}.$$

Remarks. The Sullivan-Whitney theorem [Su] [W] that the cohomology of the complex of Whitney forms is isomorphic to the cohomology of K is thus equivalent to the fact ([Bra]) that the homology of the chain complex of dual cones is isomorphic to the cohomology of

K . Whitney forms are bounded (*i.e.* in L^∞); however there are bounded forms which are not Whitney forms. In fact the cohomology of the complex of bounded shadow forms is precisely the perversity 0 intersection homology of K , which coincides with the cohomology of K if K is normal [GM1].

Proof. For notational simplicity we set $k = n - r$. By renumbering the vertices of Δ so that F is spanned by $\{e_1, e_2, \dots, e_{r+1}\}$, and using the notation of Section 4, each k dimensional simplex σ in the dual cone $D(F)$ is associated to a decomposition of the set $\{1, 2, \dots, n + 1\}$ into $k + 1$ nonempty subsets $S_1 \cup S_2 \cup \dots \cup S_{k+1}$ with $S_1 = \{1, 2, \dots, r + 1\}$ (corresponding to F) and with $s_j = |S_j| = 1$ for $2 \leq j \leq k + 1$. Thus the k simplices σ in $D(F)$ are in one to one correspondence with permutations of the set $\{r + 2, r + 3, \dots, n + 1\}$. A permutation τ corresponds to the choices $S_j = \{\tau(r + j)\}$. Let us denote this simplex by σ_τ . We have

$$\omega(\sigma_\tau) = (-1)^{k(k+1)/2} \Phi(\Sigma S_1, x_{\tau(r+2)}, \dots, x_{\tau(n+1)}; r + 1, 1, 1, \dots, 1) \frac{W(S_1)}{(\Sigma S_1)^r}.$$

In particular, the forms $\omega(\sigma_\tau)$ for various choices of τ are all proportional to $W(S_1)$. The factor of proportionality $\Phi = \Phi_\tau$ can be evaluated using the change of variables given in Section 4, and it equals

$$\Phi_\tau = (\Sigma S_1)^r \frac{x_{\tau(r+2)}}{(1 - \Sigma S_1)} \frac{x_{\tau(r+3)}}{(1 - \Sigma S_1 - x_{\tau(r+2)})} \dots \frac{x_{\tau(n+1)}}{(1 - \Sigma S_1 - x_{\tau(r+2)} - \dots - x_{\tau(n)})}.$$

Summing over all $(n - r)!$ permutations τ gives

$$\sum_\tau \Phi_\tau = (\Sigma S_1)^r$$

which completes the proof. □

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