Introduction

1. Uniform Asymptotic Properties of Discounted Zero-Sum Stochastic Games, using Semi-Algebraic Theory.

2. Uniform Polynomial Convergence Rate of V^N to V^∞ on a Bounded Set of Payoffs.

The Model

Two-Player Zero-Sum Stochastic Games.

Finite State Space: $s \in \{1 \ldots S\}$.
Finite Actions Space: $(i, j) \in I_s \times J_s$.
Payoffs: $\gamma = \left\{\gamma_{ij}^{s}\right\} \in \mathbb{R}^F$.
Transitions: $\pi = \left\{\pi_{ij,s'}^{s}\right\} \in \Delta(S)^F$.
Behavioral Strategies: σ and τ.
Initial State: s.
Measure on infinite game: $\mu_s(\sigma, \tau)$.
Payoff Stream: $X = (X_1, X_2, \ldots)$.
Evaluation on Stream: $\eta(X)$.
Value: $\inf_\tau \sup_\sigma E_{\mu_s(\sigma,\tau)}(\eta(X)) =$
$\sup_\sigma \inf_\tau E_{\mu_s(\sigma,\tau)}(\eta(X))$.

Different Games

Von Neumann / Shapley:
N-stage game: $\eta_N(X) := \frac{1}{N}\sum_{n=1}^{N} X_n$.
Value: $V^N(\gamma,\pi) = (V_1^N,\ldots,V_S^N)$.
Extension of zero-sum matrix game:
val $\gamma = \min_\tau \max_\sigma \tau^t\gamma\sigma = \max_\sigma \min_\tau \tau^t\gamma\sigma$.

Shapley (1953):
r-discount game: $\eta_r(X) := r\sum_{n=1}^{\infty}(1-r)^{n-1}X_n$.
Value: $V(\gamma,\pi,r) = (V_1(r),\ldots,V_S(r))$.
($V(r)$ solves dynamic programming equation).

Mertens and Neyman (1980):
non-discount game: $\eta_\infty(X) := \liminf_{N\to\infty}\eta_N(X)$.
Value: $V^\infty(\gamma,\pi) = (V_1^\infty,\ldots,V_S^\infty)$.

Uniform Properties of $V(\gamma, \pi, r)$

Studied by Bewley and Kohlberg (1976) using Tarski's Principle for Real Closed Fields.

By showing that $V(\gamma, \pi, r)$ is a semi-algebraic mapping, we prove the following:

Let the dimensions be fixed (S, $\{I_s\}$ and $\{J_s\}$).

1. There exists $N > 0$ such that:

$$\forall \gamma, \pi, s \; \exists r_0 > 0 \; \exists k \leq N \; \exists \{c_i\}_{i \geq 0} :$$
$$V_s(\gamma, \pi, r) = \sum_{i \geq 0} c_i r^{i/k} \;\; \forall r \in (0, r_0).$$

2. There exist $M > 0$ and (unbounded) $B(\pi) > 0$ such that:

$$\forall \gamma \in [-1, 1]^F \; \forall \pi \; \forall s \; \forall r \in (0, 1) :$$
$$\left| \partial V_s(\gamma, \pi, r) / \partial^+ r \right| \leq B(\pi) r^{-(1-1/M)}.$$

Can these results be improved?

Why doesn't #2 follow from #1?

1. $V_s(\gamma, \pi, r) = \sum_{i \geq 0} c_i r^{i/k},\ k \leq N\ r \in (0, r_0)$.

2. $\forall \gamma \in [-1,1]^F\ \forall \pi\ \left|\frac{\partial V_s(\gamma,\pi,r)}{\partial^+ r}\right| \leq B(\pi) r^{-(1-1/M)}$.

