TOROIDAL AND REDUCTIVE BOREL-SERRE COMPACTIFICATIONS OF LOCALLY SYMMETRIC SPACES

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Introduction

During the last twenty years, several topological approaches have been developed in order to study the action of Hecke operators on the cohomology of arithmetic groups and of Shimura varieties. Let $X{=}\Gamma\backslash G/K$ be a Hermitian locally symmetric space (where $G{=}\mathbf{G}(\mathbb{R})$ is the real points of a semisimple algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$, $K \subset G$ is a maximal compact subgroup, $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is a neat arithmetic subgroup, and G/K is assumed to carry a G-invariant complex structure). A central object of study is the Baily-Borel Satake compactification $\overline{X}^{BB}$: it is a complex algebraic variety which may be highly singular. One approach to understanding the topology of this space is through the toroidal compactification $\overline{X}^{tor}_{\Sigma} \to \overline{X}^{BB}$ of [AMRT]. It is not unique but involves a choice Σ of Γ-equivariant polyhedral cone decomposition of certain self adjoint homogeneous cones; if Σ is chosen sufficiently fine then the toroidal compactification is a resolution of singularities of $\overline{X}^{BB}$. Essential use of this resolution was made, for example, by E. Looijenga [L] in his proof of the Zucker conjecture.

A second approach to understanding the Baily Borel compactification involves the so-called reductive Borel-Serre compactification $\overline{X}^{RBS} \to \overline{X}^{BB}$. This compactification is neither complex algebraic, nor is it smooth. Nevertheless, its singularities are easily understand. In a series of papers ([GM1], [GM2], [GHM], [GKM]) it was shown that Arthur's L^2 Lefschetz formula for Hecke correspondences on X may be interpreted term for term as the Lefschetz fixed point formula for the weighted cohomology of $\overline{X}^{RBS}$. One is therefore led to the problem of comparing these two "resolutions" of $\overline{X}^{BB}$. In [HZ], M. Harris and S. Zucker suggested that there may be little hope in comparing these spaces, by conjecturing that their greatest common quotient is the Baily Borel compactification $\overline{X}^{BB}$. This conjecture was shown [Ji] by L. Ji to be false except in several low rank cases, but he determined that the greatest common quotient is something only slightly larger than $\overline{X}^{BB}$.

In this paper we show (assuming Σ is chosen sufficiently fine, cf. §3.1, §3.7), that the projection $g : \overline{X}^{tor}_{\Sigma} \to \overline{X}^{BB}$ is homotopic to a mapping g' which factors through $\overline{X}^{RBS}$, and that g' may be taken to coincide with g on the complement of an arbitrarily small regular neighborhood of the boundary $\partial\overline{X}^{tor}_{\Sigma} = \overline{X}^{tor}_{\Sigma} - X$. It follows, for example, that the compact support cohomology of X, and the cohomology of X and its compactifications are all related in a single sequence of compatible homomorphisms,

$$H_c^*(X) \to H^*(\overline{X}^{BB}) \to H^*(\overline{X}^{RBS}) \to H^*(\overline{X}_\Sigma^{tor}) \to H^*(X).$$

Now let us give a precise statement of the main result (§7.2). Let $\hat{X} \subset \overline{X}^{RBS} \times \overline{X}_\Sigma^{tor}$ denote the closure of the diagonal embedding of X in these two compactifications. Let θ_1 and θ_2 denote the projections to the first and second factor.

Theorem A. *If the polyhedral decomposition Σ is chosen sufficiently fine then the fibers of the projection $\theta_2 : \hat{X} \to \overline{X}_\Sigma^{tor}$ are contractible.*

The mapping $g' : \overline{X}_\Sigma^{tor} \to \overline{X}_{BB}$ is then obtained as the composition

$$\overline{X}_\Sigma^{tor} \xrightarrow{\tau} \hat{X} \xrightarrow{\theta_1} \overline{X}^{RBS} \to \overline{X}^{BB}$$

where τ is a homotopy inverse to θ_2.

This result was conjectured by R. MacPherson and M. Rapoport [R] in 1991 and was verified by them in the case that the rational rank of $\mathbf{G}$ is 1. In this case, if a point $x \in \overline{X}_\Sigma^{tor}$ corresponds to a (closed) polyhedral cone $\sigma \in \Sigma$, then the fiber $\theta_2^{-1}(x)$ is not only contractible but it may be canonically identified with the quotient of $\sigma - \{0\}$ under homotheties, i.e. it is homeomorphic to a convex polyhedron. The case of higher rank turned out to be much more difficult than we expected: the fibers are obtained from the associated convex polyhedral cones by a sequence of blowups.

The proof of Theorem A is an immediate consequence of two difficult technical results: theorem C (§4.2) (which is the key statement relating convergence in the reductive Borel-Serre compactification to convergence in the toroidal compactification) and theorem B (§2.9) (which is the main result concerning contractibility of the fibers of certain mappings between compactifications of self adjoint homogeneous cones).

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§1. Symmetric Spaces

In this section we review some basic results of Borel and Serre [BS] and establish the notation which we will use throughout this paper.

(1.1) Parabolic subgroups. Algebraic groups will be designated by bold face type ($\mathbf{G}$, $\mathbf{P}$, etc.). If an algebraic group is defined over the real numbers then its group of real points will be in Roman ($G = \mathbf{G}(\mathbb{R})$, $P = \mathbf{P}(\mathbb{R})$, etc.). The connected component of the identity is denoted with a superscript 0 ($\mathbf{G}^0$, $\mathbf{P}^0$, etc.). If $\mathbf{S}$ is an algebraic torus then the identity component of the group of real points will be denoted by $A = \mathbf{S}(\mathbb{R})^0$, and may be inaccurately referred to as a torus. If $\mathbf{G}$ is a reductive algebraic group which is defined over the rational numbers $\mathbb{Q}$, we denote by $\mathbf{S_G}$ the greatest $\mathbb{Q}$-split torus in the center of $\mathbf{G}$, and set $A_G = \mathbf{S_G}(\mathbb{R})^0$. Then the group of real points splits as a direct product

$$G = A_G \times {}^0\mathbf{G}(\mathbb{R}) \tag{1.1.1}$$

where

$${}^0\mathbf{G} = \bigcap_{\chi} \ker(\chi^2) \tag{1.1.2}$$

denotes the intersection of the kernels of all the algebraically defined rational characters $\chi \in \mathrm{Mor}(\mathbf{G}, GL_1)$. The group ${}^0G \subset G$ contains all compact and arithmetic subgroups of G.

For any parabolic subgroup $\mathbf{P} \subset \mathbf{G}$ we denote by $\boldsymbol{\mathcal{U}}_\mathbf{P}$ the unipotent radical of $\mathbf{P}$, and by $\nu_P : \mathbf{P} \to \mathbf{L_P} = \mathbf{P}/\boldsymbol{\mathcal{U}}_\mathbf{P}$ the projection to the Levi quotient. We denote by

$$M_P = {}^0L_P. \tag{1.1.3}$$

If $\mathbf{S_P} \subset \mathbf{L_P}$ denotes the greatest $\mathbb{Q}$-split torus in the center of $\mathbf{L_P}$ then $L_P = A_P M_P$ splits as a (commuting) direct product. Any lift $i : \mathbf{L_P} \to \mathbf{P}$ of $\mathbf{L_P}$ determines a *Langlands decomposition* (which is a semi-direct product),

$$P = \mathcal{U}_P i(A_P M_P). \tag{1.1.4}$$

Choose a minimal rational parabolic subgroup $\mathbf{Q}_0 \subset \mathbf{G}$ and call it *standard*. Choose a rationally defined lift $i : \mathbf{L_{Q_0}} \to \mathbf{Q_0}$ and let $\mathbf{S} = i(\mathbf{S_{Q_0}})$ be the resulting lift of the greatest $\mathbb{Q}$-split torus in the center of $\mathbf{L}(\mathbf{Q_0})$, so that $\mathbf{S} \subset \mathbf{Q_0} \subset \mathbf{G}$. Then $\mathbf{S}$ is a maximal $\mathbb{Q}$-split torus in $\mathbf{G}$. The root system $\Phi(\mathbf{S}, \mathbf{G})$ admits a linear order so that the positive roots $\Phi^+(\mathbf{S}, \mathbf{G})$ are those occurring in $\boldsymbol{\mathcal{U}}_\mathbf{P}$. Let $\Delta = \Delta(\mathbf{S}, \mathbf{G})$ denote the resulting set of simple positive roots. The elements $\phi \in \Delta$ are trivial on $\mathbf{S}_G$ and form a basis for the character module $\chi(\mathbf{S}/\mathbf{S}_G) \otimes_\mathbb{Z} \mathbb{Q}$.

The rational parabolic subgroups which contain Q_0 are in one to one correspondence with subsets $I \subset \Delta$. For a given subset $I \subset \Delta$ define

$$\mathbf{S}(I) = \left(\bigcap_{\phi \in I} \ker(\phi) \right)^0 . \tag{1.1.5}$$

Then the corresponding parabolic subgroup is

$$\mathbf{P} = \mathbf{P}(I) = Z(\mathbf{S}(I)) \boldsymbol{\mathcal{U}}_{\mathbf{Q_0}} = Z(\mathbf{S}(I)) \boldsymbol{\mathcal{U}}_\mathbf{P} \tag{1.1.6}$$

(the latter decomposition being a semidirect product). Then $\mathbf{S}(I) = \mathbf{S}_{P(I)}$ and $Z(\mathbf{S}(I))$ is a lift of the Levi quotient $\mathbf{L}_{\mathbf{P}(I)}$. We denote by

$$\Delta_P = \{\phi | \mathbf{S}(I)\}_{\phi \in \Delta - I} \tag{1.1.7}$$

the restrictions of the remaining simple roots to the torus $\mathbf{S}(I)$; they form a rational basis for the character module $\chi(\mathbf{S}(I)/\mathbf{S}_G)\otimes\mathbb{Q}$. If $I\subset J\subset\Delta$ then $\mathbf{S}(I)\supset\mathbf{S}(J)$ and $\mathbf{P}(I)\subset\mathbf{P}(J)$.

Let $K\subset G$ be a maximal compact subgroup and define $D=G/K$. The space D is referred to as a "generalized symmetric space". If $\mathbf{S}_G$ is not trivial, then we denote by $[D]=G/KA_G$ the quotient of D under the identity component of this central torus. There is a unique basepoint $x_0\in D$ with $K=\mathrm{Stab}_G(x_0)$. This choice also determines the following data:

(1) A maximal compact subgroup $K_P=K_P(x_0)=K\cap P$ and a diffeomorphism $P/K_P\to D$.
(2) A Cartan involution $\theta: G\to G$ with $G^\theta=K$
(3) A unique θ-equivariant lifting $i_{x_0}:\mathbf{L_P}\to\mathbf{P}$ of the Levi quotient. For any subset $B\subseteq L_P$ we denote its lift by $B(x_0)=i_{x_0}(B)$. The basepoint $x_0\in D$ is *rational for* P if the lift $\mathbf{L_P}(x_0)\subset\mathbf{G}$ is a rationally defined algebraic subgroup.
(4) A diffeomorphism

$$D_P=P/K_PA_P\mathcal{U}_P\cong M_P/K_P \tag{1.1.8}$$

given by $mK_P\mapsto i_{x_0}(m)K_PA_P\mathcal{U}_P$.
(5) A canonical rational Langlands' decomposition

$$P=\mathcal{U}_PA_P(x_0)M_P(x_0) \tag{1.1.9}$$

(6) A diffeomorphism

$$\mathcal{U}_P\times A_P\times D_P\to D \tag{1.1.10}$$

given by $(u,a,mK_P)\to ui_{x_0}(a)i_{x_0}(m)K_P(x_0)$ (where $mK_P\in M_P/K_P\cong D_P$).
(7) Projections $u_P: D\to\mathcal{U}_P$, $\alpha_P: D\to A_P$ and $\Phi_P: D\to D_P$ to the first, second, and third factors. (The projection Φ_P is actually independent of the basepoint.)

In the coordinates given by (1.1.10), the action of $g\in P$ on D and the geodesic action (see below) of $b\in A_P$ on D are given by

$$g(u,a,mK_P)\cdot b=(gui_{x_0}\nu_P(g^{-1}),abc,xmK_P) \tag{1.1.11}$$

where $\nu_P(g)=cx\in A_PM_P=L_P$.

The canonical Langlands decompositions (1.1.9) of two parabolic subgroups $\mathbf{Q}\subset\mathbf{P}$ are related in the following way. The image $\overline{Q}=\nu_P(Q)\subset L_P$ is parabolic in L_P. Set $\mathcal{U}_{\bar{Q}}(x_0)=i_{x_0}(\mathcal{U}_{\bar{Q}})$ and $A'(x_0)=i_{x_0}(A_Q)\cap M_P(x_0)$. Then

$$\begin{aligned}Q&=(\mathcal{U}_P\mathcal{U}_{\bar{Q}}(x_0))(A_P(x_0)A'(x_0))M_Q(x_0)\\&=\mathcal{U}_PA_P(x_0)(\mathcal{U}_{\bar{Q}}(x_0)A'(x_0)M_Q(x_0))\end{aligned} \tag{1.1.12}$$

The first is the canonical Langlands decomposition of Q while the second is the decomposition of Q which is induced from the canonical Langlands decomposition of P.

(1.2) Borel-Serre Partial Compactification. Throughout this section we suppose that $\mathbf{G}$ is a reductive algebraic group defined over $\mathbb{Q}$ with $\mathbf{S}_G = \{1\}$. Fix a basepoint $x_0 \in D$ in the associated generalized symmetric space, with stabilizer $K = K(x_0) = \mathrm{Stab}_G(x_0)$. Let $\mathbf{P} \subset \mathbf{G}$ be a rational parabolic subgroup. The torus $A_P \subset L_P$ acts on D in two ways. Write $D = P/K_P$. The action from the left is given by $a \cdot gK_P := i_{x_0}(a)gK_P$ (for any $g \in P$), while the action from the right (which is the *geodesic action* of Borel and Serre) is given by $(gK_P) \cdot a := gi_{x_0}(a)K_P$. The geodesic action is well defined since $A_P(x_0) = i_{x_0}(A_P)$ commutes with $K_P = i_{x_0}(\nu_P(K_P))$, and it is even independent of the choice of basepoint. The quotient $e_P = D/A_P$ is called the Borel-Serre boundary component or the Borel-Serre stratum corresponding to P.

The characters $\beta \in \Delta_P$ determine a diffeomorphism $A_P \cong (\mathbb{R}_{>0})^{\Delta_P}$ and we denote by $\overline{A_P}$ the partial compactification obtained by adding the point at infinity to each copy of $\mathbb{R}_{>0}$, i.e.

$$\overline{A_P} \cong (0, \infty]^{\Delta_P}. \tag{1.2.1}$$

Then the "corner" associated to P is the (noncompact) smooth manifold with corners, $D(P) = D \times_{A_P} \overline{A_P}$; it is a disjoint union

$$D(P) = D \cup \coprod_{P' \supseteq P} e_{P'} \tag{1.2.2}$$

and *it is an open neighborhood of the stratum* e_P in the Borel-Serre partial compactification $\overline{D}^{BS}$ of D. The canonical projection

$$\theta_P : D(P) \to D/A_P = e_P \tag{1.2.3}$$

is the unique continuous extension of the mapping $D = P/K_P \to P/K_P A_P = e_P$.

Each $\beta \in \Delta_P$ determines a *root function* $f_\beta^P : D \to \mathbb{R}_{>0}$ by $f_\beta^P(x) = \beta(\alpha_P(x))$. In other words, write $x = uamK_P$ by (x.1.1.1); then $f_\beta^P(x) = \beta(a)$. The root function f_β^P is equivariant with respect to the P action and the geodesic action on D in the following sense: If $g' = u'a'm' \in \mathcal{U}_P A_P(x_0) M_P(x_0)$ and if $b' \in A_P$ then

$$f_\beta^P(g'x \cdot b') = f_\beta^P(x)\beta(a'b') \tag{1.2.4}$$

which follows from (1.1.11).

It follows that f_β^P extends to a function (which we also denote in the same way),

$$f_\beta^P : D(P) \to (0, \infty] \tag{1.2.5}$$

hence the mapping

$$D(P) \to e_P \times (0, \infty]^{\Delta_P} \tag{1.2.6}$$

given by $x \mapsto (\theta_P(x), \{f_\beta^P(x)\}_{\beta \in \Delta_P})$ is a diffeomorphism of manifolds with corners. The following lemma characterizes convergent sequences in the Borel-Serre partial compactification of D.

(1.2.7) Lemma. *Let* $\mathbf{Q_0}$ *be a minimal parabolic subgroup of* G *and let* $\mathbf{P} = \mathbf{P(I)}$ *be a standard parabolic subgroup corresponding to a subset* $I \subset \Delta$. *Let* $x_\infty \in e_P$. *Then a sequence of points* $\{x_k\} \subset D$ *converges (in the Borel-Serre partial compactification) to* x_∞ *iff the following two conditions hold:*

(1) $\theta_P(x_k) \in e_P$ *converges in* e_P *to* x_∞
(2) $f_\beta^{Q_0}(x_k) \to \infty$ *for all* $\beta \in \Delta - I$.

(1.2.8) Proof. Let us consider the convergence of the sequence $\{x_k\}$ in the open corner $D(Q_0)$. By (1.2.6) (with P replaced by Q_0), we have a diffeomorphism $D(Q_0) \cong e_{Q_0} \times (0,\infty]^\Delta$ which takes $e_P \cong e_{Q_0} \times (0,\infty)^I \times \{\infty\}^{\Delta - I}$. So we need to show that

(1) $\theta_{Q_0}(x_k)$ converges to $\theta_{Q_0}(x_\infty)$
(2) $f_\alpha^{Q_0}(x_k) \to f_\alpha^{Q_0}(x_\infty)$ for all $\alpha \in I$
(3) $f_\beta^{Q_0}(x_k) \to \infty$ for all $\beta \in \Delta - I$.

Items (1) and (3) follow immediately from the hypotheses. Let us consider item (2). For any $a \in A_P = A_{P(I)}$ and for any $\alpha \in I \subset \Delta$ we have $f_\alpha^{Q_0}(x_k \cdot a) = f_\alpha^{Q_0}(x_k).\alpha(a) = f_\alpha^{Q_0}(x_k)$ by (1.2.4) and (1.1.5). But $\theta_P(x_k) = x_k (\text{mod } A_P)$ so

$$f_\alpha^{Q_0}(x_k) = f_\alpha^{Q_0}(\theta_P(x_k)) \to f_\alpha^{Q_0}(x_\infty)$$

as claimed. □

(1.3) Reductive Borel-Serre Compactification. If $\mathbf{P} \subset \mathbf{G}$ is a rational parabolic subgroup, define the *reductive Borel-Serre boundary component* $D_P = P/K_P A_P \mathcal{U}_P \cong M_P/K_P$. Let

$$\Phi_P : D = P/K_P \to P/K_P A_P \mathcal{U}_P = D_P \tag{1.3.1}$$

denote the projection. Then D_P is a "generalized symmetric space" (for the group M_P) and it inherits a basepoint $\Phi_P(x_0)$. The reductive Borel-Serre partial compactification $\overline{D}^{RBS}$ is obtained from the Borel-Serre partial compactification by collapsing the fibers of the projection $e_P = P/K_P A_P \to D_P = P/K_P A_P \mathcal{U}_P$ to points (for every proper rational parabolic subgroup $\mathbf{P} \subset \mathbf{G}$). The closure of D_P in $\overline{D}^{RBS}$ is the reductive Borel-Serre partial compactification $\overline{D_P}^{RBS}$ of $D_P = M_P/K_P$. The projections $e_P \to D_P$ fit together to give a $\mathbf{G}(\mathbb{Q})$ equivariant continuous surjection $\overline{D}^{BS} \to \overline{D}^{RBS}$ which extends the identity mapping on D. Denote by $D[P] \subset \overline{D}^{RBS}$ the image of the corner $D(P)$ under this surjection, so that $D[P] = D \cup \coprod_{Q \supseteq P} D_Q$. Then $D[P]$ is an open neighborhood in $\overline{D}^{RBS}$ of the stratum D_P.

Suppose $\mathbf{Q} \subset \mathbf{P}$ are standard rational parabolic subgroups corresponding to subset $J \subset I \subset \Delta$ of the simple rational roots, respetively. Zucker's vexatious point [Z1] (3.19) is that the root functions f_β^P and f_β^Q do not necessarily agree. Let $\overline{\mathbf{Q}} = \nu_P(\mathbf{Q}) \subset \mathbf{L_P}$ be the parabolic subgroup of $\mathbf{L_P}$ which is determined by $\mathbf{Q}$. For $x \in D$ use (1.1.12) to write

$x = uwabm.x_0$ with $u \in \mathcal{U}_P$, $w \in i_{x_0}(\mathcal{U}_{\bar{Q}})$, $a \in A_P(x_0)$, $b \in A' = A_Q(x_0) \cap M_P(x_0)$, and $m \in M_Q(x_0)$. Then for all $\beta \in \Delta$ we have

$$f_\beta^Q(x) = \beta(ab), \text{ and } f_\beta^P(x) = \beta(a) \tag{1.3.2}$$

(but $\beta \in I \implies \beta(a) = 1$; and $\beta \in J \implies \beta(a) = \beta(b) = 1$). Since $\Phi_P(x) = wbm.\Phi_P(x_0) \in M_P/K_P$, we have (for all $\beta \in I$),

$$f_\beta^{\bar{Q}}(\Phi_P(x)) = \beta(b) = f_\beta^Q(x) \tag{1.3.3}$$

It follows from (1.2.4) and (1.3.3) that, for all $\beta \in \Delta$ the root function f_β^Q admits a unique well-defined continuous extension $f_\beta^Q : D[Q] \to (0, \infty]$ by defining, for any $y \in D_{P(I)}$

$$f_\beta^Q(y) = \begin{cases} \infty & \text{for} \quad \beta \in \Delta - I \\ f_\beta^{\bar{Q}}(y) & \text{for} \quad \beta \in I - J \\ 1 & \text{for} \quad \beta \in J \end{cases} \tag{1.3.4}$$

Similarly, the projection $\Phi_Q : D \to D_Q$ factors,

$$\Phi_Q(x) = \Phi_{\bar{Q}} \circ \Phi_P(x) = m.\Phi_Q(x_0) \in D_Q \cong M_Q/K_Q \tag{1.3.5}$$

so it also has a unique continuous extension to the neighborhood $D[Q]$ which we denote by the same symbol, $\Phi_Q : D[Q] \to D_Q$.

