Gaussian Marginals of Uniformly Convex Bodies

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Setup

$K \subset \mathbb{R}^n$ convex body.
Assume Vol$(K) = 1$ and centrally symmetric $K = -K$.
Associate norm $\|\cdot\|_K$, space X_K.

$X \sim U(K)$ uniformly distributed random vector in K.

K is isotropic if $\forall \theta \in S^{n-1} \;\; Var(\langle X, \theta \rangle) = L_K^2$.

Well known (Milman-Pajor):
any K has an isotropic affine image.

Therefore we may define L_K for any body K.

"position of K" $=$ volume-preserving affine image of K.

CLT for Convex Bodies

Examples:

- $K = [-\frac{1}{2}, \frac{1}{2}]^n$, $\theta = (\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}})$, $X \sim U(K)$:
  CLT: $\langle X, \theta \rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i \to \sim N(0, \frac{1}{12})$.
  False for all $\theta \in S^{n-1}$, true for most w.r.t. σ.

- $K = r_n D_n$ (Vol $(K) = 1$, $r_n \simeq \sqrt{n}$) $\forall \theta \in S^{n-1}$:
  Maxwell, Poincaré, Borel: $\langle X, \theta \rangle \to \sim N(0, \frac{1}{2\pi e})$.

Others: *Sudakov, Diaconis-Freedman, von-Weizäcker*.

Central Limit Problem for Convex Bodies (*Anttila-Ball-Perissinaki 98, Brehm-Voigt 00*):

$\{\mathcal{K}^n\}_n$ - family of convex bodies in $\mathbb{R}^n$.
d - (pseudo) metric on 1-dim densities.

Do there exist $\varepsilon_n, \mu_n \searrow 0$ s.t.
$\forall n \ \ \forall K \in \mathcal{K}^n$ isotropic , if $X \sim U(K)$:

$$\sigma(\theta \in S^{n-1}; d(g_\theta(K), \phi_{L_K}) \leq \varepsilon_n) \geq 1 - \mu_n.$$

$g_\theta(K)$ - density of $\langle X, \theta \rangle$; ϕ_ρ - density of $N(0, \rho^2)$.

Additional contributions: Anttila, Ball, Bastero, Bernués, Bobkov, Brehm, Hinow, Klartag, E. & M. Meckes, Perissinaki, Sodin, Vogt, Voigt, Wojtaszczyk.

Known Results

Klartag 06: CLP holds for all convex bodies with:
$d_{TV}(f,g) = \|f-g\|_1 \quad \varepsilon_n \simeq \sqrt{\frac{\log\log n}{\log n}} \quad \mu_n \simeq \exp(-cn^{0.99})$.
Also treats multi-dimensional marginals.

Using additional information on $\mathcal{K}^n$:

- Bodies with symmetries:

  Unit-balls of l_p^n - ABP 98, Sodin 04.
  Unit-balls of gen. Orlicz norms - Wojtaszczyk 06.
  Bodies with certain symmetries - E&M Meckes 05.

  Unconditional bodies - Klartag 06
  $(x_1,\ldots,x_n) \in K \Leftrightarrow (\pm x_1,\ldots,\pm x_n) \in K$.

- Uniformly convex bodies with restricted diameter in isotropic position - ABP 98, Sodin 04.

Pseudo metrics d used:

- $d_{Kol}(f,g) = \sup_{t\in\mathbb{R}} \left| \int_{-\infty}^t f(s)ds - \int_{-\infty}^t g(s)ds \right|$.
  ABP 98, E. & M. Meckes 05, Wojtaszczyk 06.

- $d_{Sod}^T(f,g) = \sup_{0\le s\le T} \left| \frac{f(s)}{g(s)} - 1 \right|$, $T \simeq n^\delta$ - Sodin 04.

- Other - Brehm-Voigt, Klartag, ... ; Bobkov 03.

Concentration of Volume in K

First step towards CLT: show concentration of volume in K around thin spherical shell ($|X|$ is concentrated).

General framework for deducing CLT from concentration of volume for isotropic convex bodies (Diaconis-Freedman, von-Weizäcker, ABP, Bobkov, Sodin, etc..)

- [ABP] K satisfies ε-weak concentration ($\varepsilon < \frac{1}{2}$):

$$\exists \rho > 0 \quad Prob\left(\left|\frac{|X|}{\sqrt{n}} - \rho\right| \geq \varepsilon\rho\right) \leq \varepsilon.$$

- [Sodin] K satisfies strong concentration:

$$\begin{array}{ccc} \exists \rho, A, B, \delta, \beta > 0 & & \forall 0 < t < 1 \\ Prob\left(\left|\frac{|X|}{\sqrt{n}} - \rho\right| \geq t\rho\right) & \leq & A \exp(-Bn^{\delta}t^{\beta}). \end{array}$$

Concentration of Volume implies CLT

First observation: isotropic assumption can be removed:

$$C_{iso}(K) \;:=\; \frac{\rho_{max}(K)}{\rho_{avg}(K)}$$

$$\rho_\theta^2 = Var(\langle X, \theta \rangle) \;,\; \rho_{max} \;=\; \max_{\theta \in S^{n-1}} \rho_\theta \;,\; \rho_{avg} = \int_{S^{n-1}} \rho_\theta d\sigma(\theta).$$