Consider the game: $\begin{array}{|c|}\hline 1^* \\ \hline 0 \\ \hline \end{array} \longrightarrow \boxed{1+\gamma^*}$ $\gamma \in [0,1]$.

$$V(\gamma, r) = \max(1, (1-r)(1+\gamma))$$

$$r_0(\gamma) = \frac{\gamma}{1+\gamma}$$

- $\partial V_s(\gamma, \pi, r)/\partial r$ does not always exists.

- $c_1(\gamma, \pi)$ is not continuous in γ nor π.

- $r_0(\gamma, \pi)$ is not bounded away from 0.

Semi-Algebraic Sets

R - Real Closed field = maximal ordered field.
Ordered/Real field: $-1 \neq \sum_{i=1}^{n} x_i^2$.
Real Closed field: $\mathsf{R}[\sqrt{-1}]$ algebraically closed.
Real Puiseux Series: $f(x) = \sum_{i=-I}^{\infty} c_i x^{i/k}$, $c_i \in \mathbb{R}$, $k \in \mathbb{N}$, $f(\xi)$ converges for $0 < \xi < \epsilon$.

(All polynomials are over R)

$\mathcal{S}^n$ - The *semi-algebraic subsets* of R^n:

$$\bigcup_{k=1}^{K} \bigcap_{l=1}^{L_k} \left\{ x \in \mathsf{R}^n \,\middle|\, f_{kl}(x) \overset{>}{\underset{=}{}} 0 \right\}$$

1. $\mathcal{S}^n$ is an Algebra.
2. $\mathcal{S}^1$ - finite unions of points and open intervals.
3. Tarski-Seidenberg Theorem: $\Pi(\mathcal{S}^n) = \mathcal{S}^{n-1}$:

$$A \in \mathcal{S}^n$$
$$\Downarrow$$
$$\Pi(A) = \{(x_1, \ldots, x_{n-1}) | \exists x_n \, (x_1, \ldots, x_n) \in A\} \in \mathcal{S}^{n-1}$$

Tarski-Seidenberg Theorem

Atomic Formula - $f > 0$ or $f = 0$ over R.
Elementary Formula - $\wedge$, $\vee$, $\neg$, $\exists x$, $\forall x$ over R.
Free Variable - non-quantified ($\forall x\ y + x^2 > 0$).
Elementary Sentence - Elementary formula with no free variables.

Tarski's Principle - "An elementary sentence's validity does not depend on R".

Seidenberg's Theorem - eliminates $\exists x$ from elementary formulae.
Corollary: If $\Phi(X_1, \ldots, X_n)$ is an elementary formula over R with free variables $X_1, \ldots, X_n$, then $\{x \in \mathrm{R}^n | \Phi(x)\} \in \mathcal{S}^n$. E.g. :

$A \in \mathcal{S}^n \Rightarrow$
$\bar{A} = \{x \in \mathrm{R}^n | \forall \epsilon > 0\ \exists y \in A\ \ \|x - y\| < \epsilon\} \in \mathcal{S}^n$.

Semi-Algebraic Mappings

$A \in \mathcal{S}^m$ $B \in \mathcal{S}^n$. $f : A \to B$ is a *semi-algebraic mapping*, if Graph(f) $\in \mathcal{S}^{n+m}$. Denote $f \in \mathcal{M}(A, B)$.

1. $f \in \mathcal{M}(A, B)$ $g \in \mathcal{M}(B, C) \to$
   $g \circ f \in \mathcal{M}(A, C)$.

2. $\mathcal{M}(A, \mathrm{R})$ is a ring:

$$\text{Graph}(f+g) = \left\{ (a,z) \;\middle|\; \begin{array}{c} \exists x, y \;\; z = x + y \\ (a,x) \in \text{Graph}(f) \\ (a,y) \in \text{Graph}(g) \end{array} \right\}$$

3. Let $f \in \mathcal{M}(A, B)$. If $S \subset A$ is semi-algebraic then so is $f(S)$. If $T \subset B$ is semi-algebraic then so is $f^{-1}(T)$.