From lemma 1.2.7, we obtain a characterization for convergence in the reductive Borel-Serre compactification:

(1.3.6) Lemma RBS. *Let $\mathbf{Q_0}$ be the standard minimal parabolic subgroup of G and let $\mathbf{P} = \mathbf{P(I)}$ be the standard parabolic subgroup corresponding to a subset $I \subset \Delta$. Let $x_\infty \in D_P$. Then a sequence of points $\{x_k\} \subset D$ converges (in the reductive Borel-Serre compactification) to x_∞ iff the following two conditions hold:*

(1) *$\Phi_P(x_k) \in D_P$ converges in D_P to x_∞*
(2) *$f_\beta^{Q_0}(x_k) \to \infty$ for all $\beta \in \Delta - I$.*

Moreover, in the presence of (1), condition (2) is equivalent to the condition

(2′) *$f_\beta^P(x_k) \to \infty$ for all $\beta \in \Delta - I$.*

(1.3.7) Proof. Use (1.1.12) to write $x_k = u_k w_k a_k b_k m_k.x_0$ with $u_k \in \mathcal{U}_P$, $w_k \in i_{x_0}(\mathcal{U}_{\bar{Q}_0})$, $a_k \in A_P(x_0)$, $b_k \in A_{Q_0}(x_0) \cap M_P(x_0)$, and $m_k \in M_{Q_0}(x_0)$. Then $\Phi_P(x_k) = w_k b_k m_k.\Phi_P(x_0)$ which converges; hence the b_k converge. So by (1.3.2) we conclude that $f_\beta^P(x_k) = \beta(a_k) \to \infty$ iff $f_\beta^{Q_0}(x_k) = \beta(a_k)\beta(b_k) \to \infty$. □

We shall also need the following consequence of (1.3.5) and Lemma RBS,

(1.3.8) Corollary. *Let $\mathbf{P} = \mathbf{P(I)}$ be the standard parabolic subgroup corresponding to a subset $I \subset \Delta$, and let $x_\infty \in \overline{D}_P^{RBS} \subset \overline{D}^{RBS}$. Suppose $\{x_k\} \subset D$ is a sequence which converges in $\overline{D}^{RBS}$ to x_∞. Then the sequence $\{\Phi_P(x_k)\} \subset D_P$ also converges to x_∞ in $\overline{D}^{RBS}$.* □

§2. Linear Symmetric Spaces

The main result in this section is Theorem B (§2.9) which describes the topology of the reductive Borel-Serre compactification of certain convex polyhedral cones.

(2.1). Throughout this section, $\mathbf{G}$ denotes a connected reductive algebraic group defined over $\mathbb{Q}$, and $\rho : \mathbf{G} \to GL(\mathbf{V})$ denotes a faithful rational representation of $\mathbf{G}$ on some rational vectorspace $\mathbf{V}$. Let $G = \mathbf{G}(\mathbb{R})^0$ denote the connected component of the group of real points. We assume that G acts with an open orbit $C \subset V = \mathbf{V} \otimes_{\mathbb{Q}} \mathbb{R}$ such that the stabilizer $K = \mathrm{Stab}_G(e)$ of a chosen basepoint $e \in C$ is a maximal compact subgroup of G. Then we may identify the group $G \cong \mathrm{Aut}^0(C) \subset GL(V)$ with the connected component of the group of linear automorphisms of V which preserve the orbit C. The vectorspace V admits a rational inner product $\langle \cdot, \cdot \rangle$ so that

$$\check{C} = \{x \in V | \langle x, c \rangle > 0 \ \forall c \in \overline{C} - \{0\}\}$$

coincides with C (and $G = G^t \subset GL(V)$). Then C is a self adjoint homogeneous rational cone in V. We shall assume for simplicity that C *is irreducible over* $\mathbb{Q}$ which implies that the split component $A_G = \mathbf{S_G}(\mathbb{R})^0$ is 1-dimensional, and acts on V by homotheties. [The results of this chapter will eventually be applied to the group $\mathbf{G}_\ell$ of chapter 4.]

Fix once and for all a basepoint $e \in C$ which is rational, $e \in \mathbf{V}(\mathbb{Q})$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ denote the Cartan decomposition corresponding to the choice of maximal compact subgroup $K = \mathrm{Stab}_G(e)$ and let $\phi : \mathfrak{g} \to \mathrm{End}(V)$ denote the differential of ρ. Then ϕ determines an isomorphism $\mathfrak{p} \to V$ by $x \mapsto \phi(x)(e)$ whose inverse we denote by $a \mapsto T_a \in \mathfrak{p}$. The vectorspace V admits a Jordan algebra structure such that, for any $x \in \mathfrak{p}$, the mapping $\phi(x) \in \mathrm{End}(V)$ is Jordan multiplication by the element $\phi(x)(e) \in V$. In other words, $a \bullet b = \phi(T_a)b$ for all $a, b \in V$. It is customary to drop the explicit mention of ϕ and to write $a \bullet b = T_a(b)$. The basepoint $e \in C \subset V$ is the identity element of the Jordan algebra. For all $a, b, x \in V$ and for all $s \in \mathbb{R}$ we have $T_{a+sb}(x) = T_a(x) + sT_b(x)$. The cone C is given by $C = \{x^2 | x \in V \text{ is invertible}\}$ and its closure is $\bar{C} = \{x^2 |\ x \in V\}$.

(2.2) Standard Compactification. The standard partial compactification of C is the Satake partial compactification which corresponds to the representation ρ. It may be explicitly described as follows.

For each idempotent $\epsilon \in V$ the associated endomorphism (given by Jordan multiplication) $T_\epsilon : V \to V$ is semisimple with eigenvalues 0, $\frac{1}{2}$, and 1. The "Peirce decomposition" $V \cong V_0 \oplus V_{\frac{1}{2}} \oplus V_1$ is the corresponding eigenspace decomposition. Define boundary components $C_0(\epsilon) = \mathrm{int}(\bar{C} \cap V_0)$ and $C_1(\epsilon) = C_0(e - \epsilon) = \mathrm{int}(\bar{C} \cap V_1)$. These boundary components are *rational* if the corresponding idempotents ϵ and $e - \epsilon$ are rational elements of V. The subgroup $P = \mathrm{Norm}_G(C_1(\epsilon))$ which preserves $C_1(\epsilon)$ is a maximal parabolic subgroup of G. If ϵ is rational, then P is the real points of a rationally defined maximal parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$. Conversely, every maximal rational parabolic subgroup $P \subset G$ preserves a unique rational boundary component.

The closure $\bar{C} \subset V$ is the disjoint union of C and all its boundary components. Let $C^* \subset V$ denote the union of C and all its *rational* boundary components. Define the

Satake topology on C^* to be the unique topology so that: (1) for any Siegel set $\Omega \subset C$ its closure $\bar{\Omega} \subset V$ in V coincides with its closure $\bar{\Omega}^{Sat}$ in the Satake topology, and (2) if $y \in C^*$ is a point on the boundary, then for any arithmetic group $\Gamma \subset \mathbf{G}(\mathbb{Q})$, there exists a basis of neighborhoods U of y (in the Satake topology) such that $\Gamma_y U = U$ where Γ_y denotes the stabilizer of y in Γ. The existence of such a topology is essentially proven in [S2] §2, [BB] thm. 4.9, or [Z2] p. 340, although their proofs must be modified slightly so as to apply to our $\mathbf{G}$ which is reductive, rather than semisimple. The *standard partial compactification* $\overline{C}^{std}$ of C is the space C^* with the Satake topology. For any rational idempotent $\epsilon \in V$ the closure of $C_1(\epsilon)$ in $\bar{C}^{std}$ is the standard partial compactification of $C_1(\epsilon)$.

Let $D = C/A_G$ and $\overline{D}^{std} = \overline{C}^{std}/A_G$ (with the Satake topology) be the quotients under homotheties. For any subset $S \subset C$ we denote by $[S] \subset D$ its image in D. The group $\mathbf{G}(\mathbb{Q})$ of rational points acts on the partial compactification $\overline{D}^{std}$.

(2.2.1) Proposition. *For any arithmetic subgroup $\Gamma \subset G$ the quotient $\Gamma \backslash \overline{D}^{std}$ is compact. If Γ is neat, then $\Gamma \backslash \overline{D}^{std}$ is a stratified space with one stratum $\Gamma \cap P(\epsilon) \backslash C_1(\epsilon)/A_G$ for each Γ conjugacy class of rational boundary components $C_1(\epsilon)$.*

(2.3) Roots. Throughout the rest of this chapter we fix a "standard" minimal rational parabolic subgroup $\mathbf{Q_0} \subset \mathbf{G}$. This corresponds to a choice of a complete set of mutually orthogonal rational idempotents $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_{r+1}\}$ (in other words, $\epsilon_i \bullet \epsilon_j = 0$ for $i \neq j$, $\epsilon_i^2 = \epsilon_i$, and $\epsilon_1 + \epsilon_2 + \ldots + \epsilon_{r+1} = e$). The ϵ_i may be ordered so that Q_0 is the normalizer of the "complete" flag of rational boundary components

$$\bar{C}_1(\epsilon_1) \subset \bar{C}_1(\epsilon_1 + \epsilon_2) \subset \ldots \subset \bar{C}_1(\epsilon_1 + \ldots + \epsilon_{r+1}) = \bar{C}. \tag{2.3.1}$$

The canonical lift of A_{Q_0} (determined by the basepoint $e \in C$) is $A_{Q_0}(e) = \exp(\sum \mathbb{R}T_{\epsilon_i})$. If $\gamma_i \in \mathrm{Hom}(A_{Q_0}, \mathbb{R})$ denotes the dual basis to the elements $\exp(T_{\epsilon_i})$ then the rational roots $\Phi(A_{Q_0}, G)$ of G are $\{\frac{1}{2}(\gamma_i - \gamma_j)\}$ for $i \neq j$ and the simple roots $\Delta = \Delta_{Q_0}$ (appearing in the unipotent radical of Q_0) are

$$\alpha_i = \frac{1}{2}(\gamma_i - \gamma_{i+1}) \text{ for } 1 \leq i \leq r. \tag{2.3.2}$$

The Dynkin diagram for G is linear and it corresponds to the ordering $\alpha_1, \alpha_2, \ldots, \alpha_r$ of the simple roots (or to the ordering $\epsilon_1, \epsilon_2, \ldots, \epsilon_r, \epsilon_{r+1}$ of the idempotents).

§**2.4 Reductive Borel-Serre boundary components.** Throughout this section we fix a standard parabolic subgroup $\mathbf{Q} \supset \mathbf{Q_0}$ corresponding to a subset $I \subset \Delta$ of the simple roots as in (1.1.6). Write $\Delta - I = \{\alpha_{m_1}, \alpha_{m_2}, \ldots, \alpha_{m_q}\}$ (where $1 \leq m_1 < m_2 < \ldots < m_q \leq r$). Then we obtain orthogonal idempotents

$$\begin{aligned} d_1 &= \epsilon_1 + \epsilon_2 + \ldots + \epsilon_{m_1} \\ d_2 &= \epsilon_{m_1+1} + \epsilon_{m_1+2} + \ldots + \epsilon_{m_2} \\ &\ldots \\ d_{q+1} &= \epsilon_{m_q+1} + \epsilon_{m_q+2} + \ldots + \epsilon_{r+1} \end{aligned} \tag{2.4.1}$$

so that Q is the normalizer of the (partial rational) flag of boundary components

$$\bar{C}_1(d_1) \subset \bar{C}_1(d_1 + d_2) \subset \ldots \subset \bar{C}_1(d_1 + \ldots + d_{q+1}) = \bar{C}. \tag{2.4.2}$$

The canonical lift $A_Q(e)$ of the torus A_Q may be parametrized by elements

$$\lambda_Q(t_1, t_2, \ldots, t_{q+1}) = \exp(s_1 T_{d_1} + s_2 T_{d_2} + \ldots + s_{q+1} T_{d_{q+1}}) \tag{2.4.3}$$

where $t_i = e^{\frac{1}{2}s_i}$. The torus $A_G = \mathbf{Z_G}(\mathbb{R})^0$ is given by $\lambda_Q(t, t, \ldots, t)$. It follows from(2.3.2) that, when viewed as characters on A_Q, the restrictions of the simple roots $\Delta - I = \{\alpha_{m_1}, \alpha_{m_2}, \ldots, \alpha_{m_q}\}$ to A_Q are given by

$$\alpha_{m_j}(\lambda_Q(t_1, t_2, \ldots, t_{q+1})) = t_j t_{j+1}^{-1} \text{ for } 1 \leq j \leq q \tag{2.4.4}$$

Let

$$V = \bigoplus_{1 \leq i \leq j \leq q+1} V_{ij} \tag{2.4.5}$$

denote the (simultaneous) Peirce decomposition of V relative to this collection $d_1 + d_2 + \ldots + d_{q+1} = e$ of idempotents, where $V_{ii} = V_1(d_i)$ and $V_{ij} = V_{\frac{1}{2}}(d_i) \cap V_{\frac{1}{2}}(d_j)$ for $i \neq j$ (cf [AMRT] II, 3.8 p. 92 or [FK] thm. IV.2.1 p. 68). Then $\lambda_Q(t_1, t_2, \ldots, t_{q+1})$ acts on V_{ij} with eigenvalue $t_i t_j$ for $1 \leq i, j \leq q+1$. The Jordan algebra structure on V restricts to a rationally defined Jordan algebra structure on each $V_{ii} = V_1(d_i) \subset V$ with identity element d_i and with self adjoint homogeneous cone $C_1(d_i) = \overline{C} \cap V_1(d_i)$ (for $1 \leq i \leq q+1$). Let $G_i = \mathrm{Aut}^0(C_1(d_i), V_{ii})$. Then G_i is the real points of a rationally defined algebraic group $\mathbf{G}_i$ and $C_1(d_i) \cong G_i/K_i$ where K_i is the isotropy subgroup in G_i of the basepoint d_i. Let $p_i : V \to V_{ii}$ denote the linear projection which is determined by the Peirce decomposition. Define

$$\psi_i : L_Q(e) \to G_i \tag{2.4.6}$$

by $\psi_i(g) = g|C_1(d_i)$. Thus there are two projections $C \to C_1(d_i)$: one given by the linear projection p_i and the second, ϕ_i, given by the composition $C = Q/K_Q \to L_Q/K_Q \to G_i/K_i$. The following lemma says that these projections agree on points $x = g.e$ which are in the orbit of the Levi subgroup $L_Q(e)$; the discrepancy between these two projections is analyzed in the proof of proposition 2.6.2.

(2.4.7) Lemma. *For all $g \in L_Q(e)$ and for all i ($1 \leq i \leq q+1$) we have*

$$p_i(g.e) = \phi_i(g.e) = \psi_i(g).d_i \in C_1(d_i).$$

(2.4.8) Proof. Since $L_Q(e)$ is the centralizer of $A_Q(e)$ it follows that each V_{ii} is preserved by $L_Q(e)$ and that the projection p_i commutes with the action of $g \in L_Q(e)$ on V_{ii}. Furthermore, $p_i(e) = d_i$. □

The mapping

$$\psi_Q : L_Q(e) \to G_1 \times G_2 \times \ldots \times G_{q+1} \tag{2.4.9}$$

given by

$$\psi_Q(g) = (g|C_1(d_1), g|C_1(d_2), \ldots, g|C_1(d_{q+1}))$$

is surjective with compact kernel ([AMRT] II, 3.9 prop. 10) and it induces diffeomorphisms $C_Q = L_Q/K_Q \cong C_1(d_1) \times \ldots \times C_1(d_{q+1})$ and

$$\Psi_Q : D_Q \cong D_1 \times \ldots \times D_{q+1} \tag{2.4.10}$$

where $D_i = C_1(d_i)/\text{homotheties}$ for $1 \le i \le q+1$.

(2.4.11) Lemma. *The mapping ψ_Q induces a homeomorphism (which is smooth on each boundary component) between the reductive Borel-Serre partial compactifications,*

$$\overline{\Psi}_Q : \overline{D}_Q^{RBS} \cong \overline{D}_1^{RBS} \times \overline{D}_2^{RBS} \times \ldots \times \overline{D}_{q+1}^{RBS}. \tag{2.4.12}$$

(2.4.13) Proof. Each parabolic subgroup of $G_1 \times G_2 \times \ldots \times G_{q+1}$ is of the form $R_1 \times R_2 \times \ldots \times R_{q+1}$ with R_i parabolic in G_i. □

(2.4.14). The RBS boundary component D_Q appears as a stratum in $\overline{D}_P^{RBS}$ for every parabolic subgroup $P \supset Q$. However the ordering of the roots determines an ordering of the *maximal* parabolic subgroups containing Q. Define

$$P = Q^\dagger = \text{Norm}(C_1(d_1)) \tag{2.4.15}$$

to be the first maximal parabolic subgroup in this ordering. It corresponds to the single idempotent d_1. Set $C_0 = C_1(e - d_1)$, $G_0 = \text{Aut}(C_0, V_0(d_1))$, $D_0 = C_0/\text{homotheties}$ (and $C_1 = C_1(d_1)$, $G_1 = \text{Aut}(C_1, V_1(d_1))$, $D_1 = C_1/\text{homotheties}$). As in (2.4.9) and (2.4.11) set $\psi_P : L_P \to G_1 \times G_0$ and $\overline{\Psi}_P : \overline{D}_P^{RBS} \cong \overline{D}_1^{RBS} \times \overline{D}_0^{RBS}$.

Lemma 2.4.16. *Suppose $Q^\dagger = P = \mathcal{U}_P G_1 G_0$ as above. There exists a rational parabolic subgroup $H \subset G_0$ with corresponding reductive Borel-Serre boundary component $D_{0,H} \subset \overline{D}_0^{RBS}$ so that*

$$\overline{\Psi}_P(D_Q) = D_1 \times D_{0,H} \subset D_1 \times \overline{D}_0^{RBS}$$

(2.4.17) Proof. The image of Q under the composition

$$Q \subset P \xrightarrow[\nu_P]{} L_P \xrightarrow[\psi_P]{} G_1 \times G_0$$

is of the form $G_1 \times H$ for some parabolic subgroup $H \subset G_0$. □

(2.4.18). These spaces and mappings fit together in the following diagram. The composition across the top row is ϕ_i and the composition across the middle row is Φ_i.

$$\begin{array}{ccccccc} C = P/K_P & \xrightarrow[\nu_P]{} & C_P = L_P/K_P & \xrightarrow[\Psi_P]{} & C_1 \times C_0 & \longrightarrow & C_i \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ D = P/K_PA_P & \xrightarrow[\Phi_P]{} & D_P = L_P/K_PA_P & \xrightarrow[\Psi_P]{} & D_1 \times D_0 & \longrightarrow & D_i \\ \| & & \downarrow & & \downarrow & & \\ D = P/K_PA_P & \xrightarrow[\Phi_Q]{} & D_Q = L_Q/K_QA_Q & \xrightarrow[\overline{\Psi}_P]{} & D_1 \times D_{0,H} & & \end{array}$$

(2.5) Peirce Coordinates. The Peirce decomposition gives rise to a coordinate system on C which is analogous to the Siegel coordinate system of Piatetski-Shapiro for the case of Hermitian symmetric spaces. Let $P \supset Q_0$ be a standard maximal rational parabolic subgroup of G, (see (2.3.2)) say, $P = P(\Delta - \{\alpha_k\})$ for some simple root $\alpha_k \in \Delta = \Delta_{Q_0}$. Then $P = \mathrm{Norm}_G(C_1(\epsilon))$ where $\epsilon = \epsilon_1 + \epsilon_2 + \ldots + \epsilon_k$. Let $V = V_1(\epsilon) \oplus V_{\frac{1}{2}}(\epsilon) \oplus V_0(\epsilon)$ denote the resulting Peirce decomposition of V. Let $\lambda_P(t_1, t_0) = \exp(s_1 T_\epsilon + s_0 T_{e-\epsilon})$ denote the parametrization of A_P as in (2.4.3), with $t_i = e^{\frac{1}{2}s_i}$ for $i = 1, 0$. Let $v = (v_1, v_{\frac{1}{2}}, v_0) \in V_1 \oplus V_{\frac{1}{2}} \oplus V_0$.

