Direct Sum Fails for Zero Error Average Communication

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Information & Communication Complexities

\[ f : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z} \]

Alice has \( x \in \mathcal{X} \)  
Bob has \( y \in \mathcal{Y} \)

Want to compute \( f(x, y) \)

[Shannon, Yao]
A classic: The Transmission Problem
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- Alice receives $x \sim \mu$
- She wants to transmit $x$ to Bob
- Objective: minimize the expected number of bits sent
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[...Braverman-Rao] generalized this result for the interactive case when some error is allowed

[This paper] studies the question when no error is allowed
OUTLINE

• Communication and Information Complexity measures

• Information versus Amortized Communication

• Our Results
Communication Model

\[ \Pi = \Pi(x, y) \text{ is the transcript of the protocol} \]

\[ \Pi = \pi_1 \pi_2 \ldots \pi_m \]

\[ \pi_1(x) \]

\[ \pi_2(\pi_1, y) \]

\[ \pi_3(\pi_1 \pi_2, x) \]

\[ \vdots \]
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\[ \begin{align*}
\pi_1(x) \\
\pi_2(\pi_1, y) \\
\pi_3(\pi_1 \pi_2, x) \\
\vdots
\end{align*} \]

We also allow randomness...
Distributional Communication Model
Distributional Communication Model

• Let $\mu$ be a distribution on $\mathcal{X} \times \mathcal{Y}$
Distributional Communication Model

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- Let $\Pi$ be a protocol
Distributational Communication Model

- Let $\mu$ be a distribution on $\mathcal{X} \times \mathcal{Y}$
- Let $\Pi$ be a protocol
- $\text{CC}_{\mu}^\text{avg}(\Pi)$: The expected number of bits exchanged by Alice and Bob (w.r.t $\mu$)
Distributional Communication Model

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- $\mathbb{C}C_{\mu}^{\text{avg}}(\Pi)$: The expected number of bits exchanged by Alice and Bob (w.r.t $\mu$)
- Let $f : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z}$ be a function
Distributional Communication Model

- Let $\mu$ be a distribution on $\mathcal{X} \times \mathcal{Y}$
- Let $\Pi$ be a protocol
- $\text{CC}^\text{avg}_\mu(\Pi)$: The expected number of bits exchanged by Alice and Bob (w.r.t $\mu$)
- Let $f : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z}$ be a function
- $\Pi$ computes $f$ with $\varepsilon$-error means:
  \[ \Pr (\Pi(x, y) = f(x, y)) \geq 1 - \varepsilon \]
Distributional Communication vs Randomized Communication
Distributional Communication vs Randomized Communication

Distributional Communication Complexity:

$D_\mu(f, \epsilon)$ is the complexity of the best protocol w.r.t. $\mu$
Distributional Communication vs Randomized Communication

Distributional Communication Complexity:
\[ D_\mu(f, \epsilon) \text{ is the complexity of the best protocol w.r.t. } \mu \]

Randomized Communication Complexity:
\[ R(f, \epsilon) \text{ is } D_\mu(f, \epsilon) \text{ in the worst-case distribution } \mu \]
Information Complexity
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Let $\Pi$ be a protocol
Information Complexity

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External Information:
Information Complexity

Let $\Pi$ be a protocol

External Information:

$\text{IC}^\text{ext}_\mu(\Pi) = \text{I}(\Pi; XY)$
Information Complexity

Let $\Pi$ be a protocol

External Information:

$$IC^\text{ext}_\mu(\Pi) = I(\Pi; XY)$$

The number of bits an external observer "learns" on the input from the transcript
Information Complexity

Let $\Pi$ be a protocol

External Information: $\text{IC}_{\mu}^{\text{ext}}(\Pi) = I(\Pi; XY)$

Internal Information:

The number of bits an external observer "learns" on the input from the transcript
Information Complexity

Let $\Pi$ be a protocol

External Information:

$$IC^\text{ext}_\mu(\Pi) = I(\Pi; XY)$$

The number of bits
an external observer “learns”
on the input from the transcript

Internal Information:

$$IC^\text{int}_\mu(\Pi) = I(\Pi; X|Y) + I(\Pi; Y|X)$$
Information Complexity

Let $\Pi$ be a protocol

External Information:

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The number of bits an external observer "learns" on the input from the transcript

Internal Information:

$IC^\text{int}_\mu(\Pi) = I(\Pi; X|Y) + I(\Pi; Y|X)$

The number of bits the parties "learn" on each other's input from the transcript
Information vs Communication

\[ \text{CC}_{\mu}^{\text{avg}}(\Pi) \geq \text{IC}_{\mu}^{\text{ext}}(\Pi) \geq \text{IC}_{\mu}^{\text{int}}(\Pi). \]

- # of bits communicated
- # of bits learnt by an external observer
- # of bits learnt by Alice & Bob
Distributional Information vs Randomized Information
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Let $IC \in \{IC^{\text{ext}}, IC^{\text{int}}\}$
Distributional Information vs Randomized Information

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Distributional Information Complexity:

$IC_{\mu}(f, \epsilon)$ is the information complexity of the best protocol w.r.t $\mu$
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reveals minimum information
Distributional Information vs Randomized Information

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Distributional Information Complexity:
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Randomized Information Complexity
$IC(f, \epsilon)$ is $IC_\mu(f, \epsilon)$ in the worst case distribution $\mu$
• We defined the Communication Model and the Communication and Information Complexities

• Next: We will discuss the relations between Amortized Communication and Information Complexity
Information vs Amortized Communication
Amortized Communication
Amortized Communication

Want to understand the Communication Complexity of solving $n$ independent inputs of a function $f$
Amortized Communication

Want to understand the Communication Complexity of solving \( n \) independent inputs of a function \( f \)

Let \( D^n_{\mu}(f, \epsilon) \) be the complexity of computing \( n \) independent inputs of \( f \)
(Distributional: w.r.t \( \mu \))
Amortized Communication

Want to understand the Communication Complexity of solving $n$ independent inputs of a function $f$

Let $D^n_\mu(f, \epsilon)$ be the complexity of computing $n$ independent inputs of $f$ (Distibutional: w.r.t $\mu$)

Let $R^n(f, \epsilon)$ be the complexity of computing $n$ independent inputs of $f$ (Randomized: worst case $\mu$)
Amortized Communication

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Let $D^n_\mu(f, \varepsilon)$ be the complexity of computing $n$ independent inputs of $f$
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$$\lim_{n \to \infty} \frac{D^n_\mu(f, \varepsilon)}{n} = \text{Amortized Distributional Complexity}$$

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Information vs Amortized Communication
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$\epsilon > 0$
Information vs Amortized Communication

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[Braverman-Rao 2011]:
Information vs Amortized Communication

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[Braverman-Rao 2011]:

\[ \text{IC}_{\mu}^{\text{int}}(f, \epsilon) = \lim_{n \to \infty} \frac{D_{\mu}^{n}(f, \epsilon)}{n} \]
Information vs Amortized Communication

\[ \epsilon > 0 \]

[Braverman-Rao 2011]:

**Distributional**

\[
IC^\text{int}_\mu(f, \epsilon) = \lim_{n \to \infty} \frac{D_n^\mu(f, \epsilon)}{n}
\]

**Non-Distributional**

\[
IC^\text{int}(f, \epsilon) = \lim_{n \to \infty} \frac{R_n^\epsilon(f, \epsilon)}{n}
\]
Information vs Amortized Communication

\[ \epsilon > 0 \quad \text{Distributional} \]

\[ \text{IC}_\mu^{\text{int}}(f, \epsilon) = \lim_{n \to \infty} \frac{D_n^\mu(f, \epsilon)}{n} \]

\[ \epsilon = 0 \quad \text{Non-Distributional} \]

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\text{IC}_{\mu}^{\text{ext}}(f, 0) = \lim_{n \to \infty} \frac{D_{\mu}^n(f, 0)}{n}
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No (this paper)

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Still open…
• We discussed some known connections between Information and Amortized Communication
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Next we prove that

$$\text{IC}_{\mu}^\text{ext}(f, 0) \neq \lim_{n \to \infty} \frac{D_{\mu}^n(f, 0)}{n}$$