Semi-Algebraic Theorems

1. Partition Lemma - Let $A \in S^n$, $f \in \mathcal{M}(A, \mathsf{R})$. Then $A = \cup_{i=1}^m A_i$, $A_i \in \mathcal{S}^n$, s.t. for all i, $f \in C(A_i)$ and $\exists p_i(X_1, \ldots, X_n, Y)$ s.t. $\forall x \in A_i$, $p_i(x, Y) \neq 0$, $p_i(x, f(x)) = 0$.

2. Curve Selection Lemma - Let $A \in \mathcal{S}^n$ and $x \in \bar{A}$. $\exists f \in \mathcal{M}([0,1], \mathsf{R}^n) \cap C^\infty([0,1], \mathsf{R}^n)$ s.t. $f(0) = x$ and $f((0,1]) \subset A$.
($\mathsf{R} = \mathbb{R}$ - f is analytic and algebraic).

3. Lojasiewicz's Inequality - Let $A \in \mathcal{S}^n$ closed and bounded, $f, g \in \mathcal{M}(A, \mathsf{R}) \cap C(A, \mathsf{R})$ s.t. $f^{-1}(0) \subset g^{-1}(0)$. Then $\exists N, c > 0$ s.t. $|g|^N \leq c\,|f|$ on A.

Semi-Algebraic Lemmas over $\mathbb{R}$

1. Let $A \in \mathcal{S}^{n+1}$, $f(X_1, \ldots, X_n, Y) \in \mathcal{M}(A, \mathbb{R})$. Then $\exists N > 0\ \forall (x, y) \in A^*\ \exists \epsilon > 0\ \exists k \leq N$ s.t. $f(x, y + \xi) = \sum_{i \geq -I} c_i \xi^{i/k}\ \forall \xi \in (0, \epsilon)$.

2. Let $f \in \mathcal{M}((0,1), \mathbb{R})$:
A. $\lim_{y \to 0+} f(y)$ exists (wide sense).
B. $\exists \epsilon > 0$ s.t. $f((0, \epsilon))$ has constant sign.

3. Let $f(X_1, \ldots, X_n, Y) \in \mathcal{M}(A, \mathbb{R})$. Then $\partial f(X, Y)/\partial$
exists (wide sense) on A^*. If it is finite on a semi-algebraic subset B of A^*, it is semi-algebraic on B.

4. Let $f(Y) \in \mathcal{M}([0,1], \mathbb{R})$ be continuous, and assume $df(Y)/d^+Y$ is finite on $(0,1)$. Then $f(y_2) - f(y_1) = \int_{y_1}^{y_2} df(Y)/d^+Y dY\ \forall y_1, y_2 \in [0,1]$.

Shapley's Equation

If $x \in \mathbb{R}^S$, denote:

$$G_s^r(x) = \left\{ r\,\gamma_{ij}^s + (1-r)\sum_{s'=1}^{S} \pi_{ij,s'}^s x_{s'} \right\}_{ij}$$

$$\text{val } G^r(x) = (\text{val } G_1^r(x), \ldots, \text{val } G_S^r(x))$$

Shapley's Dynamic Programming Equation:

$$x = \text{val } G^r(x)$$

Denoting $V(\gamma, r) := V(\gamma, \pi_0, r)$, it follows that:

1. $\|V(\gamma, \pi, r)\|_\infty \leq \|\gamma\|_\infty$.
2. $\|V(\gamma, r_1) - V(\gamma, r_2)\|_\infty \leq \frac{2\|\gamma\|_\infty}{\max(r_1, r_2)} |r_1 - r_2|$.
3. $\|V(\gamma_1, r) - V(\gamma_2, r)\|_\infty \leq \|\gamma_1 - \gamma_2\|_\infty$.