(2.5.1) Lemma. *In these coordinates, the action of P on V is given by*

$$g.v = \begin{pmatrix} A & M & N \\ 0 & C & D \\ 0 & 0 & B \end{pmatrix} \begin{pmatrix} v_1 \\ v_{\frac{1}{2}} \\ v_0 \end{pmatrix}$$

(*where A, M, N, C, D, and B are linear mappings which depend on g.*) *Furthermore,*

(1) *$g \in L_P$ iff $M = 0$, $N = 0$, $D = 0$.*
(2) *The element $g = \lambda_P(t_1, t_0) \in A_P$ is given by*

$$\lambda_P(t_1, t_0) = \begin{pmatrix} t_1^2 & 0 & 0 \\ 0 & t_1 t_0 & 0 \\ 0 & 0 & t_0^2 \end{pmatrix}$$

(3) *The root function $f^P_{\alpha_k}$ is given by*

$$f^P_{\alpha_k}(u\lambda_P(t_1, t_0)me) = t_1 t_0^{-1}.$$

for any $u \in \mathcal{U}_P$, $\lambda_P(t_1, t_0) \in A_P$ and $m \in M_P$.
(4) *If $g \in \mathcal{U}_P$ then $A = I$ and $B = I$.*
(5) *The orbit of the basepoint $e = (\epsilon, 0, e - \epsilon)$ under L_P is the product $C_1(\epsilon) \times \{0\} \times C_1(e - \epsilon)$.*

(2.5.2) Proof. The 1-parameter group $\lambda(t) = \lambda(t_1, 1) = \exp(s_1 T_\epsilon)$ is in the center of the Levi quotient L_P and it has the property that (cf. [AMRT] II, 3.3 and [MF] ch. 2 §2),

$$P = \{g \in G | \lim_{t \to 0} \lambda(t) g \lambda(t)^{-1} \text{exists}\}.$$

Then $\lambda(t).(v_1, v_{\frac{1}{2}}, v_0) = (t^2 v_1, t^1 v_{\frac{1}{2}}, v_0)$. Writing

$$g = \begin{pmatrix} A & M & N \\ U & C & D \\ V & W & B \end{pmatrix}$$

we see that

$$\lambda(t) g \lambda(t)^{-1} = \begin{pmatrix} A & tM & t^2 N \\ t^{-1} U & C & tD \\ t^{-2} V & t^{-1} W & B \end{pmatrix}$$

which implies that $U = 0$, $V = 0$, and $W = 0$. Moreover, $g \in L_P(e) = Z_P(\lambda(t))$ if and only if $\lambda(t) g \lambda(t)^{-1} = g$ for all t, which implies that $M = 0$, $N = 0$, and $D = 0$. This proves parts (1) and (5). Part (2) follows from the definition of $\lambda(t_1, t_0)$ and part (3) is a restatement of (2.4.4). Part (4) follows from the observation that the projection

$$\begin{pmatrix} A & M & N \\ 0 & C & D \\ 0 & 0 & B \end{pmatrix} \mapsto (A, B)$$

is a group homomorphism $P \to G_1 \times G_0$ whose restriction to L_P agrees with ψ (cf. Lemma 2.4.7). Since $G_1 \times G_0$ is reductive, the unipotent radical $\mathcal{U}_P$ is in the kernel of this mapping. $\square$

(2.6) Convergence. Fix any point $w \in \overline{D}^{RBS}$. Then w lies in some RBS boundary component, say D_Q. Let $P = Q^\dagger$ (cf. (2.4.15)) be the maximal parabolic subgroup which is first among the ordering of all maximal parabolic subgroups containing Q. Then $P = \mathrm{Norm}(C_1(\epsilon))$ for some rational idempotent ϵ (denoted d_1 in §2.4.14) which determines a Peirce decomposition $V = V_1(\epsilon) \oplus V_{\frac{1}{2}}(\epsilon) \oplus V_1(\epsilon)$. If $v = (v_1, v_{\frac{1}{2}}, v_0) \in V$ then we will write $[v] = [v_1 : v_{\frac{1}{2}} : v_0] \in D$ for its homothety class. Set $C_i = \overline{C} \cap V_i(\epsilon)$, $D_i = C_i/\text{homotheties}$, and $G_i = \mathrm{Aut}(C_i, V_i(\epsilon))$ for $i = 1, 0$. Then (2.4.9) $\psi_P : L_P \to G_1 \times G_0$ induces $\overline{\Psi}_P : \overline{D}_P^{RBS} \cong \overline{D}_1^{RBS} \times \overline{D}_2^{RBS}$. The linear projections $p_i : V \to V_i(\epsilon)$ determine projections $p_i : C \to C_i$ and $p_i : D \to D_i$ (for $i = 1, 0$). By lemma (2.4.16) we have

$$\overline{\Psi}_P(D_Q) = D_1 \times D_{0,H} \subset D_1 \times \overline{D}_0^{RBS} \tag{2.6.1}$$

for some rational parabolic subgroup $H \subset G_0$, and we write $\overline{\Psi}_P(w) = (w_1, w_0)$.

Proposition 2.6.2. *Let $\{y_k\} \subset D$ be a sequence. If y_k converges to w in the reductive Borel Serre compactification $\overline{D}^{RBS}$ then the following two conditions hold,*

(1) *The sequence y_k converges to $w_1 \in D_1 \subset \overline{D}^{std}$ in the standard compactification of D and*
(2) *The sequence $p_0(y_k)$ converges to $w_0 \in D_{0,H} \subset \overline{D}_0^{RBS}$ in the reductive Borel-Serre compactification of D_0.*

Conversely, suppose $\{y_k\} \subset D$ is contained in a Siegel set and satisfies conditions (1) and (2) above. Then $y_k \to w$ in $\overline{D}^{RBS}$.

Corollary 2.6.3. *The identity mapping $D \to D$ has a unique continuous extension*

$$\mu = \mu_G : \overline{D}^{RBS} \to \overline{D}^{std}$$

which takes boundary components to boundary components. For any maximal rational parabolic subgroup $P = P(\epsilon) \subset G$ the restriction $\mu|\overline{D}_P^{RBS}$ is given by the composition

$$\overline{D}_P^{RBS} \underset{\overline{\Psi}_P}{\longrightarrow} \overline{D}_1^{RBS} \times \overline{D}_0^{RBS} \underset{\pi_1}{\longrightarrow} \overline{D}_1^{RBS} \underset{\mu_{G_1}}{\longrightarrow} \overline{D}_1^{std}$$

(where $D_i = C_i(\epsilon)/homotheties$ and with $\overline{\Psi}_P$ as in (2.4.18)). A RBS boundary component D_Q is contained in $\mu^{-1}(D_1)$ iff $Q^\dagger = P$. $\square$

(2.6.4) Proof of Proposition 2.6.2. For $i = 1, 0$, denote by $\Phi_i : D \to D_i$ the composition along the middle row in diagram 2.4.18. The proposition is not trivial because the mapping $\Phi_1 : C \to C_1$ does not necessarily agree with the linear projection $p_1 : (v_1, v_{\frac{1}{2}}, v_0) \mapsto v_1$ given by the Peirce decomposition. For $v \in C$ write $v = uame$ relative to the Langlands decomposition $P = \mathcal{U}_P A_P(e) M_P(e)$ of P. Using lemma 2.5.1, write

$$m = \begin{pmatrix} A & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & B \end{pmatrix}, \ a = \lambda_P(t_1, t_0) = \begin{pmatrix} t_1^2 & 0 & 0 \\ 0 & t_1 t_0 & 0 \\ 0 & 0 & t_0^2 \end{pmatrix}, \ u = \begin{pmatrix} I & M' & N' \\ 0 & C' & D' \\ 0 & 0 & I \end{pmatrix} \quad (2.6.5)$$

relative to the Peirce decomposition $V = V_1(\epsilon) \oplus V_{\frac{1}{2}}(\epsilon) \oplus V_0(\epsilon)$. Then

$$v = uam \begin{pmatrix} \epsilon \\ 0 \\ e - \epsilon \end{pmatrix} = \begin{pmatrix} t_1^2 A\epsilon + t_0^2 N' B(e-\epsilon) \\ t_0^2 D' B(e-\epsilon) \\ t_0^2 B(e-\epsilon) \end{pmatrix} \quad (2.6.6)$$

the first and last coordinates of which are $p_1(v)$ and $p_0(v)$ respectively. On the other hand by lemma (2.4.7),

$$\Psi_P \nu_P(v) = \Psi_P(ame) = (t_1^2 A\epsilon, t_0^2 B(e-\epsilon)) \in C_1(\epsilon) \times C_1(e-\epsilon).$$

Hence,

$$[p_0(v)] = \Phi_0([v]) \in D_0, \tag{2.6.7}$$

but $[p_1(v)] \neq \Phi_1([v]) \in D_1$ unless $N'B = 0$.

Suppose the sequence $\{y_k\}$ converges in $\overline{D}^{RBS}$ to $w \in D_Q$. Since $w \in D_Q \subset \overline{D}_P^{RBS}$ we have, (by corollary RBS and lemma (2.4.11)),

$$\Phi_P(y_k) \to w \text{ in } \overline{D}_P^{RBS} \cong \overline{D}_1^{RBS} \times \overline{D}_0^{RBS}$$

It follows that $\Phi_0(y_k) \to w_0$ in $\overline{D_0}^{RBS}$ (and that $\Phi_1(y_k) \to w_1$ in D_1^{RBS}). By (2.6.7) this implies that $p_0(y_k) \to w_0$ which proves (2).

To prove (1), we may assume the parabolic subgroup Q is standard, that it corresponds to a subset $I \subset \Delta$ of the simple roots Δ which occur in the unipotent radical of Q_0 (cf. §2.4), and that it normalizes the partial flag (2.4.2) corresponding to the ordered set of orthogonal idempotents $d_1 + d_2 + \ldots + d_{q+1} = e$. Write

$$y_k = u_k a_k m_k [e] \tag{2.6.8}$$

relative to the canonical Langlands decomposition $Q = \mathcal{U}_Q A_Q(e) M_Q(e)$, where $[e] \in D$ denotes the homothety class of the basepoint $e \in C$. By Lemma RBS, the sequence m_k may be chosen so as to converge to some limit $m_\infty \in M_Q(e)$. If $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is an arithmetic group, then the quotient $\mathcal{U}_Q/(\Gamma \cap \mathcal{U}_Q)$ is compact. Hence there exists a compact subset $F \subset \mathcal{U}_Q$ such that $\mathcal{U}_Q = F.(\Gamma \cap \mathcal{U}_Q)$. So we may write $u_k = \gamma_k u'_k$ where $\gamma_k \in \Gamma \cap \mathcal{U}_Q$ stabilizes the point $y_1 \in D_1$ and where u'_k lies in the fixed compact subset $F \subset \mathcal{U}_Q$. The sequence $\{y_k\}$ converges in the Satake topology to $y_\infty(1)$ iff the sequence $\gamma_k^{-1} y_k = u'_k a_k w_k b_k \ell_k x_0 \in D$ also converges to $y_\infty(1)$. So, replacing y_k by $\gamma_k^{-1} y_k$ if necessary, we may (and will) assume *the elements u_k in equation* (2.6.8) *remain within some fixed compact subset $F \subset \mathcal{U}_Q$*. Since the $\{m_k\}$ also remain within some compact set, the elements y_k are contained in a Siegel set.

Let us write

$$V = V_{11} \oplus V_{22} \oplus \ldots \oplus V_{q+1,q+1} \oplus \bigoplus_{1<i<j\leq q+1} V_{ij}$$

Then the L_Q orbit of the basepoint $e = (d_1, d_2, \ldots, d_{q+1}, 0)$ is

$$C_Q = C_1(d_1) \times C_1(d_2) \times \ldots \times C_1(d_{q+1}) \times \{0\}. \tag{2.6.9}$$

so the point $m_k e$ may be expressed as

$$m_k e = (z_{1,k}, z_{2,k}, \ldots, z_{q+1,k}, 0) \longrightarrow m_\infty e = (z_{1,\infty}, z_{2,\infty}, \ldots, z_{q+1,\infty}, 0).$$

As in (2.4.3), write $a_k = \lambda(t_{1,k}, t_{2,k}, \ldots, t_{q+1,k})$. By (2.4.4), and Lemma RBS we have

$$\frac{t_{1,k}}{t_{2,k}} \to \infty, \frac{t_{2,k}}{t_{3,k}} \to \infty, \ldots, \frac{t_{q,k}}{t_{q+1,k}} \to \infty \tag{2.6.10}$$

while

$$a_k m_k e = (t_{1,k}^2 z_{1,k}, t_{2,k}^2 z_{2,k}, \dots, t_{q+1,k}^2 z_{q+1,k}, 0).$$

But $\mathcal{U}_Q$ preserves the flag $V_{11} = V_1(d_1) \subset V_1(d_1 + d_2) \subset \dots \subset V$ hence

$$u_k a_k m_k e = (t_{1,k}^2 z_{1,k}, \text{linear combinations of } \{t_{i,k}^2 z_{i,k}\} \text{ for } i > 1) \tag{2.6.11}$$

and the coefficients (of the linear combinations) are restricted to lie in some compact set. But (2.6.10) implies that, modulo homotheties,

$$y_k \longrightarrow [z_{1,\infty} : 0 : 0 : \dots : 0] = w_1 \in D_1$$

in the "usual" topology. Since $\{y_k\}$ is contained in a Siegel set it follows that $y_k \to w_1$ in the Satake topology also. This completes the proof of (1) and (2).

Now let us prove the converse, i.e., suppose that $\{y_k\} \subset D$ is contained in a Siegel set, $y_k \to w_1 \in \overline{D}^{std}$ and $p_0(y_k) \to w_0 \in \overline{D}_0^{RBS}$. By Lemma RBS, to show that $y_k \to w \in \overline{D}^{RBS}$ we must verify that

(a) $\Phi_Q(y_k) \to w$ and

(b) $f_\alpha^Q(y_k) \to \infty$ for all $\alpha \in \Delta - I$.

In order to verify (a) it suffices (see diagram 2.4.18) to show that (i) $\Phi_1(y_k) \to w_1$ in D_1 and (ii) $\Phi_0(y_k) \to w_0$ in $\overline{D}_0^{RBS}$. But (ii) follows from (2.6.7) and the assumption that $p_0(y_k) \to w_0$. So we must verify (i). Choose any lift $v_k \in C$ of $y_k \in D$ and write $v_k = u_k a_k m_k e$ relative to the Langlands decomposition $P = \mathcal{U}_P A_P(e) M_P(e)$ of P. Since $\{v_k\}$ lie in a Siegel set, the elements u_k and m_k lie in some compact set. As in (2.6.5) we may write

$$a_k = \lambda_P(t_{1,k}, t_{0,k}) = \begin{pmatrix} t_{1,k}^2 & 0 & 0 \\ 0 & t_{1,k}t_{0,k} & \\ 0 & 0 & t_{0,k}^2 \end{pmatrix} \tag{2.6.12}$$

and

$$m_k = \begin{pmatrix} A_k & 0 & 0 \\ 0 & C_k & 0 \\ 0 & 0 & B_k \end{pmatrix}, \text{ and } u_k = \begin{pmatrix} I & M_k' & N_k' \\ 0 & C_k' & D_k' \\ 0 & 0 & I \end{pmatrix} \tag{2.6.13}$$

Each family of matrices $C_k, C_k', D_k', M_k', N_k'$ is contained in some compact set of matrices, while the A_k and B_k are contained in compact sets of invertible matrices. Since $y_k \to [w_1 : 0 : 0]$ in the Satake topology, it does so also in the usual topology, so it follows from (2.6.6) that

$$t_{1,k} t_{0,k}^{-1} \to \infty \tag{2.6.14}$$

(because the first coordinate dominates the second and third coordinates). From this it also follows that $p_1(y_k) \to w_1$ in D_1 and also that

$$p_1(u_k^{-1} y_k) = p_1(a_k m_k[e]) = [t_{1,k}^2 A_k \epsilon : 0 : t_{0,k}^2 B_k(e - \epsilon)] \longrightarrow [w_1 : 0 : 0]$$

as well. So, by lemma 2.4.7 we have

$$\Phi_1(y_k) = \Phi_1(u_k^{-1} y_k) = p_1(u_k^{-1} y_k) \longrightarrow w_1$$

which completes the proof of (i) and hence also the proof of (a).

Now let us prove part (b). Write $\Delta - I = \{\alpha_{m_1}, \alpha_{m_2}, \dots, \alpha_{m_{q+1}}\}$ as in §M, and parametrize $A_Q(e)$ by elements $\lambda_Q(t_1, t_2, \dots, t_{q+1})$ as in (2.4.3). Then

$$\lambda_P(t_1, t_0) = \lambda_Q(t_1, t_0, t_0, \dots, t_0).$$

Write $v_k = u_k a_k m_k e$ relative to the Langlands decomposition $Q = \mathcal{U}_Q A_Q(e) M_Q(e)$ and set $a_k = \lambda_Q(t_{1,k}, t_{2,k}, \dots, t_{q+1,k})$. By (2.4.4) and (2.6.14) we have $f_\beta^Q(y_k) = t_{1,k} t_{2,k}^{-1} \to \infty$ where $\beta = \alpha_{m_1}$ is the first of these simple roots. It follows from (2.6.7) that, for each of the remaining simple roots $\beta \in \Delta - I$, $\beta \neq \alpha_{m_1}$ we have $f_\beta^H(p_0(y_k)) = f_\beta^Q(y_k)$. Hence these also diverge, by hypothesis (2) and lemma RBS. This completes the verification of condition (b). □

(2.7) Polyhedral cones. A *closed convex polyhedral cone* $\sigma \subset V$ is a closed set,

$$\sigma = \{x \in V | \ell_i(x) \geq 0 \text{ with } i = 1, 2, \dots, k\}$$

for some finite collection $\{\ell_1, \ell_2, \dots, \ell_k\}$ of linear functions $\ell_i : V \to \mathbb{R}$. The *span* L_σ of σ is the smallest vector subspace of V which contains σ. A proper *face* τ of σ is the intersection of σ with a supporting (homogeneous) hyperplane. It is again a (closed) convex polyhedral cone. The "interior" σ^o of σ is the complement of its proper faces. The polyhedral cone σ is *simplicial* if $\dim(\sigma)$ equals the number of 1-dimensional faces of σ.

The vectorspace V is defined over the rationals, $V = \mathbf{V} \otimes_{\mathbb{Q}} \mathbb{R}$. A polyhedral cone σ is *rational* if it is possible to find linear functions $\{\ell_1, \ell_2, \dots, \ell_k\}$ defining σ which are defined over the rationals. In this case, all the faces of σ are rational as well.

(2.7.1) Lemma. ([AMRT] II, 4.3, Thm. 1, p. 113) *Any rational polyhedral cone $\sigma \subset C^*$ is contained in the closure of a Siegel set. The intersection $\sigma \cap C$ is contained in a Siegel set.* □

Thus, for any polyhedral cone $\sigma \subset C^*$ such that $\sigma^o \subset C$ we may identify σ with the closure $\overline{\sigma^o} \subset \overline{C}^{std}$ of σ^o (in the Satake topology).

Let $\sigma \subset C^*$ be a polyhedral cone. Let L_σ be the linear subspace of V spanned by σ and let $c \in L_\sigma^\perp$ be an element in the perpendicular complement. Let $\{v_k\} \subset C$ be a sequence of points, say $v_k = (v_{1,k}, v_{2,k}) \in L_\sigma^\perp \oplus L_\sigma$.

(2.7.2) Definition. We say the sequence $v_k \to c + \infty\sigma$ if

(1) the sequence $v_{1,k} \to c$ in $L_\sigma^\perp$ and
(2) for every $x \in L_\sigma$, there exists N so that $k \geq N \implies v_{2,k} - x \in \sigma$.

(2.7.3) Proposition. *Let $\{v_k\} \subset C$ be a sequence and suppose that $v_k \to c + \infty\sigma$ for some polyhedral cone $\sigma \subset C^*$ (with $\sigma^o \subset C$) and for some $c \in L_\sigma^\perp$. Suppose the sequence $[v_k] \in D$ converges in the reductive Borel-Serre compactification to some point $w \in \overline{D}^{RBS}$. Let $\ell \in V$ be any element and suppose $\{\ell_k\} \subset V$ is a sequence which converges to ℓ. Then for k sufficiently large, the sequence $v_k + \ell_k$ is contained in the cone C and its quotient modulo homotheties $[v_k + \ell_k] \in D$ converges in $\overline{D}^{RBS}$ to the same point w. (In particular, the sequence $[v_k + \ell] \to w$ also.)*

(2.7.4) Proof. We must show that the sequence $[v_k + \ell_k] \in D$ is contained in a Siegel set and satisfies conditions (1) and (2) of proposition 2.6.2. The interior σ^o of the polyhedral cone σ may be embedded in the *interior* $(\sigma')^o$ of a polyhedral cone $\sigma' \subset C^*$ of top dimension. (There are several ways to do this. If σ is a polyhedral cone in a polyhedral decomposition Σ of C, then σ' may be taken to be the convex hull of the star $\mathrm{St}(\sigma)$.) Then $v_k \to \infty\sigma'$. It follows that, for any $\ell \in V$, there exists N so that whenever $k \geq N$ we have $v_k + \ell \in (\sigma')^o \subset C$, so the same is true for the sequence $v_k + \ell_k$. By Lemma 2.7.1, the sequence $v_k + \ell_k$ is therefore contained in a Siegel set.

The limit point w lies in some RBS boundary component, say, D_Q. Set $P = P(\epsilon) = Q^\dagger$ as in (2.4.15) above. As in (2.4.6) and (2.4.11) the decomposition $L_P \cong G_1 \times G_0$ induces $\overline{\Psi}_P : \overline{D}_P^{RBS} \cong \overline{D}_1^{RBS} \times \overline{D}_0^{RBS}$ with $\overline{\Psi}_P(D_Q) = D_1 \times D_{0,H}$ by lemma 2.4.16 (for some rational parabolic subgroup $H \subset G_0 = \mathrm{Aut}^0(C_0(\epsilon), V_0(\epsilon))$). Set $\overline{\Psi}_P(w) = (w_1, w_0)$. Write $v_k = u_k a_k m_k$ relative to the Langlands decomposition of P. Then $a_k, u_k,$ and m_k are given by matrices (2.6.12), (2.6.13). Since $[v_k] \to [w_1 : 0 : 0]$ the same argument as that following (2.6.13) gives $t_{1,k}^2 t_{0,k}^{-2} \to \infty$. Moreover, since $v_k \to \infty\sigma'$ we have

$$t_{0,k}^2 \langle B_k(e - \epsilon), e - \epsilon \rangle = \langle v_k, e - \epsilon \rangle \to \infty.$$

But $\{B_k\}$ is contained in a compact set of invertible matrices, so we also have $t_{0,k}^2 \to \infty$. It follows from (2.6.6) that adding a constant $\ell = (\ell_1, \ell_{\frac{1}{2}}, \ell_0)$ will not affect the limiting homothety class. It follows that $[v_k + \ell_k] \to [w_1 : 0 : 0]$ in $\overline{D}^{std}$ which verifies condition (1) of proposition 2.6.2.

