Math 303. Solutions of the optional assignment – 28 Mar 2008.

Solution of problem 1.

We first consider $N_{m,f}(t)$, number of matings between the male m and the female f by time t: since

$$P\Big(N_{m,f}(t+h) = n+1 \Big| N_{m,f}(t) = n\Big) = \lambda h + \mathrm{o}(h)$$

the matrix q for the process $N_{m,f}$ is given by

$$q_{n,k} = \begin{cases} \lambda & \text{for} \quad k = n+1 \\ 0 & \text{otherwise} \end{cases}$$

We have seen in class that such a process is a Poisson process with parameter λ (see the example about the Poisson process and its Kolmogorov equation). Then, $T_{m,n}$, the time before a new mating of the pair (m, f), is an exponential r.v. with rate λ.

The continuous time M.c that we want to study is $X(t) = (n_m, n_f)$ the the number of males and females in the population at time t (for $n_m > 0$ and $n_m > 0$, otherwise there is no mating and the population remains constant). Since there are $n_m \cdot n_f$ possible male-female pairs, the time until the next mating is $\min_{m,f} T_{m,f}$ that has a rate $n_m n_f \lambda$.

Finally, $\Omega = \{(n_m, n_f) : n_m, n_f > 0\}$; the only non-zero transition probability of the embedded M.c. are

$$Q_{(n_m,n_f),(n_m,n_f+1)} = Q_{(n_m,n_f),(n_m+1,n_f)} = \frac{1}{2}$$

and the rate of the wanting times is $v_{(n_m,n_f)} = \lambda n_m n_f$

Solution of problem 3.

The only natural way to describe the system in terms of a birth-death process is to consider the stochastic process $X(t) = 0, 1, 2$ number of machines serviced by the repairman. Anyway, such a process is not a continuous time Markov chain, since some waiting times are not exponential r.v. Indeed, if X_1 and X_2 are the lifetimes of the two machines, we have that from state 0 the process can only jump to state 1 with rate at time $T_0 = \min\{X_1, X_2\}$:

hence T_0 is an exponential r.v. with rate $v_0 = \mu_1 + \mu_2$. But $T_1 = \min\{T_{1,0}, T_{1,2}\}$, where $T_{1,0}$ is the virtual time to jump into 0 (namely the time that takes to the machine in service to be repaired) and is an expon. r. v. with rate $q_{1,0} = \mu$; and $T_{1,2}$ is the virtual time to jump into state 2. We have

$$T_{1,2} = \begin{cases} X_1 - X_2 & \text{if} \quad X_1 > X_2 \\ X_2 - X_1 & \text{if} \quad X_2 > X_1 \end{cases}$$

then

$$P(T_{1,2} > t) = \frac{\mu_1}{\mu_1 + \mu_2} e^{-\mu_2 t} + \frac{\mu_2}{\mu_1 + \mu_2} e^{-\mu_1 t}$$

Accordingly, $T_{1,2}$ is not an exponential random variable. Finally,

$$P(T_1 > t) = P(T_{1,0} > t) P(T_{1,2} > t) = \frac{\mu_1}{\mu_1 + \mu_2} e^{-(\mu_2 + \mu)t} + \frac{\mu_2}{\mu_1 + \mu_2} e^{-(\mu_1 + \mu)t}$$

hence T_1 is not an exponential r.v. and $X(t)$ is not a continuous time M.c.

A possible way to describe the model in terms of a continuous time M.c. is to introduce more states: $\Omega = \{0, 1, 1', 2, 2'\}$ that mean that both machines are working, only the machine 1 is in service, only the machine 2 is in service, both machines are in service and machine 1 is currently being repaired, both machines are in service and machine 2 is currently being repaired.

Let Y_j be the lifetime of machine j counting from the break down of the other machine. It is clear that

$$T_0 = \min\{X_1, X_2\}, \quad T_1 = \min\{X, Y_2\}, \quad T_{1'} = \min\{X, Y_1\}$$

$$T_2 = X, \quad T_{2'} = X$$

then

$$v_0 = \mu_1 + \mu_2, \quad v_1 = \mu_2 + \mu_0 \quad v_{1'} = \mu_1 + \mu_0, \quad v_2 = v_{2'} = \mu$$

and the only non-zero transition probabilities for the embedded M.c are

$$Q_{0,1} = P(X_1 < X_2) = \frac{\mu_1}{\mu_1 + \mu_2} \quad Q_{0,1'} = P(X_2 < X_1) = \frac{\mu_2}{\mu_1 + \mu_2}$$

$$Q_{1,2} = P(X_2 < X) = \frac{\mu_2}{\mu + \mu_2} \qquad Q_{1,0} = P(X < X_2) = \frac{\mu}{\mu + \mu_2}$$

$$Q_{1',2'} = P(X_1 < X) = \frac{\mu_1}{\mu + \mu_1} \qquad Q_{1',0} = P(X < X_1) = \frac{\mu}{\mu + \mu_1}$$

$$Q_{2,1} = 1 \quad Q_{2',1'} = 1$$

Solution of problem 5.