$V(\gamma, \pi, r)$ is a semi-algebraic mapping

$V(\gamma, \pi, r) : \mathbb{R}^F \times \Delta(S)^F \times (0, 1] \to \mathbb{R}^S$ is the unique solution to Shapley's Equation. It is semi-algebraic because val $(G) : \mathbb{R}^{nm} \to \mathbb{R}$ is a semi-algebraic function:

$$\left\{ (G, v) \in \mathbb{R}^{nm+1} \;\middle|\; \begin{array}{l} \exists p \in \Delta(n) \;\; \exists q \in \Delta(m) \\ \forall j = 1 \ldots m \; \sum_{i=1}^{n} p_i G_{ij} \geq v \\ \forall i = 1 \ldots n \; \sum_{j=1}^{m} G_{ij} q_j \leq v \end{array} \right\}$$

$V_s(\gamma, \pi, r) : \mathbb{R}^F \times \Delta(S)^F \times (0, 1] \to \mathbb{R}$ is semi-algebraic; extend $V_s(\gamma, \pi, 0) := \lim_{r \to 0+} V_s(\gamma, \pi, r)$. Semi-algebraic on $\mathbb{R}^F \times \Delta(S)^F \times [0, 1]$.

Main Theorem 1

Fix the game's dimensions.

Local Version: There exists $N > 0$ such that:

$$\forall \gamma, \pi, s, r \in [0,1) \ \exists \epsilon > 0 \ \exists k \leq N \ \exists \{c_i\}_{i \geq 0} : \\ V_s(\gamma, \pi, r + \xi) = \sum_{i \geq 0} c_i \xi^{i/k} \ \ \forall \xi \in [0, \epsilon).$$

Note: analogue for optimal stationary strategies. $\exists N' \ \ \hat{\sigma}_s(\gamma, \pi, r + \xi) = \sum_{i \geq 0} \overrightarrow{c_i} \xi^{i/k} \ \ k \leq N'$.

Algebraic Version: Choose s. Then $\mathbb{R}^F \times \Delta(S)^F \times [0,1] = \cup_{i=1}^{n} A_i$ s.t. $V_s(\gamma, \pi, r)$ is continuous algebraic on A_i (satisfying a non-zero $g_i(\gamma, \pi, r, Z)$).

Uniform Bound on $\partial V_s(\gamma, \pi, r)/\partial^+ r$

$$\exists N > 0\ \forall \gamma, \pi\ \exists r_0(\gamma, \pi) > 0\ \exists k \leq N\ \forall r \in [0, r_0) :$$

$$V_s(\gamma, \pi, r) = \sum_{i \geq 0} c_i(\gamma, \pi) r^{i/k}$$

$$\left| \frac{\partial V_s(\gamma, \pi, r)}{\partial r} = \sum_{i \geq 1} \frac{i}{k} c_i r^{i/k - 1} \right| \leq B(\gamma, \pi) r^{-(1-1/N)}$$

Is there $B(\pi) r^{-(1-1/N)}\ \forall r \in (0,1)\ \forall \gamma \in [-1,1]^F$?

- $\partial V_s(\gamma, \pi, r)/\partial r$ does not always exists.
- $c_1(\gamma, \pi)$ is not continuous in γ nor π.
- $r_0(\gamma, \pi)$ is not bounded away from 0.

Therefore use $\partial V_s(\gamma, \pi, r)/\partial^+ r$ is semi-algebraic.

Main Theorem 2

$$C := [-1,1]^F \qquad D_s(\pi,r) := \sup_{\gamma\in C}\left|\frac{\partial V_s(\gamma,\pi,r)}{\partial^+ r}\right|$$

Theorem:
Fix the dimensions.
There exist $M > 0$ and $B(\pi) > 0$ such that:
$\forall s\ \forall\pi\ \forall r \in (0,1) : D_s(\pi,r) \leq B(\pi)r^{-(1-1/M)}$.