Condition (2) is verified by induction on the rank of G. For sufficiently large k we have

$$p_0(v_k) \in p_0((\sigma')^o) \subset C_0(\epsilon)$$

and $[p_0(v_k)]$ converges in $\overline{D}_0^{RBS}$ to the point w_0. By induction, the sequence $[p_0(v_k) + p_0(\ell_k)] \in D_0$ also converges to $\lim[p_0(v_k)] = w_0$ in $\overline{D}_0^{RBS}$. $\square$

(2.8) Blowups of Polyhedral Cones. Let $\sigma \subset C^* \subset V$ be a rational polyhedral cone with $\sigma^o \subset C$. Define $[\sigma^o] \subset D$ and $[\sigma] \subset \overline{D}^{std}$ to be the quotients modulo homotheties. The *RBS blowup* $[\sigma]^{RBS} \subset \overline{D}^{RBS}$ is the closure of $[\sigma^o] \subset D$ in the reductive Borel-Serre compactification of D. The restriction of the mapping $\mu : \overline{D}^{RBS} \to \overline{D}^{std}$ to $[\sigma]^{RBS}$ will be denoted

$$\mu_\sigma : [\sigma]^{RBS} \to [\sigma]. \tag{2.8.1}$$

Now let $\sigma \subset C^*$ be a rational polyhedral cone with $\sigma^o \subset C$. The interior of each face of σ is either contained in C or it is contained in some rational boundary component of C. Suppose $\tau_1 \subset \sigma \cap C_1(\epsilon)^*$ is a proper face of σ whose interior τ_1^o is contained in the proper rational boundary component $C_1(\epsilon)$ with normalizing parabolic subgroup $P = P(\epsilon)$. Let $p_i : V \to V_i(\epsilon)$ denote the linear projection (for $i = 1, 0$). Set $D_i = C_i(\epsilon)/\text{homotheties}$ (for $i = 1, 0$) and let $\Psi_P : D_P \to D_1 \times D_0$ denote the diffeomorphism of (2.4.18). Define the rational polyhedral cone

$$\tau_0 = p_0(\sigma) \subset C_0(\epsilon)^*. \tag{2.8.2}$$

(2.8.3) Proposition. *For any $z \in [\tau_1^o] \subset D_1$ we have $\mu_\sigma^{-1}(z) \subset \overline{D}_P^{RBS}$ and*

$$\overline{\Psi}_P((\mu_\sigma)^{-1}(z)) = \{z\} \times \overline{[\tau_0]}^{RBS} \subset D_1 \times \overline{D}_0^{RBS}$$

where $\overline{[\tau_0]}^{RBS}$ denotes the closure of $[\tau_0^o] \subset D_0$ in the reductive Borel-Serre compactification of D_0.

(2.8.4) Proof. First let us show that $\overline{\Psi}_P(\mu_\sigma)^{-1}(z)) \subset \{z\} \times \overline{[\tau_0]}^{RBS}$. Fix $w \in \mu_\sigma^{-1}(z) \subset \overline{D}^{RBS}$. Then w lies in some RBS boundary component D_Q for which $P = P(\epsilon) = Q^\dagger$ (cf (2.4.15) and Corollary 2.6.3). Put $\overline{\Psi}_P(w) = (z, w_0) \in D_1 \times \overline{D}_0^{RBS}$. Since $w \in \overline{[\sigma]}^{RBS}$, it is a limit of points $y_k \in [\sigma^o]$. By Proposition 2.6.2, $y_k \to z$ in $\overline{D}^{std}$ and the sequence $p_0(y_k) \in [\tau_0^o]$ converges in $\overline{D}_0^{RBS}$ to the point w_0. This proves that $w_0 \in \overline{[\tau_0]}^{RBS}$ as claimed.

Now let us verify the reverse inclusion. We will show that $\overline{\Psi}_P(\mu_\sigma^{-1}(z)) \supset \{z\} \times [\tau_0^o]$; then the full statement follows from the fact that $\mu_\sigma^{-1}(z)$ is a closed subset of $\overline{D}^{RBS}$. So choose $w_0 \in [\tau_0^o]$. Then there is a point $v_0 \in \sigma^o$ so that $[p_0(v_0)] = w_0$. Also, choose any lift $z' \in C_1(\epsilon)$ of $z \in D_1$ and let $z_k \in \sigma^o$ be any sequence so that $z_k \to z'$ in the Satake topology of C^* and hence also in the usual topology of V. Then $p_0(z_k) \to 0$. Now consider the sequence

$$v_k = \frac{z_k}{\sqrt{||p_0(z_k)||}} + v_0$$

Then $v_k \in \sigma^o$ because σ^o is a convex cone; in particular this sequence lies in a Siegel set. Moreover, $[v_k] \to z$ in $\overline{D}^{std}$ because the homothety class is dominated by the first term, while $[p_0(v_k)] \to [p_0(v_0)] = w_0$ in D_0. By proposition 2.6.2 this implies that the sequence $[v_k] \in [\sigma^o]$ converges in $\overline{D}^{RBS}$ to the point $\overline{\Psi}_P^{-1}(z, w_0)$ as claimed. □

(2.9) Theorem B. *Let $C \subset V$ be a rational self adjoint homogeneous cone in a rationally defined real vectorspace. Let $\sigma \subset C^*$ be a rational polyhedral cone with $\sigma^o \subset C$. Then both $\overline{[\sigma]}^{RBS}$ and $\overline{[\sigma]}$ are compact and contractible. Moreover, each admits the structure of a cell complex which is the closure of the single cell $[\sigma]$, such that $(\mu_\sigma) : \overline{[\sigma]}^{RBS} \to \overline{[\sigma]}$ is a cellular mapping with contractible fibers.*

(In fact, we believe that $\overline{[\sigma]}^{RBS}$ is homeomorphic to a closed ball.)

(2.9.1) Proof. Since $\sigma \subset V$ is a closed convex polyhedral cone, it admits the structure of a subanalytic set, and its quotient modulo homotheties $[\sigma]$ is compact and is subanalytically homeomorphic to a convex polyhedron. The subset $[\sigma]^{RBS}$ also admits the structure of a subanalytic set so that the mapping $\mu_\sigma : [\sigma]^{RBS} \to [\sigma]$ is subanalytic. (To see this, it is necessary to check that all the mappings involved in the definition of the topology on $\overline{D}^{RBS}$ are locally subanalytic). Hence, both sets may be Whitney stratified so that the mapping μ_σ is a "weakly" stratified map.

Let $z \in [\sigma]$. Then z lies in the interior $[\tau_1^o]$ of some face $[\tau_1] \subset [\sigma]$. If $\tau_1^o \subset C$ then the fiber $\mu_\sigma^{-1}(z)$ consists of a single point. Otherwise, τ_1^o lies in some proper boundary component $C_1(\epsilon)$, in which case the fiber $\mu_\sigma^{-1}(z)$ has been identified by (2.8.3) with a certain subset $[\tau_0]^{RBS}$ (which is the closure in $\overline{D}_0^{RBS}$ of the interior $[\tau_0^o]$ of a certain polyhedral cone τ_0 modulo homotheties). By induction, this fiber is compact and contractible. It follows that the mapping μ_σ is proper, that $[\sigma]^{RBS}$ is compact, and by proposition (8.2) it is contractible.

Finally, we sketch a proof that $[\sigma]^{RBS}$ admits the structure of a cell complex. First we obtain a "paving" of $[\sigma]^{RBS}$ by subanalytic cells. If τ_1 is a face of σ then either $\tau_1^o \subset C$ (in which case $\mu_\sigma^{-1}([\tau_1^o]) \cong \tau_1^o$ is a cell) or else τ_1^o lies in some proper boundary component $C_1(\epsilon)$, in which case $\mu_\sigma^{-1}(\tau_1^o) \cong \tau_1^o \times [\tau_0]^{RBS}$ which is, by induction, paved by cells of the form $\tau_1^o \times \text{cell of } [\tau_0]^{RBS}$. It can be shown that the closure of each of these cells is a subanalytic set. Then it follows from stratification theory that whenever W is a compact subanalytic set which is paved by subanalytic cells, then these are in fact the cells of a cell complex: the attaching maps may be constructed from Thom-Mather tubular neighborhood data.

§3 Admissible Polyhedral Decompositions

In this section we define the notion of a "sufficiently fine Γ-admissible polyhedral decomposition" Σ of a self-adjoint homogeneous cone C, and we show (Thm. 3.7) that they are cofinal in the collection of all Γ-admissible polyhedral decompositions of C.

(3.1). As in §2, we suppose that $C \subset V = \mathbf{V} \otimes_{\mathbb{Q}} \mathbb{R}$ is a rationally defined self adjoint homogeneous cone with automorphism group $G = \mathbf{G}(\mathbb{R})$, and quotient under homotheties $D = [C]$. Let $C^* \subset V$ denote the union of C and all its rational boundary components, and let $\overline{D}^{std}$ denote its quotient under homotheties, with the Satake topology. Let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be an arithmetic group. Let $\Sigma = \{\sigma\}$ be a collection of rational polyhedral cones $\sigma \subset C^*$. The collection Σ is a *polyhedral cone decomposition* of C^* provided

1. Each σ is a closed polyhedral cone in the (rationally defined) vectorspace spanned by the 1-dimensional faces of σ_α
2. If $\sigma \in \Sigma$ then each face τ of σ is also an element of Σ
3. Each intersection $\sigma \cap \tau$ is $\{0\}$ or else it is a common face of each
4. the cone C^* is the disjoint union

$$C^* = \coprod_{\sigma \in \Sigma} \sigma^o$$

of the interiors of the cones in Σ

Such a polyhedral decomposition is Γ-*admissible* provided

(1) For all $\gamma \in \Gamma$ and for any polyhedral cone $\sigma \in \Sigma$, we have $\gamma\sigma \in \Sigma$.
(2) the collection $\{\sigma\}/\Gamma$ is finite.

Let us say that a closed subset $S \subset \overline{D}^{std}$ is Γ-*small* if, for each $\gamma \in \Gamma$, either $S \cap \gamma S = \phi$ or γ acts as the identity on $S \cap \gamma S$. A polyhedral cone $\sigma \subset C^*$ is Γ-small if its homothetic quotient $[\sigma] \subset \overline{D}^{std}$ is Γ-small. The cone decomposition Σ is Γ-*fine* if every cone $\sigma \in \Sigma$ is Γ-small. Let us say that a closed subset $S \subset \overline{D}^{RBS}$ is Γ-*sufficiently small* if, for each $\gamma \in \Gamma$, either $S \cap \gamma S = \phi$ or else γ acts as the identity on $S \cap \gamma S$. A polyhedral cone $\sigma \subset C^*$ with $\sigma^o \subset C$ is Γ-sufficiently small if the closed subset $\overline{[\sigma]}^{RBS}$ is Γ-sufficiently small. A polyhedral cone $\sigma \subset C^*$ with $\sigma^o \subset C_1(\epsilon)$ is Γ-sufficiently small if the closed subset $[\sigma]^{RBS} \subset \overline{[C_1(\epsilon)]}^{RBS}$ is Γ_P-sufficiently small, (where $\Gamma_P = \Gamma \cap P$ is the intersection of Γ with the parabolic subgroup P which preserves the boundary component $C_1(\epsilon)$). The cone decomposition Σ is Γ-*sufficiently fine* if every cone $\sigma_\alpha \in \Sigma$ is Γ-sufficiently small.

If $\pi : \overline{C}^{std} \to \Gamma \backslash \overline{C}^{std}$ denotes the quotient mapping and if $\sigma \subset C^*$ is Γ-small, then $\pi|\sigma$ is a homeomorphism onto its image; in other words the quotient under Γ does not introduce any identifications on σ. Hence, *a Γ-fine polyhedral decomposition of C^* induces a ("flat") regular cell decomposition of the standard compactification* $\overline{\Gamma \backslash D}^{std} = \Gamma \backslash C^*/A_G$. If the polyhedral decomposition Σ is simplicial (meaning that it consists of simplicial cones), then the induced regular cell decomposition of $\overline{\Gamma \backslash D}^{std}$ is a ("flat") triangulation.

(3.2) Lemma. *Suppose $\sigma \subset C^*$ is a closed polyhedral cone. Let $\gamma \in \Gamma$ and suppose that $\gamma\sigma = \sigma$. Then γ acts as the identity on σ.*

(3.3) Proof. The interior of σ is contained in some boundary component C' (possibly $C' = C$). Let $P = \mathcal{U}_P G_1 G_0$ denote the maximal parabolic subgroup which normalizes C' where $G_1 = \mathrm{Aut}^0(C')$. Let $\nu_1 : P \to G_1$ denote the projection. Then the element γ acts on σ through its projection $\nu_1(\gamma) \subset \Gamma' = \nu_1(\Gamma)$. Consider the subgroup

$$\Gamma'_\sigma = \{\gamma' \in \Gamma' | \gamma'\sigma = \sigma\}$$

This group is finite since it is contained in the subset

$$\{\gamma' \in \Gamma' | \gamma'\sigma \cap \sigma \cap C' \neq \phi\}$$

(which is finite by lemma 2.7.1). Since Γ is neat, we conclude that $\Gamma'_\sigma = \{1\}$ so $\gamma \in \ker \nu_1 = Z(C')$. In other words, γ acts trivially on C', hence also on σ. $\square$

(3.4) Refinements. If Σ is a (rational) polyhedral cone decomposition of C^*, a (rational) first barycentric subdivision of Σ is determined by a choice of (rational) 1-dimensional cone (which is usually called "a barycenter") $\widehat{\sigma} \in \sigma^o$ in the interior of each cone $\sigma \in \Sigma$, and consists of simplicial cones which are spanned by 1-dimensional cones $\widehat{\sigma}_1, \widehat{\sigma}_2, \ldots, \widehat{\sigma}_k$ for each chain $\sigma_1 \subset \sigma_2 \subset \ldots \subset \sigma_k$ in Σ.

If Σ is a Γ-admissible polyhedral cone decomposition of C^* and if Σ' is a (rational) first barycentric subdivision of Σ then Σ' is Γ-admissible iff the choices $\widehat{\sigma}$ are Γ-compatible, i.e. for all $\gamma \in \Gamma$ and for all $\sigma \in \Sigma$ we have $\widehat{\gamma\sigma} = \gamma\widehat{\sigma}$. If Σ is Γ-admissible then by Lemma (3.2) there exists a Γ-compatible set of choices of (rational) barycenters.

(3.5) Lemma. *Suppose Σ is a Γ-admissible polyhedral decomposition of C^*. Let L be a closed subcomplex of Σ such that its support $|L| \subset C^*$ is Γ-small. Let L' be a choice of first (rational) barycentric subdivision of L. Then there is a choice of first (rational) barycentric subdivision Σ' of Σ which is Γ-admissible and which contains L' as a subcomplex.*

(3.6) Proof. For each cone $\sigma \in L$ and for each $\gamma \in \Gamma$ the choice of barycenter $\widehat{\sigma}$ of σ determines a unique choice of barycenter $\gamma\widehat{\sigma}$ of $\gamma\sigma$ because $|L|$ is Γ-small. Modulo Γ, there are finitely many remaining cones $\sigma \notin \Gamma.L$. Choose a single representative cone σ from each equivalence class, choose its barycenter $\widehat{\sigma}$ arbitrarily, and translate by Γ. By lemma (3.2), this gives a well defined Γ-invariant family of barycenters, so the resulting first barycentric subdivision of Σ is Γ-admissible. □

(3.7) Theorem. *Let Σ be a rational cone decomposition of C^*. Let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be a neat arithmetic group.*

1. *If Σ is Γ-sufficiently fine, then Σ is Γ-fine.*
2. *If Σ is a Γ-admissible (resp. Γ-fine, resp. Γ-sufficiently fine) decomposition of C^* and if $\Gamma' \subset \Gamma$ is an arithmetic subgroup, then Σ is a Γ'-admissible (resp. Γ'-fine, resp. Γ'-sufficiently fine) decomposition of C^*.*
3. *Suppose Σ is a Γ-admissible decomposition of C^*, and if σ' is a Γ-admissible refinement of Σ. If Σ is Γ-fine (resp. Γ-sufficiently fine) then Σ' is also Γ-fine (resp. Γ-sufficiently fine).*
4. *If Σ is Γ-admissible, then there is a refinement Σ' of Σ which is Γ-fine.*
5. *If Σ is Γ-fine, then there is a refinement Σ' of Σ which is Γ-sufficiently fine.*

(3.8) Proof. Part (1) follows from the fact that the mapping $\mu : \overline{D}^{RBS} \to \overline{D}^{std}$ commutes with the action of Γ. Parts (2) and (3) follow directly from the definitions.

Let us consider part (4). We shall prove that any rational first barycentric subdivision of any Γ-admissible decomposition Σ is a Γ-fine decomposition.

Let Σ be a Γ- admissible polyhedral decomposition of C^* and let Σ' be a Γ-admissible first barycentric subdivisioin of Σ. Let $\tau \in \Sigma'$ be a simplicial cone in the first barycentric subdivision of some simplicial cone σ. We claim that τ is Γ-small.

Write $\tau = \langle \hat{s}_0 \hat{s}_1 \ldots \hat{s}_r \rangle$ where $s_0 < s_1 < \ldots < s_r$ are faces of σ (and where $\langle v_0 v_1 \ldots v_r \rangle$ denotes the simplicial cone spanned by 1-dimensional cones $v_0, v_1, \ldots, v_r$). So

$$\gamma\tau = \langle (\gamma\hat{s}_0)(\gamma\hat{s}_1) \ldots (\gamma\hat{s}_r) \rangle.$$

Since γ acts linearly and takes 1-dimensional cones of Σ' to 1-dimensional cones of Σ', the intersection $\tau \cap \gamma\tau$ is precisely the simplex which is spanned by the 1-dimensional cones which are in common among the two sets $\{\hat{s}_0, \hat{s}_1, \ldots, \hat{s}_r\}$ and $\{(\gamma\hat{s}_0), (\gamma\hat{s}_1), \ldots, (\gamma\hat{s}_r)\}$. However, if $\hat{s}_i = \gamma\hat{s}_j$ then $s_i = \gamma s_j$ and $\dim(s_i) = \dim(s_j)$. Since the chain $s_0 < s_1 < \ldots < s_r$ is strictly increasing we conclude that $s_i = s_j$. In other words, any 1-dimensional cone $\hat{s}_i = \gamma\hat{s}_j$ which is in common between these two sets, is actually preserved by γ. Therefore, $\tau \cap \gamma\tau$ is spanned by 1-dimensional cones, each of which is preserved by γ. The last 1-dimensional cone $\hat{s}_r$ lies in (the interior of) the boundary component C' whose closure contains τ. Since Γ is neat, and γ preserves $\hat{s}_r$ it follows (as in the proof of lemma 3.2) that γ acts as the identity on C' and hence it acts as the identity on τ. This completes the proof that τ is Γ-small.

(3.9). The rest of §3 is devoted to proving part (5) of theorem 3.7. Let us say that a polyhedral cone $\sigma \subset C$ is *flag like* if its intersection with the rational boundary components of C form a flag, i.e. if there is a partial flag of boundary components

$$C_1 \subset \overline{C_2} \subset \ldots \subset \overline{C_m} \tag{3.9.1}$$

and a (partial) chain of faces

$$\tau_1 \subset \tau_2 \subset \ldots \subset \tau_m = \sigma$$

such that for each i (with $1 \le i \le m$), the interior of τ_i is contained in C_i and the intersection $\tau_i \cap \partial C_i^* = \tau_{i-1}$ where $\partial C_i^* = C_i^* - C_i$ denotes the proper rational boundary components of C_i. If $\sigma \in \Sigma$ is a flag-like polyhedral cone with associated partial flag (3.9.1), then there are associated rational idempotents $d_1, d_2, \ldots, d_m$ so that $C_1 = C_1(d_1), C_2 = C_1(d_1+d_2), \ldots, C_m = C_1(d_1+\ldots+d_m)$. Let $p_0^{(i)} : C_d \to C_0(d_1+d_2+\ldots+d_i)$ and $p_1^{(i)} : C_d \to C_1(d_1+d_2+\ldots+d_i)$ denote the linear projections which are determined by the Peirce decompositions for these idempotents.

(3.10) Lemma. *If σ is Γ-small and flag-like and if each for each i (with $1 \le i \le m-1$) the image*

$$p_0^{(i)}(\sigma) \subset C_0(d_1 + \ldots + d_i)^* \text{ is } \Gamma\text{-small} \tag{3.10.1}$$

then σ is Γ-sufficiently small.

The proof is by induction on the $\mathbb{Q}$-rank of the linear symmetric space C_m which contains σ (with the rank 0 case being trivial). For notational convenience, let us assume that $\sigma \subset C = C_m$. Let $\gamma \in \Gamma$ and suppose that σ satisfies (3.10.1). Let $x \in [\sigma]^{RBS} \cap \gamma[\sigma]^{RBS}$. We must show that $\gamma x = x$. Let $\mu : \overline{D}^{RBS} \to \overline{D}^{std}$ denote the projection. Then $x_1 := \mu(x)$ lies in some rational boundary component which is associated to σ, say

$$x_1 = \mu(x) \in [C_1(\epsilon)]$$

where $\epsilon = d_1 + d_2 + \ldots + d_k$ for some k, $1 \le k \le m$. Since μ commutes with the action of Γ, we have $x_1 = \mu(x) \in \sigma \cap \gamma\sigma$ hence (since σ is Γ-small),

$$\gamma x_1 = x_1. \tag{3.10.2}$$

It also follows that $\gamma \in \Gamma_P = \Gamma \cap P$ where P denotes the maximal parabolic subgroup which normalizes this boundary component. (Hence $x \in \overline{D}_P^{RBS}$.)