Let $N_i(t)$ the number of contacts by time t that cause an infection, and $N_n(t)$ be the number of contacts by time t that do not cause infection. If $N(t)$ is the total number of contacts, we have

$$N(t) = N_1(t) + N_n(t)$$

and by the theory of the mixture Poisson processes, the interarrival times of $N_i(t)$ are exponential random variable with rate $\lambda p_i(n)$, where $p_i(n)$ is the probability that the contact gives a new infection when the population is made of n people.

The process that we want to consider is $X(t) =$'number of infected people'. This is a birth-death process, with rate of death $\mu_n = 0$, and rate of birth $\lambda_0 = 0$, $\lambda_n = \lambda p_i(n)$.

The time that the infection takes to spread over all the population is $T = T_2 + T_3 + \cdots + T_N$, where T_j is the time that it takes to to infect a new individual if $j - 1$ individuals are already infected: in the notation used in class for the birth-death processes, $T_j = A_{j-1}$. Finally, we find:

$$E[T] = \sum_{n=1}^{N-1} E[A_n] = \sum_{n=1}^{N-1} \frac{1}{\lambda p_i(n)}$$

We only need to find $p_i(n)$: if the number of infected people is n, we find

$$\begin{aligned} p_i &= P(\text{infection transmission}) \cdot P(\text{contact infected - non infected}) \\ &= p \cdot n(N-n) \Big/ \frac{N!}{2!(N-2)!} = 2p\frac{n(N-n)}{N(N-1)} \end{aligned}$$

Solution of problem 6.

Let F_i the first time that, starting in state i the process enters the state $i+1$: in a birth-death process the time to go from state i to state f is given by

$$T_{i\to f} = \sum_{j=i}^{f-1} F_j$$

By linearity of the expectation, an independence of the F's, we find

$$E[T] = \sum_{j=i}^{f-1} E[F_j] \qquad Var[T] = \sum_{j=i}^{f-1} Var[F_j]$$

We can compute the expectation and the variance of F_j by a recursive formula. We have

$$F_j = \begin{cases} A_j & \text{if} \quad A_j < D_j \\ \\ D_j + F_{j-1} + F'_j & \text{if} \quad D_j < A_j \end{cases}$$

where F'_j has the same distribution of F_j, but they are independent r.v. Then, since F'_j is independent from A_j and D_j, but has the same expectation of F_j,

$$\begin{aligned} E[F_j] &= E[F_j | A_j < D_j] P(A_j < D_j) + E[F_j | D_j < A_j] P(D_j < A_j) \\ &= E[A_j | A_j < D_j] P(A_j < D_j) + E[D_j | D_j < A_j] P(D_j < A_j) \\ &+ \Big(E[F_{j-1}] + E[F_j]\Big) P(D_j < A_j) \\ &= E[\min\{A_j, D_j\}] + \Big(E[F_{j-1}] + E[F_j]\Big) P(D_j < A_j) \\ &= \frac{1}{\lambda_j + \mu_j} + \Big(E[F_{j-1}] + E[F_j]\Big) \frac{\mu_j}{\lambda_j + \mu_j} \end{aligned}$$

This is an equation for $E[F_j]$, that has solution

$$E[F_j] = \frac{1}{\lambda_j} + \frac{\mu_j}{\lambda_j} E[F_{j-1}]$$

Hence we find

$$E[F_0] = \frac{1}{\lambda_0} + 0$$

$$E[F_1] = \frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1} E[F_0]$$

$$E[F_2] = \frac{1}{\lambda_2} + \frac{\mu_2}{\lambda_2} E[F_1]$$

$$E[F_3] = \frac{1}{\lambda_3} + \frac{\mu_3}{\lambda_3} E[F_2]$$
$$E[F_4] = \frac{1}{\lambda_4} + \frac{\mu_4}{\lambda_4} E[F_3]$$

and the solution of question (a) is $E[F_3]+E[F_2]+E[F_1]+E[F_0]$, while the solution to question (b) is $E[F_4]+E[F_3]+E[F_2]$.

In a similar way we can find the solution to question (c) by a recursive formula for the variance. The developments are given in the textbook, at pag 374 and 375:

$$Var[F_j] = \frac{1}{\lambda_j(\lambda_i + \mu_i)} + \frac{\mu_j}{\lambda_j} Var[F_{j-1}] + \frac{\mu_i}{\mu_i + \lambda_i}\left(E[T_{i-1}] + E[T_i]\right)^2$$