Remarks:

- V_s and D_s are positively homogeneous in γ.
- $D_s(\pi,r) \leq 2/r$ is S.A. on $\Delta(S)^F \times (0,1)$.
- We show:

$$\forall s\ \exists M > 0\ \forall\pi\ \exists B'(\pi) > 0\ \exists r_0(\pi) :$$
$$D_s(\pi,r) \leq B'(\pi)r^{-(1-1/M)}\ \ \forall r \in (0,r_0).$$

Construction of M

$$G_\epsilon(\pi, r) = \left\{ \gamma \in C \,\middle|\, \left| \frac{\partial V_s(\gamma, \pi, r)}{\partial^+ r} \right| > D_s(\pi, r) - \epsilon \right\}$$

$G(\pi, r) = \bigcap_{\epsilon > 0} \overline{G_\epsilon(\pi, r)} \neq \emptyset$ (C compact).

Let $g(\pi, r)$ be minimal (lexicographic) in $G(\pi, r)$. S.A. on $\Delta(S)^F \times (0, 1)$.

$\exists g(\pi, 0) := \lim_{r \to 0+} g(\pi, r)$ - S.A. extension to $\Delta(S)^F \times [0, 1)$.

There exists $M > 0$ s.t.:

$$\|g(\pi, r) - g(\pi, 0)\|_\infty = \sum_{i \geq 0} c_i(\pi) r^{i/k} \; \forall \pi \forall r \in [0, r_0(\pi))$$

$$\Downarrow (k \leq M, g \in C)$$

$$\|g(\pi, r) - g(\pi, 0)\|_\infty \leq L(\pi) r^{1/M} \; \forall \pi \forall r \in [0, 1)$$

Assume $M \geq N$.

Construction of $B'(\pi)$

Fix $\pi = \pi_0$. $\gamma_0 := g(\pi_0, 0)$.

$\exists T(\pi_0)\ |V_s(\gamma_0, r) - V_s(\gamma_0, 0)| \leq T r^{1/N}\ \forall r \in [0, 1]$

$B' := ((F+1)(L+1)+T+1)/M$ satisfies:

$\exists r_0 > 0\ D_s(r) \leq 2B' r^{-(1-1/M)}\ \forall r \in (0, r_0)$

Otherwise:

$\exists r_1 > 0\ D_s(r) > 2B' r^{-(1-1/M)}\ \forall r \in (0, r_1)$

Contradiction - Technical Finale

$$H := \left\{ (\gamma, r) \,\middle|\, \begin{array}{l} r \in (0, r_1) \wedge \|\gamma - g(r)\|_\infty \leq r^{1/M} \\ \wedge \left|\partial V_s(\gamma, r)/\partial^+ r\right| \geq D_s(r)/2 \end{array} \right\}$$

Then $H \cap \{r = r_0\} \neq \emptyset$ and $(\gamma_0, 0) \in \bar{H}$.

$\exists h(t) = (\gamma_1^h(t), \ldots, \gamma_F^h(t), r^h(t)) : [0,1] \to \bar{H}$ S.A. and analytic s.t. $h(0) = (\gamma_0, 0)$, $h((0,1)) \subset H$. Assume $dr^h(t)/dt$ does not change signs, therefore increasing, $t = t(r)$ and $h = (h(r), r)$ are S.A. and continuous.

$$h(0) = \gamma_0 \quad \|h(r) - \gamma_0\|_\infty \leq (L+1) r^{1/M}$$

Contradiction by examining $W(r) := V_s(h(r), r)$ S.A. and continuous:

$$\begin{aligned} |W(r) - W(0)| &\geq \int_0^r \frac{D_s(\rho)}{2} d\rho - F \|h(r) - \gamma_0\|_\infty \\ |W(r) - W(0)| &\leq \|h(r) - \gamma_0\|_\infty + T r^{1/N} \end{aligned}$$

$(T+1) r^{1/M} \leq T r^{1/N} \forall r \in (0, r_2)$ but $M \geq N$ $\quad\square$

Can We Improve?