Recall (§2.4) that the Levi quotient splits as an almost direct product, $L_P \cong G_1 \times G_0$ where $G_i = \mathrm{Aut}^0(C_i(\epsilon))$ and that this splitting induces a homeomorphism

$$\overline{D}_P^{RBS} \cong \overline{D}_1^{RBS} \times \overline{D}_0^{RBS}$$

(where D_i is the linear symmetric space associated with the self adjoint homogeneous cone $C_i(\epsilon)$, $i = 0, 1$). Write $x = (x_1, x_0)$ relative to this product decomposition. The element γ acts on the RBS boundary component D_P via its projection, $\nu_P(\gamma) = (\gamma_1, \gamma_2) \in L_P$ to the Levi quotient. In summary,

$$\gamma x = (\gamma_1 x_1, \gamma_0 x_0) \in D_1 \times \overline{D}_0^{RBS}$$

By (3.10.2), since Γ is neat, we have $\gamma_1 = 1$. Therefore γ preserves the fiber $\mu^{-1}(x_1) = \{x_1\} \times [\tau_0]^{RBS}$ (by §2.4) where

$$\tau_0 = p_0(\sigma) \subset C_0(\epsilon)^*.$$

(For simplicity we write p_0 and p_1 rather than $p_0^{(k)}$ and $p_1^{(k)}$.) But $x_0 \in [\tau_0]^{RBS} \cap \gamma_0[\tau_0]^{RBS}$ The assumption (3.10.1) is that τ_0 is Γ-small. Since τ_0 is contained in the lower rank symmetric space $C_0(\epsilon)$, it follows from the induction hypothesis that $\gamma_0 x_0 = x_0$. In summary, $\gamma x = (\gamma_1 x_1, \gamma_0 x_0) = (x_1, x_0) = x$ as claimed. $\square$

(3.11). If $\sigma \subset C^*$ is a closed polyhedral cone, then any first barycentric subdivision of σ is both flag-like and simplicial. If $\sigma \in \Sigma$ is flag-like with respect to a chain of boundary components $C_1 \subset \overline{C}_2 \subset \ldots \subset \overline{C}_k$ then any polyhedral cone τ in any first barycentric subdivision of σ is flag-like with respect to some sub-chain $C_{i_1} \subset \overline{C}_{i_2} \subset \ldots \subset \overline{C}_{i_r}$. Let us say that a (closed) polyhedral cone $\sigma \subset C^*$ is *compatible* with Γ if, for each $\gamma \in \Gamma$, either $\sigma \cap \gamma\sigma = \phi$ or else it is a face of σ. If $\sigma \subset C^*$ is compatible with Γ, then (as in the proof of part(4) of theorem 3.7) its first barycentric subdivision consists of small cells.

(3.12) Lemma. *Suppose $\sigma \subset C^*$ is a rational (closed) polyhedral cone. Then it admits a rational refinement, each of whose polyhedral cones is compatible with Γ.*

(3.13) Proof. By taking a rational first barycentric subdivision if necessary, we may assume that σ is flag-like with respect to some chain $C_1 \subset \overline{C}_2 \subset \ldots \subset \overline{C}_m$ of rational boundary components. For $i = 1, 2, \ldots, m$ let $P_i = P(C_i)$ denote the corresponding normalizing maximal parabolic subgroups and define

$$\Gamma_{\sigma,i} = \{\gamma \in \Gamma |\ \gamma\sigma \cap \sigma \cap C_i \neq \phi \text{ and } \gamma\sigma \cap \sigma C_j = \phi \text{ for all } j > i\}$$

The flag-like assumption on σ guarantees that $\Gamma_{\sigma,i} \subset P_i$. (Since the C_i form a chain of boundary components, for any $\gamma \in \Gamma$ and for any $i \neq j$ we have $\gamma C_i \cap C_j = \phi$. Since σ is flag-like, it is the disjoint union of the subsets $\sigma \cap C_j$. So, if $\gamma \in \Gamma_{\sigma,i}$ then $\gamma(\sigma \cap C_i) \cap (\sigma \cap C_i) \neq \phi$. Hence $\gamma C_i \cap C_i \neq \phi$ so $\gamma \in P(C_i)$.)

The discrete group $\Gamma'_i = \nu_i(\Gamma \cap P_i)$ acts on the boundary component C_i, where ν_i is the composite projection

$$P_i \to L_{P_i} = G_{1,i}G_{0,i} \to G_{1,i}/(G_{1,i} \cap G_{0,i})$$

By [AMRT] II§4.3 p.116, the set

$$\Gamma'_{\sigma,i} = \nu_i(\Gamma_{\sigma,i}) \subset \Gamma'_i$$

is finite: it is a subset of $\{\tau \in \Gamma'_i |\ \tau(\sigma \cap C_i) \cap (\sigma \cap C_i) \neq \phi\}$ which is finite. Let $\Gamma''_{\sigma,i} \subset \Gamma_{\sigma,i}$ be a set of lifts of these finitely many elements. Define $\Gamma''_\sigma = \cup_{i=1}^m \Gamma''_{\sigma,i}$. This is a finite set of elements which complete captures the possible nontrivial intersections $\sigma \cap \gamma\sigma$ for $\gamma \in \Gamma$. As in [AMRT] II§4.3, choose finitely many rational hyperplanes $H_1, H_2, \ldots, H_m$ which define σ (and its faces), i.e. so that $L_\sigma = H_1 \cap H_2 \cap \ldots \cap H_k$ and so that $\sigma = L_\sigma \cap H^+_{k+1} \cap H^+_{k+2} \ldots \cap H^+_m$ (where H^+_j denotes a chosen halfspace on one side of the hyperplane H_j). Then the connected components of the complement

$$\sigma - \bigcup \{\gamma H_j |\ \gamma \in \Gamma''_\sigma \text{ and } 1 \leq j \leq m\}$$

(and all their faces) form a Γ-compatible refinement of σ. $\square$

(3.14) Corollary. *Let $\sigma \subset C^*$ be a closed ratioinal polyhedral cone. Then σ has a rational refinement, each of whose cones are Γ-small.*

(3.15) Proof. First choose a rational refinement which is compatible with Γ. Now choose a rational Γ-invariant first barycentric subdivision of that. Now the same argument as in the proof of part (4) of theorem 3.7 shows that each cell in this barycentric subdivision is Γ-small.

(3.16) Proof of part (5). We will need to use the following lemmas from P.L. topology (e.g. [Hu] §3, Cor. 1.6; §4, 1.8; and §4, 1.9):

1. (1) If K and L are simplicial complexes and if $|K| \subset |L|$ then for some r there exists an r-th barycentric subdivsion $L^{(r)}$ of L which contains a subdivision of K.
2. (2) Let K and L be simplicial complexes and $f : K \to L$ a simplicial mapping. Given any subdivision L' of L there exists a subdivision K' of K so that $f : K' \to L'$ is simplicial.
3. (3) Let K and L be simplicial complexes, and $f : |K| \to |L|$ a continuous mapping whose restriction to each cell of K is linear. Then there are subdivisions K' of K and L' of L so that $f : K' \to L'$ is simplicial.

If $\sigma \subset C^*$ is a flag-like polyhedral cone with respect to a flag (3.9.1) of boundary components, we will say that the resulting projections $p_0^{(i)} : \sigma \to C_0(d_1 + \ldots + d_i)$ (for $1 \le i \le m-1$) are *relevant* for σ. If σ is Γ-small and flag-like and if, for each relevant projection p the image $p(\sigma)$ is Γ-small, then the same is true for every translate $\gamma\sigma$ (for any $\gamma \in \Gamma$).

Now suppose that Σ is a Γ-fine, flag-like decomposition of C^*. Modulo Γ, there are finitely many pairs (σ, p) where $\sigma \in \Sigma$ and where p is a relevant projection for σ. Order a collection of unique representatives (modulo Γ) of these pairs in any way, $(\sigma_1, p_1), (\sigma_2, p_2), \ldots, (\sigma_n, p_n)$ (so a given polyhedral cone may be repeated many times in this ordering). Let us suppose by induction that we have found a refinement Σ' of Σ with the following property:

(P_{m-1}) Whenever $\eta \in \Sigma'$ is a polyhedral cone which is contained in some σ_i (where $1 \le i \le m-1$) then $p_i(\eta)$ is Γ-small. (It follows that η is Γ-sufficiently small because the projections which are relevant for η are a subset of the projections which are relevant for σ_i).

Now let us further refine Σ' so that the same holds for all polyhedral cones contained in σ_m. Let $K \subset \Sigma'$ be the simplicial complex consisting of all simplices in σ_m. Let L be the simplicial complex consisting of the polyhedral cone $p_m(\sigma_m)$ together with all its faces. By (3) above, there are subdivisions K' of K and L' of L so that the projection $p_m : K' \to L'$ is simplicial. By Corollary 3.14 there is a different subdivision L'' of L so that the cones in L'' are Γ-small. Let L''' be the common refinement of L' and L''. By (2) above, there is a subdivision (let us call it K''') of K' so that $p_m : K''' \to L'''$ is simplicial. By (1) above, there is a barycentric refinement $K^{(r)}$ which contains a subdivision of K''' as a subcomplex. By Lemma 3.5 this barycentric refinement may be extended to a Γ-invariant barycentric refinement $\Sigma^{(r)}$ of Σ. Since the property of being Γ-small is inherited by closed subsets, every simplex in $\Sigma^{(r)}$ whose support is contained in $\sigma_1 \cup \ldots \cup \sigma_m$ is small and all of their relevant projections are also small. This verifies condition (P_m) above and completes the inductive step. $\square$

§4 Hermitian symmetric spaces

The main result in this chapter is Theorem C (§4.2). This, together with Theorem B (§2.9) are the main technical results which are needed for the proof of Theorem A.

In this chapter we suppose that $\mathbf{G}$ is semisimple over $\mathbb{Q}$ and that the associated symmetric space D is Hermitian. We may assume that D is a bounded symmetric domain in some $\mathbb{C}^N$. Denote by $\overline{D}$ its closure in $\mathbb{C}^N$; it is a disjoint union of boundary components. The action of G on D extends continuously to the closure $\bar{D}$. As above, let us fix a basepoint $x_0 \in D$ and a standard minimal rational parabolic subgroup $\mathbf{Q}_0 \in \mathbf{G}$.

(4.1). Suppose $F \subset \overline{D}$ is a rational boundary component. Let $\mathbf{P} \subset \mathbf{G}$ be its normalizing subgroup: it is a maximal rational parabolic subgroup of $\mathbf{G}$. After conjugating by some element of $\mathbf{G}(\mathbb{Q})$ (if necessary), we may assume that $\mathbf{P}$ is standard, i.e. $\mathbf{P} \supset \mathbf{Q}_0$. The Levi quotient $L_P = P/\mathcal{U}_P$ splits as an almost direct product $L_P = G_h G_\ell$ where G_h acts transitively on F, and where G_ℓ acts transitively on a certain self adjoint homogeneous cone $C_P \subset \mathfrak{z}$. (Here, $\mathfrak{z} \subset \mathfrak{u}_P$ is the center of the Lie algebra $\mathfrak{u}_P$ of the unipotent radical $\mathcal{U}_P$ of P.) Furthermore, $A_P \subset G_\ell$. The choice of basepoint $x_0 \in D$ determines basepoints $z_0 \in F$, $e \in C_P$; it determines maximal compact subgroups $K_h = K \cap G_h(x_0) = \mathrm{Stab}_{G_h}(z_0) \subset G_h$, $K_\ell = K \cap G_\ell(x_0) = \mathrm{Stab}_{G_\ell}(e)$; and it determines diffeomorphisms $G_h/K_h \cong F$, $G_\ell/K_\ell \cong C_P$. The mapping

$$\psi : (G_h/K_h) \times (G_\ell/K_\ell) = F \times C_P \to L_P/K_P \cong P/K_P\mathcal{U}_P \tag{4.1.1}$$

given by

$$(g_hK_h, g_\ell K_\ell) \mapsto g_h g_\ell K_P \tag{4.1.2}$$

is a diffeomorphism.

Let $[C_P]$ denote the quotient of C_P under the torus of homotheties; hence $[C_P] \cong G_\ell/K_\ell A_P$, and let $D_P = P/K_PA_P\mathcal{U}_P$ denote the stratum of the reductive Borel-Serre compactification $\overline{D}^{RBS}$ corresponding to the parabolic subgroup P. Then the above diffeomorphism ψ induces a diffeomorphism

$$\Psi_P : F \times [C_P] \xrightarrow{\cong} P/K_PA_P\mathcal{U}_P = D_P \tag{4.1.3}$$

which extends to a stratum preserving homeomorphism (which is smooth on each stratum),

$$\overline{\Psi}_P : \overline{F}^{RBS} \times \overline{[C_P]}^{RBS} \xrightarrow{\cong} \overline{D}_P^{RBS} \subset \overline{D}^{RBS} \tag{4.1.4}$$

on the reductive Borel-Serre partial compactifications. Composing the canonical projection $D = P/K_P \to P/K_P\mathcal{U}_P$ with the diffeomorphism ψ^{-1} and with projection to the two factors F and C_P defines smooth mappings $\Phi_h : D \to F$, $\phi_\ell : D \to C_P$, and $\Phi_\ell : D \to [C_P]$, i.e.

$$\begin{aligned} \Phi_h(gK_P) &= gK_P\mathcal{U}_PG_\ell \in P/K_P\mathcal{U}_PG_\ell \cong F \\ \phi_\ell(gK_P) &= gK_P\mathcal{U}_PG_h \in P/K_P\mathcal{U}_PG_h \cong C_P \\ \Phi_\ell(gK_P) &= gK_PA_P\mathcal{U}_PG_h \in P/K_PA_P\mathcal{U}_PG_h \cong [C_P] \end{aligned} \tag{4.1.5}$$

for any $gK_P \in P/K_P \cong D$. Then the following diagram commutes; the composition across the top row is ϕ_ℓ and the composition across the bottom row is Φ_ℓ,

$$\begin{array}{ccccccc} D & \longrightarrow & L_P/K_P & \overset{\cong}{\underset{\psi}{\longleftarrow}} & F \times C_P & \longrightarrow & C_P \\ \| & & \downarrow & & \downarrow & & \downarrow \\ D & \underset{\Phi_P}{\longrightarrow} & D_P & \overset{\cong}{\underset{\Psi_P}{\longleftarrow}} & F \times [C_P] & \longrightarrow & [C_P] \end{array} \tag{4.1.6}$$

If $\{y_k\} \subset C_P \subset \mathfrak{z}$ is a sequence of points, we say that $y_k \to \infty C_P$ if, for all $c \in C_P$ there exists $N = N(c)$ so that $k \geq N \implies y_k - c \in C_P$.

(4.2) Theorem C. *Suppose* $\mathbf{G}$ *is a semisimple algebraic group over* $\mathbb{Q}$*, that* $D = G/K$ *is a Hermitian symmetric space,* $F \subset \overline{D}$ *is a rational boundary component,* $\mathbf{P} \subset \mathbf{G}$ *is the maximal parabolic subgroup which normalizes* F*,* $L_P = G_h G_\ell$ *is its Levi quotient, and* $C_P = G_\ell/K_\ell \subset \mathfrak{z}$ *is the associated self adjoint homogeneous cone. Let* $\{x_k\} \subset D$ *be a sequence of points. Assume that*

(1) $\phi_\ell(x_k) \to \infty C_P$
(2) $\Phi_\ell(x_k)$ *converges to some point* $c \in \overline{[C_P]}^{RBS}$ *in the reductive Borel-Serre compactification of* $[C_P]$
(3) $\Phi_h(x_k)$ *converges to some point* $t \in F$.

Then the sequence $\{x_k\}$ *converges in* $\overline{D}^{RBS}$ *to the point* $x_\infty = \overline{\Psi}_P(t,c)$.

(4.3) Preliminaries to the proof. Write $x_k = u'_k b'_k m'_k K_P \in P/K_P = D$ relative to the canonical rational Langlands decomposition $P = \mathcal{U}_P A_P(x_0) M_P(x_0)$ of P. There is a canonical positive generator $\beta \in \chi(A_P)$ of the (1-dimensional) character module. We would like to say that the sequence $\{x_k\}$ converges in $\overline{D}^{RBS}$ provided

(a$'$) $\beta(b'_k) = f^P_\beta(x_k) \to \infty$ and
(b$'$) the sequence $\Phi_P(x_k)$ converges in $\overline{D}_P^{RBS}$.

since it is easy to verify that (1) $\Longrightarrow$ (a$'$) and that (2),(3) $\Longrightarrow$ (b$'$). Unfortunately it is not true that conditions (a$'$) and (b$'$) guarantee convergence in the reductive Borel-Serre compactification. Instead, we must verify the criteria of Lemma RBS §1.3.6.

(4.4) First reduction. The limit point c lies in some RBS boundary component D_{Q_ℓ} of $[C_P]$ which corresponds to some rational parabolic subgroup, say, $Q_\ell \subseteq G_\ell$. Then $Q := \mathcal{U}_P i_{x_0}(Q_\ell G_h)$ is independent of the choice of basepoint, and it is the parabolic subgroup which corresponds to the RBS boundary component $\overline{\Psi}_P(F \times D_{Q_\ell})$ of D which contains the limit point x_∞. After conjugating by an element of $\mathbf{G}(\mathbb{Q})$ (if necessary), we may assume that $\mathbf{Q} \supset \mathbf{Q}_0$, i.e., that $\mathbf{Q}$ is standard. We may also assume the basepoint $x_0 \in D$ is rational for $\mathbf{Q}_0$. Then $\mathbf{S} = i_{x_0}\mathbf{S}(\mathbf{Q_0})$ is a rationally defined maximal $\mathbb{Q}$-split torus in $\mathbf{G}$ and

$$\mathbf{S} \subset \mathbf{Q_0} \subset \mathbf{Q} \subset \mathbf{P} \subset \mathbf{G}. \tag{4.4.1}$$

Let $\Delta = \Delta(S,G)$ denote the resulting set of simple roots. Then the (maximal) parabolic subgroup $\mathbf{P}$ corresponds to the subset $\Delta - \{\beta\}$ (for some $\beta \in \Delta$), and the parabolic subgroup Q corresponds to some subset $I \subset \Delta - \{\beta\}$. By Lemma RBS we need to show

(a) $\Phi_Q(x_k) \in D_Q$ converges to the point $x_\infty = \overline{\Psi}_P(t,c) \in D_Q$
(b) $f^{Q_0}_\alpha(x_k) \to \infty$ for all $\alpha \in \Delta - I$

The mapping $\Phi_Q : D \to D_Q$ factors as the composition in the following diagram, which is easily seen (using 1.1.12) to be commutative,

$$\begin{array}{ccccc} D & \xrightarrow[\Phi_P]{} & D_P & \overset{\cong}{\underset{\Psi}{\longleftarrow}} & F \times [C_P] \\ & & \Phi_{\bar{Q}} \big\downarrow & & \big\downarrow I \times \Phi_{Q_\ell} \\ & & D_Q & \overset{\cong}{\underset{\overline{\Psi}}{\longleftarrow}} & F \times D_{Q_\ell} \end{array}$$

(where $\overline{Q} = \nu_P(Q) = G_h Q_\ell$ is the image of Q in the Levi quotient of P). Hypothesis (2) and Corollary 1.3.8 guarantee that $\Phi_{Q_\ell}(\Phi_\ell(x_k)) \to c \in D_{Q_\ell}$. Together with hypothesis (3), this implies that $\Phi_Q(x_k) \to x_\infty$. Moreover, it implies that

$$\Phi_P(x_k) \to x_\infty \text{ in } \overline{D}_P^{RBS} \tag{4.4.2}$$

Now let us verify condition (b). By (4.4.2) and Lemma RBS (§1.3.6) we have

$$f_\alpha^{\bar{Q}_0}(\Phi_P(x_k)) \to \infty \text{ for all } \alpha \in \Delta - (I \cup \{\beta\})$$

where $\overline{Q}_0 = \nu_P(Q_0) \subset L_P$. Now apply 1.3.3 (with x replaced by x_k, I replaced by $\Delta - \{\beta\}$, and Q replaced by Q_0). This gives

$$f_\alpha^{Q_0}(x_k) \to \infty \text{ for all } \alpha \in \Delta - (I \cup \{\beta\}).$$

Thus, in order to prove Theorem C, it remains to show that the hypotheses imply:

$$f_\beta^{Q_0}(x_k) \to \infty. \tag{4.4.3}$$

The splitting $L_P = G_h G_\ell$ induces a splitting $\nu_P(Q_0) = Q_{0h} Q_{0\ell}$ such that Q_{0h} is a minimal rational parabolic subgroup of G_h and $Q_{0\ell}$ is a minimal rational parabolic subgroup of G_ℓ, each of which has an associated canonical rational Langlands decomposition:

$$Q_{0h} = \mathcal{U}_{0h} A_{0h}(z_0) M_{0h}(z_0) \qquad Q_{0\ell} = \mathcal{U}_{0\ell} A_{0\ell}(e) M_{0\ell}(e) \tag{4.4.4}$$

Hence the canonical rational Langlands decomposition of Q_0 is given by

$$Q_0 = \mathcal{U}_P i_{x_0}(\mathcal{U}_{0h}\mathcal{U}_{0\ell}) i_{x_0}(A_{0h}(z_0) A_{0\ell}(e)) i_{x_0}(M_{0h}(z_0) M_{0\ell}(e))$$

so $x_k \in D = Q_0/K_{Q_0}$ may be expressed as follows,

$$x_k = u_P^{(k)} u_{0h}^{(k)} u_{0\ell}^{(k)} a_{0h}^{(k)} a_{0\ell}^{(k)} m_{0h}^{(k)} m_{0\ell}^{(k)} K_{Q_0}.$$

Then

$$f_\beta^{Q_0}(x_k) = \beta(a_{0h}^{(k)} a_{0\ell}^{(k)})$$

Since $\Phi_h(x_k) = u_{0h}^{(k)} a_{0h}^{(k)} m_{0h}^{(k)} \in G_h/K_h \cong F$ converges, we see that *the sequence $a_{0h}^{(k)}$ converges to some element $a_{0h}^\infty \in A_{0h}$*. Hence, in order to verify (4.4.3) it suffices to prove that $\beta(a_{0\ell}^{(k)}) \to \infty$.