• Need to construct $f, \mu$ for which the two quantities disagree
• We discussed some known connections between Information and Amortized Communication

• Next we prove that
\[ \text{IC}^\text{ext}_\mu (f, 0) \neq \lim_{n \to \infty} \frac{D^n_\mu (f, 0)}{n} \]

• Need to construct \((f, \mu)\) for which the two quantities disagree

• We will show something stronger...
Our Results
Theorem:

\exists \text{ a sequence } \{f_k, \mu_k\} \text{ s.t. } \lim_{k \to \infty} \text{IC}^\text{ext}_{\mu_k}(f_k, 0) = \infty \text{ and } \lim_{k \to \infty} \text{Amortized}_{\mu_k}(f_k, 0) = 0.
Theorem:

∃ a sequence \( \{f_k, \mu_k\} \) s.t.

\[
\text{IC}_{\mu_k}^{\text{ext}}(f_k, 0) \xrightarrow{k \to \infty} \infty \quad \text{and} \quad \text{Amortized}_{\mu_k}(f_k, 0) \xrightarrow{k \to \infty} 0.
\]

Specifically, let \( f = f_k, \mu = \mu_k \):
Theorem:

\[ \exists \text{ a sequence } \{f_k, \mu_k\} \text{ s.t. } \]

\[ \text{IC}_{\mu_k}^{\text{ext}}(f_k, 0) \overset{k \to \infty}{\to} \infty \text{ and Amortized}_{\mu_k}(f_k, 0) \overset{k \to \infty}{\to} 0. \]

Specifically, let \( f = f_k, \mu = \mu_k \):

\[ \text{IC}_{\mu}^{\text{ext}}(f, 0) \geq 0.99k, \text{ while } \frac{D^n_{\mu}(f, 0)}{n} \leq 10k2^{-k} + \frac{5k}{n}. \]
Sketch of proof

- Let $f : \{0, 1\}^k \times \{0, 1\}^k \rightarrow \{0, 1\}$ be the equality function
- Interpret $x, y$ as integers in $\{0, \ldots, 2^k - 1\}$
- The distribution $\mu$ is supported on $\{(x, y) : x \leq y\}$
- The distribution $\mu$ puts $1 - 2^{-k}$ weight on $\{(x, y) : x = y\}$
External Information is almost k
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- On inputs of the form \((x, x)\), \(x\) can be recovered from the transcript by an external observer (Because there is no error)
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\[
\text{IC}_{\mu}^{\text{ext}}(EQ) \geq (1 - 2^{-k}) \cdot k
\]
Can “save” a lot in amortized
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- For simplicity, consider two independent inputs \((x_1, y_1), (x_2, y_2) \sim \mu\)
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- Recall: \(\mu\) is supported on \(\{(x, y) : x \leq y\}\)
Can “save” a lot in amortized

- For simplicity, consider two independent inputs \((x_1, y_1), (x_2, y_2) \sim \mu\)

- Recall: \(\mu\) is supported on \(\{(x, y) : x \leq y\}\)

- Thus, with \(\mu\)-probability 1:
  \[x_1 + x_2 = y_1 + y_2 \iff x_1 = y_1 \land x_2 = y_2\]
Can “save” a lot in amortized!!
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The Protocol:
Can “save” a lot in amortized!!

The Protocol:

- Alice computes $x = x_1 + x_2$,
- Bob computes $y = y_1 + y_2$
  (“+” = integer addition)
Can “save” a lot in amortized!!

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- Alice sends $x$ to Bob
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- Alice sends \( x \) to Bob
- Bob Compares \( x \) and \( y \),
  if they are equal then \( x_1 = x_2 \) and \( y_1 = y_2 \)
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- Alice computes \( x = x_1 + x_2 \),
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  if they are equal then \( x_1 = x_2 \) and \( y_1 = y_2 \)

- else Alice send all her input
  and Bob answers accordingly
Can “save” a lot in amortized!!

The Protocol:

- Alice computes $x = x_1 + x_2$
- Bob computes $y = y_1 + y_2$ 0 bits
  ("+" = integer addition)
- Alice sends $x$ to Bob
- Bob compares $x$ and $y$,
  if they are equal then $x_1 = x_2$ and $y_1 = y_2$
- else Alice sends all her input
  and Bob answers accordingly
Can “save” a lot in amortized!!