1. $V_s(\gamma, \pi, r) = \sum_{i \geq 0} c_i r^{i/k}$, $k \leq N$.

2. $\forall \gamma \in [-1,1]^F \; \forall \pi \; \left| \frac{\partial V_s(\gamma,\pi,r)}{\partial^+ r} \right| \leq B(\pi) r^{-(1-1/M)}$.

- Any N can appear (as dimensions increase).

- If $B(\pi) \leq B$, $V_s(\gamma_0, \pi, r) \to V_s(\gamma_0, \pi, 0)$ uniformly in π, implying $V_s(\gamma_0, \pi, 0)$ $(= V_\infty(\gamma_0, \pi))$ is continuous in π, which contradicts:

$$\boxed{0} \longrightarrow \boxed{1^*}$$

- $M \geq N$. Can we use $M = N$?
  Consider continuous semi-algebraic f:

$$f(\gamma, r) = \max(r/\gamma, \gamma) \;=\; \begin{cases} r/\gamma & r \in [0, \gamma^2) \\ \gamma & r \in [\gamma^2, 1] \end{cases}$$

$$\sup_{\gamma \in [0,1]} \partial f(\gamma, r)/\partial^+ r \;=\; r^{-1/2}$$

- Consider the semi-algebraic:

  $$B(\pi) := \sup_{\gamma \in [-1,1]^F, r \in (0,1)} \frac{\left\| \partial V(\gamma, \pi, r) / \partial^+ r \right\|_\infty}{r^{-(1-1/M)}} \ .$$

  What are the continuity sets?

  Conjecture: these are the continuity sets of $V^\infty(\gamma, \pi)$ - the sets on which $\left\{ s, i, j, t \mid \pi^s_{ij,t} = 0 \right\}$ remains constant (Solan 2000).

Uniform Convergence of V^N to V^∞

Mertens and Neyman (1981) - **Generalization:**
Let C be a bounded set of parameters. If:

$$\left\|\partial V(\gamma,\pi,r)/\partial^+ r\right\|_\infty \leq f(r) \in L_1[0,1]\ \forall(\gamma,\pi) \in C.$$

Then: $\forall\epsilon > 0\ \exists N_\epsilon\ \forall(\gamma,\pi) \in C\ \exists\sigma\ \forall s,\tau, N \geq N_\epsilon$

$$E_{\mu_s(\sigma,\tau)}\left(\frac{1}{N}\sum_{i=1}^{N} X_n\right) \geq V_s^\infty(\gamma,\pi) - \epsilon.$$

If $f(r) = Br^{-(1-1/M)}$ then $N_\epsilon = O\left(\epsilon^{-2\max(M,2)}\right)$.

Reason: convergence rate for (γ_0,π_0) depends on bound on $\|\gamma_0\|_\infty$ and a function $\psi \in L_1[0,1]$ for which:
$\|V(\gamma_0,\pi_0,r_1) - V(\gamma_0,\pi_0,r_2)\|_\infty \leq \int_{r_1}^{r_2}\psi(r)dr$.

Main Theorem 3

Corollary:
Fix dimensions. There exists $E > 0$ s.t. for any π and bounded C of payoffs, $\exists K = K(\pi, C) > 0$ s.t.:

$$\forall \epsilon > 0 \; \exists N_\epsilon \leq K\epsilon^{-E} \; \forall \gamma \in C \; \exists \sigma \; \forall s, \tau, N \geq N_\epsilon$$
$$E_{\mu_s(\sigma,\tau)} \left(\frac{1}{N} \sum_{i=1}^{N} X_n \right) \geq V_s^\infty(\gamma, \pi) - \epsilon$$

In particular:
$\forall \gamma \in C \; \forall N \geq N_\epsilon \; \left\| V^N(\gamma, \pi) - V^\infty(\gamma, \pi) \right\|_\infty \leq \epsilon.$