Set $y_k = \Phi_\ell(x_k) = u_{0\ell}^{(k)} a_{0\ell}^{(k)} m_{0\ell}^{(k)} K_{0\ell} \in G_{0\ell}/K_{0\ell} \cong C_P$. Then $\beta(a_{0\ell}^{(k)}) = f_\beta^{Q_{0\ell}}(y_k)$ so it suffices to prove the following:

(4.5) Proposition. *Suppose* $\mathbf{G}$ *is a semisimple algebraic group over* $\mathbb{Q}$, *that* $D = G/K$ *is a Hermitian symmetric space,* $\mathbf{Q_0} \subset G$ *is a minimal rational parabolic subgroup and* $x_0 \in D$ *is a basepoint which is rational for* Q_0. *Let* $S = S_{Q_0}(x_0) \subset Q_0$ *be the resulting maximal* $\mathbb{Q}$*-split torus. Let* $\Delta = \Delta(S, G)$ *denote the simple rational roots of* G *occurring in* $\mathcal{U}_{Q_0}$. *Fix* $\beta \in \Delta$ *and let* $\mathbf{P} = \mathbf{P}(\Delta - \{\beta\}) \subset \mathbf{G}$ *be the maximal rational proper parabolic subgroup corresponding to the subset* $\Delta - \{\beta\}$, *with Levi quotient* $\nu_P : P \to L_P = G_h G_\ell$ *and corresponding self adjoint homogeneous cone* $C_P = G_\ell / K_\ell \subset \mathfrak{z}$ *with induced basepoint* $e \in C_P$. *Let* $Q_{0\ell} = \nu_P(Q_0) \cap G_\ell$ *be the corresponding minimal rational parabolic subgroup of* G_ℓ *and let* $f_\beta^{Q_{0\ell}} : C_P \to (0, \infty)$ *be the resulting root function on* C_P. *Let* $\{y_k\} \in C_P$ *be a sequence of points and suppose that* $y_k \to \infty C_P$. *Then* $f_\beta^{Q_{0\ell}}(y_k) \to \infty$.

We do not know any simple proof of this fact, although it is easy to verify in special cases (e.g. $G = Sp(2n, \mathbb{R}), SU(n, 1)$, or the Hilbert modular cases). The general proof requires explicit formulae for the roots of $\mathbf{G}$. In §6.7 we will prove the analog of Proposition 4.5 for algebraic groups $\mathbf{G}$ which are simple over $\mathbb{R}$, and in §6.6 we will prove proposition 4.5 for algebraic groups $\mathbf{G}$ which are simple over $\mathbb{Q}$. The general case follows from this.

§5. Real Roots of G

Throughout this chapter we suppose that $\mathbf{G}$ is a semisimple algebraic group defined over $\mathbb{R}$ (and for much of the chapter, $\mathbf{G}$ is assumed to be simple over $\mathbb{R}$). Most of the chapter consists of the explicit description of the roots of $\mathbf{G}$ and their relationship to the associated Jordan algebras; these facts are recalled from [BB], [AMRT], and [He]. We use this description to prove a special case of Proposition 4.5 at the end of the chapter.

(5.1). Let $\mathbf{G}$ be a semisimple algebraic group defined over $\mathbb{R}$. Let $G = \mathbf{G}(\mathbb{R})^0$ be the connected component of the group of real points, and let $K \subset G$ be a maximal compact subgroup. This corresponds to a choice of basepoint $x_0 \in D = G/K$ and a Cartan involution on G. Let $\mathfrak{g} = \mathfrak{g}_\mathbb{R} = \mathrm{Lie}(G)$ denote the Lie algebra of G, with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. We suppose the symmetric space $D = G/K$ is Hermitian. So there is an invariant complex structure $J : \mathfrak{p} \to \mathfrak{p}$ with $J^2 = -1$. Extending J to a complex linear involution on the complexification $\mathfrak{p}_\mathbb{C}$ determines a decomposition $\mathfrak{p}_\mathbb{C} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ into $\pm i$ eigenspaces of J; hence $\mathfrak{g}_\mathbb{C} = \mathfrak{k}_\mathbb{C} \oplus \mathfrak{p}_+ \oplus \mathfrak{p}_-$. Let $\mathfrak{t} \subset \mathfrak{k}$ be a (compact) Cartan subgroup and $\Phi_\mathbb{C} = \Phi(\mathfrak{t}_\mathbb{C}, \mathfrak{g}_\mathbb{C})$ be the roots of $\mathfrak{t}_\mathbb{C}$ in $\mathfrak{g}_\mathbb{C}$. Then we have the root decomposition

$$\mathfrak{g}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \sum_{\phi \in \Phi_\mathbb{C}} \mathfrak{g}_\mathbb{C}^\phi$$

As in [BB], [He] §VI (3.1) and §VIII (7.1), it is possible to choose root vectors $e_\phi \in \mathfrak{g}_\mathbb{C}^\phi$ and vectors $h_\phi \in i\mathfrak{t}$ such that

(1) $[e_\phi, e_{-\phi}] = h_\phi$
(2) $\bar{e}_\phi = e_{-\phi}$ whenever $e_\phi \in \mathfrak{p}_\pm$.
(3) $\psi(h_\phi) = \frac{2\langle \psi, \phi \rangle}{\langle \phi, \phi \rangle}$ for all $\psi \in \Phi_\mathbb{C}$.

Set $x_\phi = e_\phi + e_{-\phi}$, and $y_\phi = i(e_\phi - e_{-\phi})$ and let $\pi_+ = \{\phi \in \Phi_{\mathbb{C}} | e_\phi \in \mathfrak{p}_+\}$. Then $\{x_\phi, y_\phi\}_{\phi\in\pi_+}$ form a basis of $\mathfrak{p}$. For any $\phi \in \Phi_{\mathbb{C}}$ set $\epsilon_\phi = \frac{1}{2}(y_\phi - ih_\phi)$. The proof of the following is a direct computation:

(5.1.1) Lemma. *For all $\phi \in \Phi_{\mathbb{C}}$ we have $[\frac{1}{2}x_\phi, \epsilon_\phi] = \epsilon_\phi$.* □

(5.2). In this section we recall the explicit description of the roots of G relative to a real split torus, under the additional assumption that G *is simple over* $\mathbb{R}$. As in [He] §VIII (7.4), choose a maximal set of strongly orthogonal roots $\{\gamma'_1, \gamma'_2, \dots, \gamma'_r\} \subset \Phi_{\mathbb{C}}$ such that $\mathfrak{a} = \sum_{i=1}^r \mathbb{R}x_i$ is a maximal abelian subgroup of $\mathfrak{p}$. (Here, and in what follows, we use the notation $x_i = x_{\gamma'_i}$, $y_i = y_{\gamma'_i}$, etc.) Let $\Phi_{\mathbb{R}} = \Phi(\mathfrak{a}, \mathfrak{g})$ denote the roots of $\mathfrak{a}$ in $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}}$, so $\mathfrak{g} = Z(\mathfrak{a}) + \sum_{\phi\in\Phi_{\mathbb{R}}} \mathfrak{g}^\phi$. (Here, $Z(\mathfrak{a})$ denotes the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$.) Define $\gamma_1, \gamma_2, \dots, \gamma_r \in \mathfrak{a}^*$ by

$$\gamma_i(x_j) = 2\delta_{ij} \text{ (for } 1 \le i, j \le r) \tag{5.2.1}$$

Then each $\gamma_i \in \Phi_{\mathbb{R}}$ is a root of multiplicity one ([BB] §1.15, [AMRT] p. 185) and there is an explicit description of $\Phi_{\mathbb{R}}$ in terms of these elements. If $D = G/K$ is irreducible then there are two possibilities. The roots are the nonzero elements in the collection

$$\begin{array}{rcll} \Phi_{\mathbb{R}} &=& \left\{\frac{\pm\gamma_i\pm\gamma_j}{2}\right\}_{1\le i\le j\le r} & \text{case } \mathrm{C}_r \\ \Phi_{\mathbb{R}} &=& \left\{\frac{\pm\gamma_i\pm\gamma_j}{2}, \frac{\pm\gamma_i}{2}\right\}_{1\le i\le j\le r} & \text{case } \mathrm{BC}_r \end{array}$$

Choose a linear order on the set of roots $\Phi_{\mathbb{R}}$. We assume that the ordering $\{\gamma_1, \gamma_2, \dots, \gamma_r\}$ is chosen so that $i < j$ iff $\gamma_i > \gamma_j$. Then the resulting set of simple roots $\Delta_{\mathbb{R}} = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$ is given as follows:

$$\begin{array}{rcll} \alpha_i &=& \frac{1}{2}(\gamma_i - \gamma_{i+1}) & \text{for } 1 \le i \le r-1 \\ \alpha_r &=& \gamma_r & \text{case } \mathrm{C}_r \\ \alpha_r &=& \frac{1}{2}\gamma_r & \text{case } \mathrm{BC}_r. \end{array} \tag{5.2.2}$$

and the positive roots $\Phi_{\mathbb{R}}^+$ are given by

$$\begin{array}{rcll} \Phi_{\mathbb{R}}^+ &=& \left\{\frac{\gamma_i+\gamma_j}{2}\right\}_{1\le i\le j\le r} \cup \left\{\frac{\gamma_i-\gamma_j}{2}\right\}_{1\le i<j\le r} & \text{case } \mathrm{C}_r \\ \Phi_{\mathbb{R}}^+ &=& \left\{\frac{\gamma_i+\gamma_j}{2}\right\}_{1\le i\le j\le r} \cup \left\{\frac{\gamma_i-\gamma_j}{2}\right\}_{1\le i<j\le r} \cup \left\{\frac{\gamma_i}{2}\right\}_{1\le i\le r} & \text{case } \mathrm{BC}_r \end{array}$$

(5.3) Parabolic subgroups. Let Q_0 be the minimal (real) parabolic subgroup of G which is generated by $N = \exp(\mathfrak{n})$ and by $Z(A)$ where $A = \exp(\mathfrak{a})$ and $\mathfrak{n} = \sum\{\mathfrak{g}^\phi | \ \phi \in \Phi_{\mathbb{R}}^+\}$ The parabolic subgroups which contain Q_0 are called *standard*. Fix n with $1 \le n \le r$. The *maximal* proper (real) standard parabolic subgroup $P = P(\Delta - \{\alpha_n\})$ may be described as follows. Set $w_n = x_1 + x_2 + \ldots + x_n$. Define

$$\mathfrak{u} = \sum \left\{\mathfrak{g}^\phi | \phi \in \Phi_{\mathbb{R}}^+ \text{ and } \phi(w_n) > 0\right\} \tag{5.3.1}$$

The sum in (5.3.1) is over

$$\left\{\tfrac{\gamma_i+\gamma_j}{2}\right\}_{1\le i\le j\le n} \quad \cup \quad \left\{\tfrac{\gamma_i\pm\gamma_j}{2}\right\}_{\substack{1\le i\le n\\ j>n}} \qquad \text{case } C_r$$

$$\left\{\tfrac{\gamma_i+\gamma_j}{2}\right\}_{1\le i\le j\le n} \quad \cup \quad \left\{\tfrac{\gamma_i\pm\gamma_j}{2}\right\}_{\substack{1\le i\le n\\ j>n}} \quad \cup \quad \left\{\tfrac{\gamma_i}{2}\right\}_{1\le i\le n} \qquad \text{case } BC_r$$

(The possible values $\phi(w_n)$ are $0, \pm1, \pm2$ so the 1-parameter subgroup generated by w_n acts on $\mathfrak{u}$ with eigenvalues 0,1, and 2.) Set

$$\mathcal{U}_P = \exp(\mathfrak{u}) \text{ and } A_P = \bigcap_{j\neq n} \ker(\alpha_j) \tag{5.3.2}$$

Then $P = P(\Delta - \{\alpha_n\}) = Z(A_P)\mathcal{U}_P$ has unipotent radical $\mathcal{U}_P$ and Levi factor $L_P = Z(A_P) = G_hG_\ell$ which we now describe.

Let $\mathfrak{m}(\mathfrak{a}) = Z(\mathfrak{a}) \cap \mathfrak{k}$ be the intersection of $\mathfrak{k}$ with the centralizer of $\mathfrak{a}$. This "compact factor" appears in the minimal parabolic subgroup Q_0 and hence in each standard parabolic subgroup. Write

$$\mathfrak{a}_\ell = \sum_{i=1}^{n} \mathbb{R}x_i \text{ and } \mathfrak{a}_h = \sum_{i=n+1}^{r} \mathbb{R}x_i$$

Then $\mathfrak{a}_\ell$ is the Lie algebra of the torus

$$A_\ell = \left(\bigcap_{i=n+1}^{r} \ker(\gamma_i)\right)^0 = \left(\bigcap_{i=n+1}^{r} \ker(\alpha_i)\right)^0 \tag{5.3.3}$$

by (5.2.2), and $\mathfrak{a}_h$ is the Lie algebra of the torus

$$A_h = \left(\bigcap_{i=1}^{n} \ker(\gamma_i)\right)^0 \tag{5.3.4}$$

(It is not true that $A_h = \bigcap_{1\le i\le n} \ker(\alpha_i)$ nevertheless, by (5.2.2), $A_h \subset \bigcap_{1\le i\le n-1} \ker(\alpha_i)$.) Then

$$\mathrm{Lie}(G_\ell) = \mathfrak{a}_\ell + \sum_{\substack{\phi=\pm(\gamma_i-\gamma_j)/2\\ 1\le i<j\le n}} (\mathfrak{g}^\phi + [\mathfrak{g}^\phi, \mathfrak{g}^{-\phi}] \cap \mathfrak{m}(\mathfrak{a}))$$

$$\mathrm{Lie}(G_h) = \mathfrak{a}_h + \sum_{\phi} (\mathfrak{g}^\phi + [\mathfrak{g}^\phi, \mathfrak{g}^{-\phi}] \cap \mathfrak{m}(\mathfrak{a}))$$

where the second sum is taken over all nonzero ϕ in the collection

$$\phi \in \left\{\tfrac{\pm\gamma_i\pm\gamma_j}{2}\right\}_{n<i<j\le r} \qquad \text{in case } \mathrm{C}_r$$

$$\phi \in \left\{\tfrac{\pm\gamma_i\pm\gamma_j}{2}\right\}_{n<i<j\le r} \cup \left\{\tfrac{\pm\gamma_j}{2}\right\}_{n<j} \qquad \text{in case } \mathrm{BC}_r$$

(5.4) Jordan Algebra. Let

$$\mathfrak{z} = \sum \left\{ \mathfrak{g}^{\phi} | \phi = \frac{\gamma_i + \gamma_j}{2}, \text{for } 1 \leq i \leq j \leq n \right\}$$

denote the center of $\mathfrak{u}$. The parabolic group P acts on $\mathfrak{z}$ via the adjoint action. The subgroup G_ℓ acts with an open orbit $C = C_P = G_\ell \cdot e$ which is an open self adjoint homogeneous cone in $\mathfrak{z}$ with respect to the positive definite inner product $\langle x, y \rangle = -B(x, \sigma(y))$ where B denotes the Killing form and σ denotes the Cartan involution. Then $K_\ell = K \cap G_\ell$ is the stabilizer of the basepoint $e \in C_P$ and we obtain a diffeomorphism

$$G_\ell / K_\ell \cong C_P. \tag{5.4.1}$$

If we denote by $\mathfrak{g}_\ell = \mathfrak{k}_\ell \oplus \mathfrak{p}_\ell = (\mathfrak{k} \cap \mathfrak{g}_\ell) \oplus (\mathfrak{p} \cap \mathfrak{g}_\ell)$ the Cartan decomposition of $\mathfrak{g}_\ell = \mathrm{Lie}(G_\ell)$, then the differential of (5.4.1) gives an isomorphism

$$\mathfrak{p}_\ell \cong T_1(G_\ell/K_\ell) \cong T_e C_P \cong \mathfrak{z} \tag{5.4.2}$$

which is given by $x \mapsto \phi(x)(e)$ where

$$\phi : \mathfrak{g}_\ell \to \mathrm{End}(\mathfrak{z}) \tag{5.4.3}$$

is the differential of the adjoint action $G_\ell \to GL(\mathfrak{z})$. The vectorspace $\mathfrak{z}$ has the following Jordan algebra structure: for any $a \in \mathfrak{z}$ there is (by 5.4.2) a unique element $T_a \in \mathfrak{p}_\ell$ such that $\phi(T_a)(e) = a$. Then $a \bullet b = \phi(T_a)(b)$ for $a, b \in \mathfrak{z}$. It follows that the closure $\bar{C}_P = \{x^2 | x \in \mathfrak{z}\}$.

For each j $(1 \leq j \leq n)$ set $\epsilon_j = \frac{1}{2}(y_j - ih_j)$. It follows from Lemma 5.1.1 that

$$\epsilon_j \in \mathfrak{g}^{\gamma_j} \text{ and } [\frac{1}{2} x_j, \epsilon_j] = \epsilon_j \tag{5.4.4}$$

(5.5) Proposition. *The collection* $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$ *is a complete set of mutually orthogonal idempotents of* $\mathfrak{z}$*, and* $e = \sum_{j=1}^n \epsilon_j$ *is the identity element, relative to this Jordan algebra structure.*

(5.6) Proof. By (5.4.4), $T_{\epsilon_j} = \frac{1}{2} x_j$ because $[\frac{1}{2} x_j, \epsilon] = [\frac{1}{2} x_j, \epsilon_j] = \epsilon_j$ so $\epsilon_j \bullet \epsilon_j = \epsilon_j$ and $\epsilon_i \bullet \epsilon_j = 0$ (for $i \neq j$). The collection is complete since this is a maximal set of idempotents and $\sum_{j=1}^n \mathbb{R} x_j$ is a maximal abelian subalgebra in $\mathfrak{p}_\ell \cong \mathfrak{z}$. The identification of e with the basepoint is in [AMRT] p. 242. $\square$

(5.7) Root function. The minimal parabolic subgroup $Q_0 \subset P$ determines a minimal parabolic subgroup $Q_{0\ell} = \nu_P(Q_0) \cap G_\ell \subset G_\ell$ (where $\nu_P : P \to P/\mathcal{U}_P = L_P$ denotes the projection to the Levi quotient). Associated to the simple root $\alpha_n \in \Phi_{\mathbb{R}}$ (cf. (5.3.2) and §1.2) we have the root function

$$f = f_{\alpha_n}^{Q_{0\ell}} : C_P \to (0, \infty)$$

If $g \in Q_{0\ell}$ is given by $g = uam \in \mathcal{U}_{Q_{0\ell}} A_{Q_{0\ell}} M_{Q_{0\ell}}$ relative to the canonical real Langlands decomposition of $Q_{0\ell}$, then $f(ge) = a^{\alpha_n}$. It follows from (5.2.2) and (5.3.3) that

$$f(ge) = \begin{cases} a^{\gamma_n} & \text{if} \quad n < r \\ a^{\gamma_n} & \text{if} \quad n = r \quad \text{in case } \mathrm{C}_r \\ a^{\gamma_n/2} & \text{if} \quad n = r \quad \text{in case } \mathrm{BC}_r \end{cases}$$

(5.8) Proposition. *Suppose G is $\mathbb{R}$-simple. Let $\{y_k\} \subset C_P$ be a sequence of points in C_P and suppose that $y_k \to \infty C_P$. Then $f(y_k) \to \infty$.*

(5.9) Proof. Since ϵ_n is an idempotent, we have $\epsilon_n \in \partial C_P$. Therefore $\langle y_k, \epsilon_n\rangle \to \infty$. Write $y_k = g_k e$ with $g_k \in Q_{0\ell}$. Then $\langle g_k e, \epsilon_n\rangle = \langle e, g_k^\tau \epsilon_n\rangle \to \infty$ where g^τ denotes the transpose of g. Now let $f_1 = \epsilon_1$, $f_2 = \epsilon_1 + \epsilon_2$, $\ldots$, $f_n = \epsilon_1 + \ldots + \epsilon_n = e$. The parabolic subgroup $Q_{0\ell}$ is the subgroup of G which normalizes the flag

$$0 \subset V_1(f_1) \subset V_1(f_2) \subset \ldots \subset V_1(f_n) = \mathfrak{z}$$

In fact, for each m with $1 \le m \le n$ the maximal parabolic subgroup P_m (which corresponds to the simple root α_m) preserves the subspace

$$\mathfrak{z}_m = \sum_{1 \le i,j \le m} \mathfrak{g}^{(\gamma_i + \gamma_j)/2}$$

since it is the Lie algebra of the center of the unipotent radical of P_m. Hence $P_1 \cap P_2 \cap \ldots \cap P_n$ preserves the flag $0 \subset \mathfrak{z}_1 \subset \mathfrak{z}_2 \subset \ldots \subset \mathfrak{z}_n$. So $Q_{0\ell}^\tau$ is the subgroup of H which normalizes the flag

$$\mathfrak{z} \supset V_0(f_1) \supset V_0(f_2) \supset \ldots \supset V_0(f_n) = \{0\}$$

But $V_0(f_{n-1}) = V_1(\epsilon_n) = \mathfrak{g}^{\gamma_n}$ is one dimensional ([BB] §1.15, [AMRT] p. 185). Let $g^\tau \in Q_{0\ell}^\tau$. Then $g^\tau \cdot \epsilon_n = \lambda \epsilon_n$ for some $\lambda = \lambda(g^\tau) \in \mathbb{R}$. Then λ is a real character of $Q_{0\ell}^\tau$, it is trivial on $\mathcal{U}_{Q_{0\ell}^\tau}$ and on $M_{Q_{0\ell}^\tau}$ and it coincides with γ_n on $A_{Q_{0\ell}^\tau} = A_{Q_{0\ell}}$ (because $\mathfrak{g}^{\gamma_n}$ is the subspace of $\mathfrak{g}$ on which $A_{Q_{0\ell}}$ acts with weight γ_n). Hence $\langle e, g_k^\tau \epsilon_n\rangle = \gamma_n(a_k)\langle e, \epsilon_n\rangle = a_k^{\gamma_n}.1 \to \infty$ where $g_k = u_k a_k m_k \in \mathcal{U}_{Q_{0\ell}} A_{Q_{0\ell}} M_{Q_{0\ell}}$. It follows that $f(y_k) = a_k^{\gamma_n} \to \infty$. □

§6. Rational Theory

(6.1). In this chapter we complete the proof of Proposition 4.5. Throughout the chapter we assume that $\mathbf{G}$ is an algebraic group defined over $\mathbb{Q}$ which is semisimple over $\mathbb{Q}$, and that $D = G/K$ is a Hermitian symmetric space. We refer to the statement of Proposition 4.5 for the definitions and choices of the following items, which will be fixed throughout this chapter: $\mathbf{Q_0} \subset \mathbf{G}$, $x_0 \in D$, $\mathbf{S} = \mathbf{S_{Q_0}}(x_0) \subset \mathbf{Q_0}$, $\Delta_{\mathbb{Q}} = \Delta(S, G)$, $\beta \in \Delta_{\mathbb{Q}}$, $\mathbf{P} = \mathbf{P}(\Delta_{\mathbb{Q}} - \{\beta\})$, $\nu_P : P \to L_P = G_h G_\ell$, $C_P = G_\ell / K_\ell \subset \mathfrak{z} \subset \mathfrak{u} = \mathrm{Lie}(\mathcal{U}_P)$, $e \in C_P$, $Q_{0\ell} = \nu_P(Q_0) \cap G_\ell$, and $f = f_\beta^{Q_{0\ell}} : C_P \to (0, \infty)$. Let $\{y_k\} \subset C_P$ and suppose that $y_k \to \infty C_P$. We must show that $f(y_k) \to \infty$.