The Protocol:

- Alice computes $x = x_1 + x_2$,
  Bob computes $y = y_1 + y_2$ \(0 \text{ bits}\)
  \( (+ = \text{integer addition})\)

- Alice sends $x$ to Bob \(k+1 \text{ bits}\)

- Bob Compares $x$ and $y$,
  if they are equal then $x_1 = x_2$ and $y_1 = y_2$

- else Alice send all her input
  and Bob answers accordingly
Can “save” a lot in amortized!!

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- Alice computes $x = x_1 + x_2$,
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- Bob Compares $x$ and $y$,
  if they are equal then $x_1 = x_2$ and $y_1 = y_2$ \(1\) bit

- else Alice send all her input
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- Bob Compares $x$ and $y$,
  if they are equal then $x_1 = x_2$ and $y_1 = y_2$ \(1\) bit
- else Alice send all her input
  and Bob answers accordingly \(2k+2\) bit
Can “save” a lot in amortized!!

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- Alice computes \( x = x_1 + x_2 \),
  Bob computes \( y = y_1 + y_2 \) \(0\) bits
  \( ("+" = \text{integer addition}) \)

- Alice sends \( x \) to Bob \( k+1\) bits

- Bob Compares \( x \) and \( y \),
  if they are equal then \( x_1 = x_2 \) and \( y_1 = y_2 \) \(1\) bit \( \text{prob} \geq 1 - 2 \cdot 2^{-k} \)

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The Protocol:

- Alice computes \( x = x_1 + x_2 \)
  Bob computes \( y = y_1 + y_2 \) \(0\) bits
  (“+” = integer addition)

- Alice sends \( x \) to Bob \(k+1\) bits

- Bob compares \( x \) and \( y \),
  if they are equal then \( x_1 = x_2 \) and \( y_1 = y_2 \) \(1\) bit \( \text{prob} \geq 1 - 2 \cdot 2^{-k} \)

- else Alice sends all her input
  and Bob answers accordingly \(2k+2\) bit \( \text{prob} \leq 2 \cdot 2^{-k} \)
Can “save” a lot in amortized!!

The Protocol: \[ \text{CC}_{\mu}^{\text{avg}}(\Pi) \leq k + 1 + (2k + 2) \cdot (2 \cdot 2^{-k}) \]

- Alice computes \( x = x_1 + x_2 \), Bob computes \( y = y_1 + y_2 \) \[ \leq k + 2 \] \( 0 \) bits ("+" = integer addition)

- Alice sends \( x \) to Bob \[ k+1 \) bits

- Bob Compares \( x \) and \( y \), if they are equal then \( x_1 = x_2 \) and \( y_1 = y_2 \) \[ 1 \) bit \( \text{prob} \geq 1 - 2 \cdot 2^{-k} \]

- else Alice send all her input and Bob answers accordingly \[ 2k+2 \) bit \( \text{prob} \leq 2 \cdot 2^{-k} \]
Can “save” a lot in amortized!!

The Protocol: \[ CC_{\mu}^{\text{avg}}(\Pi) \leq k + 1 + (2k + 2) \cdot (2 \cdot 2^{-k}) \leq k + 2 \]

- Alice computes \( x = x_1 + x_2 \), 0 bits
  Bob computes \( y = y_1 + y_2 \) (“+” = integer addition) saved \( \approx k \) bits

- Alice sends \( x \) to Bob \( k+1 \) bits

- Bob Compares \( x \) and \( y \),
  if they are equal then \( x_1 = x_2 \) and \( y_1 = y_2 \) 1 bit \( \text{prob} \geq 1 - 2 \cdot 2^{-k} \)

- else Alice send all her input
  and Bob answers accordingly \( 2k+2 \) bit \( \text{prob} \leq 2 \cdot 2^{-k} \)
Open Question
Open Question

\[ IC^{\text{ext}}(f,0) \stackrel{?}{=} \lim_{n \to \infty} \frac{R^n(f,0)}{n} \]
Questions? 😊
😊 Thanks! 😊