Throughout §6.2 to §6.7 we assume that $\mathbf{G}$ is simple over $\mathbb{R}$.

(6.2). We must compare the real roots and the rational roots. Let $\mathbf{T}$ be a maximal $\mathbb{R}$ split torus in $\mathbf{G}$ with $\mathbf{S} \subset \mathbf{T}$. We assume that $\mathbf{T}$ is defined over $\mathbb{Q}$. Choose a minimal real parabolic subgroup $\mathbf{P}_0 \subset \mathbf{G}$ such that $\mathbf{S} \subset \mathbf{T} \subset \mathbf{P}_0 \subset \mathbf{Q}_0$. Let us denote the corresponding groups of real points by

$$A_{Q_0} = \mathbf{S}(\mathbb{R})^0 \subset A_{P_0} = \mathbf{T}(\mathbb{R})^0 \subset P_0 \subset Q_0.$$

Associated to these choices there are root systems $\Phi_{\mathbb{Q}} = \Phi(S, G)$ and $\Phi_{\mathbb{R}} = \Phi(T, G)$, positive roots $\Phi_{\mathbb{Q}}^+$, $\Phi_{\mathbb{R}}^+$ (which occur in the unipotent radical of P_0 and Q_0 respectively), and simple roots $\Delta_{\mathbb{Q}} \subset \Phi_{\mathbb{Q}}^+$, $\Delta_{\mathbb{R}} \subset \Phi_{\mathbb{R}}^+$. A fundamental result of Baily and Borel [BB] states:

(6.3) Lemma. *For each $\phi \in \Delta_Q$ there exists a unique simple root $\phi' \in \Delta_{\mathbb{R}}$ such that $\phi = \phi'|\mathbf{S}$. Let $\Delta'_{\mathbb{R}} \subset \Delta_{\mathbb{R}}$ denote the resulting subset of $\Delta_{\mathbb{R}}$. If $\phi \in \Delta_{\mathbb{R}}$ then*

$$\phi \notin \Delta'_{\mathbb{R}} \text{ iff } \phi|\mathbf{S} = 1 \tag{6.3.1}$$

(*i.e. the remaining simple roots are trivial on* $\mathbf{S}$). □

(6.4). The root $\beta \in \Delta_{\mathbb{Q}}$ corresponds (by Lemma 6.3) to a unique real root $\beta' \in \Delta_{\mathbb{R}}$. It follows from (6.3.1) that β' gives rise to the same maximal parabolic subgroup,

$$\mathbf{P} = \mathbf{P}(\Delta_{\mathbb{Q}} - \{\beta\}) = \mathbf{P}(\Delta_{\mathbb{R}} - \{\beta'\})$$

because

$$\mathbf{S}(\Delta_{\mathbb{R}} - \{\beta'\}) = \bigcap_{\phi \in \Delta_{\mathbb{R}} - \{\beta'\}} \ker(\phi) \subset \mathbf{T}(\mathbb{R})$$

coincides with

$$\mathbf{S}(\Delta_{\mathbb{Q}} - \{\beta\}) = \bigcap_{\phi \in \Delta_{\mathbb{Q}} - \{\beta\}} \ker(\phi) \subset \mathbf{S}(\mathbb{R}).$$

We obtain a minimal real parabolic subgroup $P_{0\ell} = \nu_P(P_0) \cap G_\ell \subset G_\ell$ and an associated root function $f_{\beta'}^{P_{0\ell}} : C_P \to (0, \infty)$.

(6.5) Lemma. *The root functions $f_{\beta'}^{P_{0\ell}}$ and $f_{\beta}^{Q_{0\ell}}$ coincide.*

(6.6) Proof. Define

$$\mathbf{T_{an}} = \bigcap_{\phi \in \chi(\mathbf{T})_{\mathbb{Q}}} \ker(\phi) \subset \mathbf{T}$$

to be the maximal $\mathbb{Q}$-anisotropic torus in $\mathbf{T}$, where $\chi(\mathbf{T})_{\mathbb{Q}}$ denotes the group of rationally defined characters of $\mathbf{T}$. Then $A_{an} = \mathbf{T_{an}}(\mathbb{R})^0 \subset M_{Q_0}$. Since $\mathbf{S} \subset \mathbf{T}$ is a maximal $\mathbb{Q}$-split torus in $\mathbf{T}$, we have an almost direct product (cf. [B] §8.14), $\mathbf{T} = \mathbf{S}.\mathbf{T_{an}}$ from which it follows that

$$A_{P_0} = A_{Q_0} A_{an}. \tag{6.6.1}$$

In fact, this is a direct product since the intersection is finite but each factor is torsion free. If $a \in A_{P_0}$ decomposes as $a = bb_{an}$ in (6.6.1), then b *is the unique element in* A_{Q_0} *such that*

$$\phi'(a) = \phi'(b) = \phi(b) \text{ for all } \phi \in \Delta_{\mathbb{Q}} \tag{6.6.2}$$

since the corresponding elements $\phi' \in \Delta'_{\mathbb{R}}$ form a rational basis of the module $\chi(\mathbf{T})_{\mathbb{Q}} \otimes \mathbb{Q}$ of rationally defined characters of $\mathbf{T}$.

Now let $y = ge$ for some $g \in P_{0\ell} \subset Q_{0\ell}$. Decompose

$$g = ua_{\mathbb{R}}m_{\mathbb{R}} = ua_{\mathbb{Q}}m_{\mathbb{Q}} \tag{6.6.3}$$

relative to the canonical *real* Langlands decomposition $P_{0\ell} = \mathcal{U}(P_{0\ell})A(P_{0\ell})M(P_{0\ell})$ and relative to the canonical *rational* Langlands decomposition $Q_{0\ell} = \mathcal{U}(Q_{0\ell})A(Q_{0\ell})M(Q_{0\ell})$. Using (6.6.1), write $a_{\mathbb{R}} = bb_{an}$ with $b \in A_{Q_0}$ and $b_{an} \in A_{an}$. We claim that $b \in A_{Q_{0\ell}}$ and that $b_{an} \in M(Q_{0\ell})$. It suffices to show that $b \in G_\ell$ (from which it follows that $b_{an} = a_{\mathbb{R}}b^{-1} \in G_\ell$ as well). By (5.3.3) the linear part $A_{Q_{0\ell}}$ is the intersection of the kernels of certain real simple roots $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset \Delta_{\mathbb{R}}$. If $\alpha_i \notin \Delta'_{\mathbb{R}}$ then $\alpha_i(b) = 1$ by (6.3.1). If $\alpha_i \in \Delta'$ then by (6.6.2), $\alpha_i(b) = \alpha_i(a_{\mathbb{R}}) = 1$, which completes the proof of the claim.

Now it follows that $g = u.b.(b_{an}m)$ is the rational Langlands decomposition of g, in other words, that $b = a_{\mathbb{Q}}$. We conclude (again using (6.6.2)) that

$$f_{\beta'}^{P_{0\ell}}(y) = \beta'(a_{\mathbb{R}}) = \beta'(b)\beta'(b_{an}) = \beta(a_{\mathbb{Q}}).1 = f_{\beta}^{Q_{0\ell}}(y)$$

which completes the proof of the lemma. □

(6.7) Proof of Proposition 4.5 when G is $\mathbb{R}$-simple. Let $\{y_k\} \subset C(\beta)$ be a sequence of points in the cone C_P with $y_k \to \infty C_P$. Then by Proposition 5.8, we have $f_{\beta'}^{P_{0\ell}}(y_k) \to \infty$, but by Lemma 6.5, $f_{\beta'}^{P_{0\ell}}(y_k) = f_{\beta}^{Q_{0\ell}}(y_k)$. □

(6.8) The $\mathbb{Q}$-simple case. Throughout §6.8 to §6.9 we assume that $\mathbf{G}$ is simple over $\mathbb{Q}$ but is not necessarily simple over $\mathbb{R}$. By Baily-Borel [BB] and Borel-Tits [BT], there exists an algebraic group $\mathbf{G}'$ defined over a totally real number field k such that $\mathbf{G}'$ is absolutely simple (and hence is simple over $\mathbb{R}$) and

$$\mathrm{Res}_{k/\mathbb{Q}}\mathbf{G}' = \mathbf{G}$$

(where Res denotes Weil's restrictions of scalars). Let $\sigma_1, \ldots, \sigma_d$ denote the d different embeddings of k into $\mathbb{R}$ and write $k_i = \sigma_i(k) = k^{\sigma_i}$. Set $\mathbf{G}_i = (\mathbf{G}')^{\sigma_i}$. Then there is an isomorphism of real algebraic groups,

$$\mathbf{G} \cong \mathbf{G}_1 \times \ldots \times \mathbf{G}_d$$

We have chosen a maximal $\mathbb{Q}$-split torus $\mathbf{S} \subset \mathbf{G}$. There is an isomorphism over k of tori, $\mathbf{S} \cong \mathbf{S}'$, where $\mathbf{S}' \subset \mathbf{G}'$ is a maximal k-split torus (and so $\mathbf{S}$ is isomorphic to a maximal $\mathbb{Q}$ split torus in $\mathrm{Res}_{k/\mathbb{Q}}(\mathbf{S}')$). Let $S_i = (S_i)^{\sigma_i}$ be the corresponding maximal k_i-split torus in $\mathbf{G}_i$. These tori are all isomorphic over $\mathbb{R}$ so we may identify the root systems

$$\Phi(\mathbf{S}, \mathbf{G}) \leftrightarrow \Phi(\mathbf{S}', \mathbf{G}') \leftrightarrow \Phi(\mathbf{S}_i, \mathbf{G}_i) \tag{6.8.1}$$

and corresponding subsets of positive and simple roots $\Delta_{\mathbb{Q}} = \Delta(\mathbf{S}, \mathbf{G})$, $\Delta(\mathbf{S}', \mathbf{G}')$, $\Delta(\mathbf{S}_i, \mathbf{G}_i)$ respectively. The chosen simple root $\beta \in \Delta(\mathbf{S}, \mathbf{G})$ corresponds to simple roots $\beta_i \in \Delta(\mathbf{S}_i, \mathbf{G}_i)$.

The minimal and maximal rational parabolic subgroups $\mathbf{Q}_0 \subset \mathbf{P} = \mathbf{P}(\Delta_{\mathbb{Q}} - \{\beta\}) \subset \mathbf{G}$ correspond to minimal and maximal k_i-parabolic subgroups $\mathbf{Q}_{0i} \subset \mathbf{P}_i \subset \mathbf{G}_i$. Then, as real parabolic subgroups, we have an isomorphism

$$P \cong P_1 \times \ldots \times P_d.$$

Write $C = C_P = G_\ell / K_\ell$ for the self adjoint homogeneous cone associated to P, and write $C_i = C_{P_i} = G_{\ell,i}/K_{\ell,i}$ for the corresponding cones associated to P_i. Then we also have a diffeomorphism,

$$C \cong C_1 \times C_2 \times \ldots \times C_d$$

Each minimal parabolic subgroup $\mathbf{Q}_{0i} \subset \mathbf{G}_i$ determines a corresponding minimal parabolic subgroup $\mathbf{Q}_{0\ell,i} \subset \mathbf{G}_{\ell,i}$ with root function

$$f_i = f_{\beta_i}^{Q_{0\ell,i}} : C_i \to (0, \infty).$$

(6.9) Proof of Proposition 4.5 when G is $\mathbb{Q}$-simple. If $y_k \in C$ is a sequence of points, with $y_k \to \infty C$ then the corresponding factors $y_k^{(i)} \in C_i$ also satisfy $y_k^{(i)} \to \infty C_i$. Since G_i is $\mathbb{R}$-simple, we may apply (6.7) to each factor (replacing $\mathbb{Q}$ by k_i in the statement and proof of Case 1) to conclude that each of the root functions $f_i(y_k^{(i)}) \to \infty$.

On the other hand, we claim that $f_\beta(y_k) = f_i(y_k^{(i)})$ for each i, from which the result follows. Write $y_k = g_k e$ with $g_k \in Q_0$. Then $g_k = (g_k^{(1)}, \ldots, g_k^{(d)}) \in Q_{0,1} \times \ldots \times Q_{0,d}$, each of which may be decomposed according to compatibly chosen Langlands decompositions,

$$g_k^{(i)} = u_k^{(i)} a_k^{(i)} m_k^{(i)}$$

However, by our identification (6.8.1) of the roots and the torus we have,

$$f_\beta(y_k) = a_k^\beta = (a_k^{(i)})^{\beta_i} \text{ for each } i.$$

This completes the proof of Proposition 4.5. □

§7 Statement and Proof of Theorem A

(7.1). Let $\mathbf{G}$ be a semisimple algebraic group defined over $\mathbb{Q}$. Let $G = \mathbf{G}(\mathbb{R})$ be the group of real points. Let us assume that $G = \mathrm{Aut}^0(D)$ is the connected component of the group of automorphisms of a Hermitian symmetric space D. Let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be a neat arithmetic subgroup. Set $X = \Gamma\backslash D$. Let $\{\Sigma_F\}$ be a Γ-admissible collection of polyhedral cone decompositions of the homogeneous cones C_F (as F runs over all the rational boundary components of D). In [AMRT] this data is used to construct a *toroidal compactification* $\overline{X}_\Sigma^{tor}$ of X. Let us say that the collection of polyhedral decompositions Σ is Γ-*sufficiently fine* if, for every rational parabolic subgroup $\mathbf{P} = \mathbf{P}(F)$ the polyhedral decomposition

Σ_F of C_F^* is Γ_ℓ-sufficiently fine (where $\Gamma_\ell = G_\ell \cap \nu_P(\Gamma \cap P)$, with $\nu_P : P \to L_P = G_h G_\ell$ the projection to the Levi quotient).

Let $\overline{X}^{RBS}$ denote the reductive Borel-Serre compactification and let $\overline{X}^{BB}$ denote the Baily-Borel Satake compactification of X. Let

$$X \xrightarrow{d} X \times X \hookrightarrow \overline{X}^{RBS} \times \overline{X}^{tor}$$

denote the diagonal inclusion of X into the product of two compactifications of X, and let $\widehat{X}$ denote the closure of the image of X in $\overline{X}^{RBS} \times \overline{X}^{tor}$. Let $\bar{a} \in \overline{X}^{tor}$, and define

$$\widehat{X}(\bar{a}) = \{\bar{b} \in \overline{X}^{RBS} \mid (\bar{b}, \bar{a}) \in \widehat{X}\} \subset \overline{X}^{RBS}.$$

Then $\widehat{X}(\bar{a}) = \theta_1 \theta_2^{-1}(\bar{a}) \cong \theta_2^{-1}(\bar{a})$ where θ_1, θ_2 denote the projections from $\widehat{X}$ to $\overline{X}^{RBS}$ and $\overline{X}^{tor}$ respectively.

(7.2) Theorem A. *Suppose the collection $\{\Sigma_F\}$ of polyhedral cone decompositions is Γ-sufficiently fine. Then for any $\bar{a} \in \overline{X}^{tor}$ the set $\widehat{X}(\bar{a})$ is contractible.*

In fact we will show that the set $\widehat{X}(\bar{a})$ is canonically homeomorphic to a set of the form $[\sigma]^{RBS}$ which, by theorem B, is contractible.

(7.3) Corollary. *If the collection $\{\Sigma_F\}$ of polyhedral decompositions is Γ-sufficiently fine, then the resolution $g : \overline{X}^{tor} \to \overline{X}^{BB}$ is homotopic to a mapping $g' : \overline{X}^{tor} \to \overline{X}^{BB}$ which factors through the projection $\overline{X}^{RBS} \to \overline{X}^{BB}$. Moreover the mapping g' may be taken to be the identity on the complement K of a regular neighborhood of the boundary $\partial\overline{X}^{tor} = \overline{X}^{tor} - X$.*

Such a closed subset $K \subset X$ is described in [Le] and [Sa] where it is shown to be homotopy equivalent to X.

(7.4) Proof of Corollary. By (8.2) there exists a homotopy inverse $\tau : \overline{X}^{tor} \to \hat{X}$ to the projection θ_2. Take g' to be the composition

$$\overline{X}^{tor} \xrightarrow{\tau} \hat{X} \xrightarrow{\theta_2} \overline{X}^{RBS} \longrightarrow \overline{X}^{BB}.$$

By (8.3) the mapping τ may be taken to be the identity on $K \subset X$. $\square$

(7.5). The rest of §7 is devoted to the proof of Theorem A. Let us recall some of the notation which is involved in the construction of the toroidal resolution. If we suppose that $\bar{a} \in \partial\overline{X}^{tor}$ then its image in the Baily-Borel compactification $\overline{X}^{BB}$ lies in some stratum which we may take to be an arithmetic quotient of some rational boundary component F. Let $P = \mathrm{Norm}(F)$ be the maximal (proper) rational parabolic subgroup which normalizes F. Let $\mathcal{U}_F = \mathcal{U}_P$ denote its unipotent radical and $Z_F = Z_P$ denote the center of $\mathcal{U}_P$. Set $\Lambda_F = Z_F \cap \Gamma$. The exponential mapping $\mathfrak{z}_F \to Z_F(\mathbb{R})$ determines a vectorspace structure on the group $Z_F(\mathbb{R})$ which in turn admits an integral structure so that the inclusion $\Lambda_F \hookrightarrow Z_F(\mathbb{R})$ induces a vectorspace isomorphism $\Lambda_F \otimes \mathbb{R} \cong Z_F(\mathbb{R})$ which takes Λ_F isomorphically to $Z_F(\mathbb{Z})$. Then $T(F) = Z_F(\mathbb{C})/\Lambda_F$ is an algebraic torus with cocharater group $\Lambda_F \cong \chi_*(T(F))$: each $\lambda \in \Lambda_F$ determines a mapping $\mathbb{C} \to Z_F(\mathbb{C})$ by $t \mapsto \lambda t$ and this determines a cocharater $\mathbb{C}^* = \mathbb{C}/\mathbb{Z} \to Z_F(\mathbb{C})/\Lambda_F$. In summary we will use these isomorphisms to make the canonical identifications

$$\chi_*(T(F)) \otimes \mathbb{R} \cong Z_F(\mathbb{R}) \cong \mathfrak{z}_F(\mathbb{R}).$$

The Levi quotient $L_P = P/\mathcal{U}_P$ decomposes as an almost direct product, $L_P = G_\ell \times G_h$ of a "linear" and a Hermitian factor. The action by conjugation of G_ℓ on Z_F induces a diffeomorphism between G_ℓ/K_ℓ and the open orbit $C_F \subset Z_F$. The adjoint action of G_ℓ on $\mathfrak{z} = \mathrm{Lie}(Z_F)$ induces a diffeomorphism between G_ℓ/K_ℓ and the open orbit $C_P \subset \mathfrak{z}$. The exponential mapping $\exp : C_P \to C_F$ is compatible with these diffeomorphisms. Abusing the notation of (4.1.5), we will write $\phi_\ell : D \to C_F$ and $\Phi_\ell : D \to [C_F]$ for the resulting projections. (And, as in (4.1.5) we also have a projection $\Phi_h : D \to F$.)

Let Σ_F be an admissible rational polyhedral cone decomposition of the standard compactification $\overline{C}_F^{std}$. Set $D(F) = Z_F(\mathbb{C}).D$ (the product taken in the compact dual symmetric space). Then there is a holomorphic isomorphism given by Siegel coordinates, [AMRT] p. 235, [S] §III (Thm. 7.1), [KW]

$$D(F) \cong F \times (\mathcal{U}_F/Z_F)(\mathbb{C}) \times Z_F(\mathbb{C}) \tag{7.5.1}$$

with the following properties:

(1) There exists a real bilinear form h_t which depends real analytically on the parameter $t \in F$ such that

$$D = \{(t, w, z) |\ \mathrm{Im}(z) - h_t(w, w) \in C_F\}.$$

(2) For all $(t, w, z) \in D$ we have $\Phi_h(t, w, z) = t$ and

$$y = \phi_\ell(t, w, z) = \mathrm{Im}(z) - h_t(w, w). \tag{7.5.2}$$

(Abusing notation, we will write $(\mathcal{U}_F/Z_F)(\mathbb{C}) = \mathbb{C}^k$ for simplicity.) Then the torus $T(F)$ acts freely on the quotient $\Lambda_F \backslash D(F)$ with quotient $D(F)' = F \times \mathbb{C}^k$. So $\Lambda(F)\backslash D(F)$ is a "torus bundle" over $D(F)'$, the fibers of which will be compactified in the following paragraph.

The choice Σ_F of polyhedral cone decomposition determines a partial compactification $(T(F))_\Sigma$ of the torus $T(F)$. This gives rise to a partial compactification

$$(\Lambda_F\backslash D(F))_\Sigma = \Lambda_F\backslash D(F) \times_{T(F)} (T(F))_\Sigma \tag{7.5.3}$$

Let $(\Lambda_F\backslash D)_\Sigma$ denote the interior of the closure of $\Lambda_F\backslash D(F)$ in $(\Lambda_F\backslash D(F))_\Sigma$. The action of $\Lambda_F\backslash\Gamma_F$ on $\Lambda_F\backslash D(F)$ extends to an action of $\Lambda_F\backslash\Gamma_F$ on the partial compactification $(\Lambda_F\backslash D(F))_\Sigma$. (Here, $\Gamma_F = \Gamma \cap P$.)

(7.5.4) Fact. ([AMRT] p. 250) There exists a collection of maps $\pi_F : (\Lambda_F\backslash D)_{\Sigma_F} \to \overline{X}^{tor}$ such that

1. each π_F is open and analytic with discrete fibers.
2. The mapping π_F commutes with the action of $\Lambda_F\backslash\Gamma_F$ and, near the boundary it induces an embedding $(\Lambda_F\backslash D)_\Sigma/(\Lambda_F\backslash\Gamma_F) \hookrightarrow \overline{X}^{tor}$
3. The union, taken over all Γ conjugacy classes of rational boundary components F,
$$\coprod_F (\Lambda_F\backslash D)_{\Sigma_F} \to \overline{X}^{tor}$$
is surjective.

(7.6) Notation. For simplicity we will write Z for Z_F. Let $\mathbb{T} = Z(\mathbb{R})/\Lambda$ be the compact torus in $T(F) = Z(\mathbb{C})/\Lambda$. The map ord : $Z(\mathbb{C}) \to Z(\mathbb{R})$ given by $\mathrm{ord}(x+iy) = y$ induces a map $T(F) \to Z(\mathbb{R})$. Let $T(F)_\Sigma$ be the torus embedding associated to Σ. From [AMRT] I §1.1 the map ord extends to a map

$$\mathrm{ord} : T(F)_\Sigma \to Z_\Sigma$$

where

$$Z_\Sigma = Z(\mathbb{R}) \coprod_{\sigma\in\Sigma} \mathcal{O}_\sigma$$

with $\mathcal{O}_\sigma = Z(\mathbb{R})/L_\sigma$. Each point in $\mathcal{O}_\sigma$ can be expressed as $v + L_\sigma$ with $v \in L_\sigma^\perp$. Following [AMRT] we use the (confusing) notation $v + \infty\sigma$ to denote such a point. This notation may be justified by considering the topology on Z_Σ which is defined as follows: if $\{y_n\} \subset Z$ is a sequence, then $y_n \to v + \infty\sigma$ provided the following holds: write $y_n = y_n' + y_n''$ with $y_n' \in L_\sigma^\perp$ and $y_n'' \in L_\sigma$. Then $y_n' \to v$ and for any $w \in L_\sigma$, if n is sufficiently large then $y_n'' - w \in \sigma$ (cf §2.7.2).

The map ord induces a homeomorphism $\mathbb{T}\backslash T(F)_\Sigma \cong Z_\Sigma$ Hence we may express every element in $T(F)_\Sigma - T(F)$ as $x' + i(y + \infty\sigma)$ where $x' \in \mathbb{T}$. However x' is only uniquely determined modulo the sub compact torus $\mathbb{T}_\sigma = \mathcal{O}_\sigma/\Lambda_\sigma$ where $\Lambda_\sigma = \Lambda/(\Lambda \cap L_\sigma)$.

Using Siegel coordinates (7.5.1), if $a = (t, w, z) \in D(F)$ then we will write $a' = (t, w, z')$ for its image in $\Lambda_F\backslash D(F) \cong F \times (\mathcal{U}_F/Z_F)(\mathbb{C}) \times T(F)$. The projections Φ_h and ϕ_ℓ pass to this quotient with $\Phi_h(a') = t$ and $\phi_\ell(a') = \mathrm{Im}(z') - h_t(w, w) \in C_F \subset Z(\mathbb{R})$. Now suppose $a' \in (\Lambda_F\backslash D)_\Sigma$ is a point on the boundary corresponding to some (closed) polyhedral cone $\sigma \in \Sigma_F$ with $\sigma^o \subset C_F$. We will write

$$a' = (t, w, x' + iv + i\infty\sigma) \tag{7.6.1}$$

if there is a sequence $a_n' = (t_n, w_n, z_n') \in \Lambda_F\backslash D$ such that

1. $t_n \to t \in F$
2. $w_n \to w \in (\mathcal{U}_F/Z_F)(\mathbb{C}) \cong \mathbb{C}^k$
3. $\mathrm{Re}(z_n') \to x' \in \Lambda_F\backslash Z_\mathbb{R}$
4. $\phi_\ell(a_n') \to v + \infty\sigma$

where $x' \in \Lambda_F \backslash Z_F(\mathbb{R})$ and $v \in L_\sigma^\perp$, the perpendicular complement to the linear space L_σ spanned by σ.

(7.7) Lift to the partial compactification. The purpose of this section is to lift the subset $\widehat{X}(\bar{a}) \subset \overline{X}^{RBS}$ to the reductive Borel-Serre partial compactification $\overline{D}^{RBS}$. Suppose, as above, that $\bar{a} \in \partial\overline{X}^{tor}$ projects to some stratum $X_F \subset \overline{X}^{BB}$ which is an arithmetic quotient of some rational boundary component F with normalizing maximal parabolic subgroup P. Fix any point $a' \in (\Lambda_F \backslash D)_\Sigma$ so that $\pi_F(a') = \bar{a}$. Let us say that a point $b \in \overline{D}^{RBS}$ is *closure related* to $a' \in (\Lambda_F \backslash D)_\Sigma$ if there exists a sequence $a_n \in D$ with the following two properties:

(1) $a'_n \to a'$ in $(\Lambda_F \backslash D)_\Sigma$ and

(2) $a_n \to b$ in $\overline{D}^{RBS}$.

(Here, $a'_n \in \Lambda_F \backslash D$ denotes the image of a_n mod Λ_F.)

Then $b \in \overline{D}_P^{RBS}$ is in the closure of the reductive Borel-Serre boundary component D_P. Let $\bar{a}_n \in \Gamma\backslash D = X$ denote the image of a_n modulo Γ (so $a_n \in D$, $a'_n \in \Lambda_F \backslash D$, and $\bar{a}_n \in \Gamma \backslash D$). Then we also have

(3) $\bar{a}_n \to \bar{a} = \pi_F(a')$ in $\overline{X}^{tor}$ and

(4) $\bar{a}_n \to \bar{b}$ in $\overline{X}^{RBS} = \Gamma \backslash \overline{D}^{RBS}$

since the following diagram commutes,

$$\begin{array}{ccccccc} a_n \in & D & \hookrightarrow & \overline{D}^{RBS} \times (\Lambda_F \backslash D)_\Sigma & \ni & (b, a') \\ & {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi\times\pi_F} & & \\ \bar{a}_n \in & X & \hookrightarrow & \overline{X}^{RBS} \times \overline{X}^{tor} & \ni & (\bar{b}, \bar{a}) \end{array}$$

Define

$$\Delta(a') = \{ b \in \overline{D}^{RBS} \mid b \text{ is closure related to } a' \} \subset \overline{D}_P^{RBS}.$$

If $\widehat{D}$ denotes the closure of the diagonal embedding of D in $\overline{D}^{RBS} \times (\Lambda_F \backslash D)_\Sigma$ and if θ_1 and θ_2 denote the projections of $\widehat{D}$ to the first and second factors respectively, then $\Delta(a') = \theta_1 \theta_2^{-1}(a') \cong \theta_2^{-1}(a')$.

The following proposition immediately implies Theorem A.

(7.8) Proposition. Let $\bar{a} \in \partial\overline{X}^{tor}$. Choose F, P, and $a' = (t, w, x' + iv + i\infty\sigma)$ as in (7.6.1) with $\pi_F(a') = \bar{a}$ and with $\sigma^o \subset C$. Let $[C_F]$ denote the quotient of C_F under homotheties, and (4.1.4) $\overline{\Psi} : \overline{D}_P^{RBS} \cong \overline{F}^{RBS} \times \overline{[C_F]}^{RBS}$ the resulting homeomorphism. Let $[\sigma]^{RBS} \subset \overline{[C]}_F^{RBS}$ be the closure of the quotient $[\sigma^o] \subset [C_F]$ in the reductive Borel Serre compactification. Then

(1) the projection to the second factor restricts to a homeomorphism

$$\overline{\Psi}_2 | \Delta(a') : \Delta(a') \xrightarrow{\cong} [\sigma]^{RBS}.$$

(2) if Σ is Γ-sufficiently fine then the mapping $\pi : \overline{D}^{RBS} \to \overline{X}^{RBS}$ restricts to a homeomorphism

$$\pi|\Delta(a') : \Delta(a') \xrightarrow{\cong} \widehat{X}(\bar{a})$$

(7.9) Proof. It is clear from properties (3) and (4) above that the restriction $\pi|\Delta(a')$ takes $\Delta(a')$ into $\widehat{X}(\bar{a})$. First let us show that $\pi : \Delta(a') \to \widehat{X}(\bar{a})$ is surjective. Fix any $\bar{b} \in \widehat{X}(\bar{a})$. Then $(\bar{b}, \bar{a}) \in \widehat{X}$ so there exists a sequence $\{\bar{a}_n\} \subset X$ so that $\bar{a}_n \to \bar{a}$ in $\overline{X}^{tor}$ and so that $\bar{a}_n \to \bar{b}$ in $\overline{X}^{RBS}$. Since π_F is a local homeomorphism, there is a unique lift $a'_n \in (\Lambda_F \backslash D)_\Sigma$ of the sequence $\bar{a}_n$ so that $a'_n \to a'$ in $(\Lambda_F \backslash D)_\Sigma$. As in (7.5.1) write

$$a'_n = (t_n, w_n, z'_n).$$

It follows that $t_n \to t$, $w_n \to w$ and that $y_n := \phi_\ell(a'_n) \in C_F$.

(7.9.1) Claim. There exists a subsequence (which we also denote by a'_n) so that the corresponding elements $[y_n] \in [C_F]$ converge to some point $c \in [\sigma]^{RBS}$. (Here, $[y]$ denotes the image of y under the quotient by homotheties, $C_F \to [C_F]$.)

(7.9.2) Proof of Claim. Since $y_n \to v + \infty\sigma$, and $v \in L_\sigma^\perp$ we may write $y_n = y'_n + y''_n$ with $y'_n \in L_\sigma^\perp$ and $y''_n \in \sigma^o$ for n sufficiently large. Therefore $y'_n \to v$ and $y''_n \to \infty\sigma$. Since $[y''_n] \in [\sigma^o] \subset [\sigma]^{RBS}$ which is compact, there exists a subsequence (which we denote by by $[y''_n]$ as well) which converges, $[y''_n] \to c \in [\sigma]^{RBS}$. By the proposition 2.7.3, the sequence $[y_n] = [y'_n + y''_n]$ converges in $\overline{[C]}_F^{RBS}$ to the same point c. □

(7.9.3). Using this claim, and choosing any lift $a_n \in D$ of this subsequence we see that

(1) $\phi_\ell(a_n) = y_n \to \infty C$
(2) $\Phi_h(a_n) = t_n \to t$
(3) $\Phi_\ell(a_n) = [y_n] \to c$ in $\overline{[C]}_F^{RBS}$

Here, $\Phi_h : D \to F$ and $\Phi_\ell : D \to [C_F]$ are the canonical projections (4.1.5). By Theorem X, this implies that the sequence $a_n \to b := (t, c) \in \overline{D}_P^{RBS} \cong \overline{F}^{RBS} \times \overline{[C]}^{RBS}$. In summary, $b \in \Delta(a')$ (since $a_n \to b$ and $a'_n \to a'$) and $\pi(b) = \bar{b}$ (by (4) above) which proves that $\pi|\Delta(a')$ is surjective to $\hat{X}(\bar{a})$.

(7.9.4). At this point, it is most convenient to prove part (2) of proposition (7.8): that $\psi(\Delta(a')) = \{t\} \times [\sigma]^{RBS}$ (where $a' = (t, w, x' + iv + i\infty\sigma)$). Let $b \in \Delta(a') \subset \overline{D}_P^{RBS}$, say $b = (t, c)$. Choose $\{a_n\} \subset D$ as above, with $a_n \to b$ in $\overline{D}^{RBS}$ and $a'_n \to a'$ in $(\Lambda_F \backslash D)_\Sigma$ as described above. Then (by the same argument as in the proof of the claim above), $\psi_2(b) = c \in [\sigma]^{RBS}$. Since $t \in F$ is fixed, we conclude that $\psi_2|\Delta(a')$ is injective. To see that it is surjective, let $c \in [\sigma]^{RBS}$ and choose $y_n \in \sigma$ so that $[y_n] \to c$ and define

$$a_n = (t, w, x' + iv + i\lambda_n y_n + h_t(w, w))$$

where $\lambda_n \in \mathbb{R}$ and $\lambda_n \to \infty$. Then $\Phi_h(a_n) = t$, $\Phi_\ell(a_n) \to c$ (again by proposition 2.7.3), and $\phi_\ell(a_n) \to \infty C_F$. So by Theorem C, the sequence a_n converges in $\overline{D}^{RBS}$ to the limit $b = (t, c) \in \Delta(a')$ which completes the proof of part (2).

(7.9.5). Now let us prove that the mapping $\pi : \Delta(a') \to \widehat{X}(\bar{a})$ is injective. Since

$$\begin{array}{ccccc} \Delta(a') \cong \{t\} \times [\sigma]^{RBS} & \subset & \overline{D}_P^{RBS} & \subset & \overline{D}^{RBS} \\ \downarrow & & \downarrow & & \downarrow \\ \widehat{X}(\bar{a}) & \subset & \Gamma_P \backslash \overline{D}_P^{RBS} & \subset & \overline{X}^{RBS} \end{array}$$

commutes, we see that π is injective if and only if the quotient under Γ_P does not introduce any identifications on $[\sigma]^{RBS}$. However this is precisely the assumption that σ is *sufficiently small*. It is guaranteed by the assumption that Σ is Γ-sufficiently fine. This concludes the proof of theorem 7.8 and hence also of Theorem A. □

§8. Contractible Cell Complexes

(8.1). In this section we review some standard facts from homotopy theory which are needed for the proof of Theorem B (§2.9). Throughout this section we suppose that $f : X \to Y$ is a weakly stratified mapping between two compact (finite dimensional) Whitney stratified spaces. (This means that f takes strata to strata by a smooth submersion). Let Y_n denote the (closed) union of all strata with dimension $\leq n$.

(8.2) Proposition. *Suppose that f is surjective and that each fiber $f^{-1}(y)$ is contractible. Then f is a homotopy equivalence. Moreover there exists a homotopy inverse $g : Y \to X$ such that, for each n,*

(1) *$fg(Y_n) \subset Y_n$ and*
(2) *the restriction $g|Y_n : Y_n \to f^{-1}(Y_n)$ is a homotopy inverse to the restriction $f^{-1}(Y_n) \to Y_n$.*

In particular, if Y is contractible then X is also contractible.

(8.3) Addendum. *Suppose that $K \subset Y$ is a closed union of simplices of some triangulation of Y which refines the stratification. Suppose that $f^{-1}(K) \to K$ is a homeomorphism. Then the homotopy inverse $g : Y \to X$ may be chosen so as to agree with f^{-1} on points of K.*

(8.4) Proof. This follows from standard results in homotopy theory, however in what follows we will indicate how to construct the homotopy inverse g explicitly. Fix a triangulation of Y which refines the stratification. We will find a mapping $g : Y \to X$ (which agrees with f^{-1} on K) and a homotopy $H : X \times I \to X$ between $H_0 = I$ and $H_1 = gf$ so that H is the constant homotopy (from the identity to the identity) on $K \times I$ and so that, for each simplex $\sigma \subset Y$ we have

(1) $fg(\sigma) \subset \sigma$
(2) $H(f^{-1}(\sigma) \times I) \subset f^{-1}(\sigma)$.

This is accomplished by induction on the dimension of σ. For $\dim(\sigma) = 0$ let $g(\sigma)$ be any point in the fiber $f^{-1}(\sigma)$ and let H be a homotopy which contracts the fiber $f^{-1}(\sigma)$ to the point $g(\sigma)$. For the inductive step, suppose $\sigma \subset Y$ is an n-dimensional simplex with boundary $\partial\sigma$, that $g : \partial\sigma \to f^{-1}(\partial\sigma)$ has been defined and that a homotopy $H : f^{-1}(\partial\sigma) \times [0,1] \to f^{-1}(\partial\sigma)$ between $H_0 = I$ and $H_1 = gf$ has been constructed. We wish to extend both g and H to all of σ. If $\sigma \subset K$, these extensions have already been defined: $g = f^{-1}$ and H is the constant homotopy. So we may assume that $\text{int}(\sigma) \subset Y - K$. Since the extensions will be made simplex by simplex without changing choices which were made on previous simplices, we may (for the sake of notational convenience) replace X by $f^{-1}(\sigma)$.

It is possible to triangulate X so that $f^{-1}(\partial\sigma)$ is a union of simplices. In particular, the inclusion $f^{-1}(\partial\sigma) \subset X$ is a *cofibration*, i.e. it satisfies the homtopy extension property, which we apply to the following situation:

$$\begin{array}{ccccc} & & X \times \{0\} & & \\ & & \downarrow & & \\ f^{-1}(\partial\sigma) \times \{0\} & \longrightarrow & X \times [0,1] & \overset{F}{\cdots\to} & X \\ & & \uparrow & & \\ & & f^{-1}(\partial\sigma) \times [0,1] & & \end{array}$$

Thus, we obtain a homotopy F between $F_0 = I$ and some mapping $F_1 : X \to X$ which collapses $f^{-1}(\partial\sigma)$ to the section $g(\partial\sigma)$.

Choose a trivialization

$$f^{-1}(\sigma^o) \cong \sigma^o \times A \tag{8.4.1}$$

of the mapping f over the interior σ^o of the simplex σ (such a trivialization exists by Thom's first isotopy lemma) and let $a_0 \in A$ be a point in the fiber. Let $\phi_t : A \to A$ be a contracting homotopy from $\phi_0 = I$ to the constant mapping $\phi_x(a) = a_0$. Define $g' : \sigma^o \to f^{-1}(\sigma^o)$ by $g'(y) = (y, a_0)$. Then g' is a homotopy inverse for f on the interior of σ, but we must patch together g' on the interior with g on the boundary.

(8.5) Definition. The section $g : \sigma \to X$ is given by

$$g(y) = \begin{cases} F_1 g'(y) \text{ for } y \in \sigma^o \\ g(y) \text{ for } y \in \partial\sigma \end{cases}$$

To verify that g is continuous, choose a strong deformation retraction

$$\psi : N(f^{-1}(\partial\sigma)) \to f^{-1}(\partial\sigma)$$

from a regular neighborhood of $f^{-1}(\partial\sigma)$ to $f^{-1}(\partial\sigma)$. Suppose $y_i \in \sigma^o$ is a sequence of points converging to some point $y_0 \in \partial\sigma$. Then for any choice of metric on X we have

$$\text{dist}(F_1 g'(y_i), g(y_0)) \leq \text{dist}(F_1 g'(y_i), F_1 \psi g'(y_i)) + \text{dist}(gf\psi g'(y_i), g(y_0)).$$

(since $F_1 = gf$ on $f^{-1}(\partial\sigma)$.) The first term goes to 0 because F_1 is continuous and $\text{dist}(g'(y_i), \psi g'(y_i)) \to 0$ (since ψ is a strong deformation retraction which approaches the identity as the point $g'(y_i)$ approaches the boundary $f^{-1}(\partial\sigma)$). To show that the second term goes to 0, it suffices to show that $f\psi g'(y_i) \to y_0$. But

$$\mathrm{dist}(f\psi g'(y_i), y_0) \leq \mathrm{dist}(f\psi g'(y_i), fg'(y_i)) + \mathrm{dist}(fg'(y_i), y_0).$$

Again the first term goes to 0 because $\psi \to I$ on $f^{-1}(\partial\sigma)$ while the second term is $\mathrm{dist}(y_i, y_0) \to 0$.

Now let us define the extension of the homotopy H. Choose a collaring of the boundary of the simplex,

$$r : \sigma \to [0,1]$$

so that $r^{-1}(0) = \partial\sigma$ and define the function $T : [0,1] \times [0,1] \to [0,1]$ by

$$T(r,t) = \begin{cases} t & \text{if } r+t \geq 1 \\ \frac{rt}{1-t} & \text{if } r+t \leq 1 \end{cases}$$

With respect to the trivialization (8.4.1) define

$$H_t(y,a) = F_t(y, \phi_{T(r(y),t)}(a))$$

Although the function T fails to be continuous at $(r=0, t=1)$ it is easy to see (using the same argument involving $\psi : N(f^{-1}(\partial\sigma)) \to f^{-1}(\partial\sigma))$ as above) that the function H_t is continuous as $y \to \partial\sigma$. Furthermore for $(y,a) \in f^{-1}(\sigma^o)$,

$$H_0(y,a) = F_0(y, \phi_0(a)) = (y,a)$$

$$H_1(y,a) = F_1(y, \phi_1(a)) = F_1(g'(y)) = gf(y,a)$$

as desired. This completes the proof of proposition 8.2. □

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