Th (M. 06, generalized from ABP):

If K is ε-weakly-concentrated around ρS^{n-1}, $\forall 0 < \delta < c$:

$$\sigma \left\{ \theta \in S^{n-1}; d_{Kol}(g_\theta(K), \phi_\rho) \leq \delta + 4\epsilon + \frac{c_1}{\sqrt{n}} \right\}$$
$$\geq \;\; 1 - C_1 C_{iso}(K) \sqrt{n} \log n \exp\left(- \frac{c_2 n \delta^2}{C_{iso}(K)^2} \right).$$

Th (Sodin, generalized formulation):

If K is strongly-concentrated (A, B, δ, β) around ρS^{n-1}, $\forall \varepsilon \in (0, c)$ $\forall \mu > 0$:

$$\sigma \left\{ \theta \in S^{n-1}; d^T_{Sod}(g_\theta(K), \phi_\rho) \leq \epsilon \right\} \geq 1 - \mu,$$

with $\gamma := \delta / (2 \max(\beta, 1))$ and:

$$T = \rho \min \left(\left(\frac{c n C_{iso}(K)^{-2} \epsilon^4}{\log n + \log \frac{1}{\epsilon} + \log \frac{1}{\mu}} \right)^{\frac{1}{6}}, (c(A, B, \delta, \beta)\epsilon)^{\gamma/\delta} n^\gamma \right).$$

Application: Uniformly Convex Bodies

Modulus of convexity of K (affine invariant):

$$\delta_K(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\|_K \; ; \; \|x\|_K, \|y\|_K \leq 1, \|x-y\|_K \geq \varepsilon\right\}.$$

Uniformly convex: $\delta_K(\varepsilon) > 0 \; \forall \varepsilon > 0$.
p-convex ($p \geq 2$) with constant α: $\delta_K(\varepsilon) \geq \alpha\varepsilon^p$.

Gromov-Milman Concentration (Arias-de-Reyna-Ball-Villa):
For any 1-Lip. (w.r.t. $\|\cdot\|_K$) function f on p-convex K:

$$\mathrm{Vol}\left\{x \in K; \left|f(x) - \int_K f(x)dx\right| \geq t\right\} \leq 4\exp(-2c^p \alpha n t^p).$$

Use for $|x|/\sqrt{n}$ - Lip. constant is $R := diam(K)/\sqrt{n}$.

$$\mathrm{Vol}\left\{x \in K; \left|\frac{|x|}{\sqrt{n}} - \frac{\int_K |x|dx}{\sqrt{n}}\right| \geq tL_K\right\} \leq 4\exp(-2c^p \alpha n (tL_K/R)^p).$$

Meaningful concentration if $R \leq C\left(\frac{\alpha n}{\log n}\right)^{1/p} L_K$.

Problem: isotropic position only guarantees $R \leq Cn^{1/2}L_K$.

2-convex bodies (Klartag, M. 06)

- Example: construct an isotropic 2-convex body with $diam(K) \geq cn \;\; \Rightarrow$ no concentration.
- Observation: In Löwner's minimal diameter position, $diam(K) \leq n^{1-\lambda}/\lambda$, where $\lambda = \lambda(\alpha) > 0$.
- Idea: use Löwner's position for CLT (bound $C_{iso}(K)$).

CLT for p-convex bodies with d_{Kol}

Denote: $\rho = \int_K |x| dx / \sqrt{n}$.

We want to show:

$$(*) \qquad \sigma(\theta \in S^{n-1}; d_{Kol}(g_\theta(K), \phi_\rho) \leq \varepsilon) \geq 1 - \mu.$$

Th (Klartag, M. 06):
If K is 2-convex (α) in Löwner's position, (*) holds with:

$$\varepsilon = C\lambda^{-1}\alpha^{-\frac{1}{2}}n^{-\frac{2}{3}\lambda} \;\; ; \;\; \mu = \exp(-cn^{\frac{2}{3}\lambda}) \;\; ,$$

with $\lambda = \lambda(\alpha) > 0$.

Th (M. 06): If K is p-convex (α) with type $s > \frac{2p}{p+2}$ in Löwner's position, (*) holds with:

$$\varepsilon = CT_s(X_K)\alpha^{-\frac{1}{p}}n^{-\frac{2}{3}v} \;\; ; \;\; \mu = \exp(-cn^{\frac{2}{3}v}) \;\; ,$$

with $v = \frac{1}{2} + \frac{1}{p} - \frac{1}{s} > 0$.

Cor (M. 06):
If $K \in SQ(L_p)$ $(1 < p < \infty)$ in Löwner's position, (*) holds with:

$$\varepsilon = C\sqrt{qr}n^{-\frac{2}{3r}} \;\; ; \;\; \mu = \exp(-cn^{\frac{2}{3r}}) \;\; ,$$

with $r = \max(p, q)$, $q = p^*$.

CLT for p-convex bodies with d^T_{Sod}

Denote: $\rho = \int_K |x| dx / \sqrt{n}$.

We want to show:

$$(*) \qquad \sigma(\theta \in S^{n-1}; d^T_{Sod}(g_\theta(K), \phi_\rho) \leq \varepsilon) \geq 1 - \mu.$$

<u>Th</u> (M. 06):
If K is 2-convex (α) in Löwner's position, (*) holds with:

$$T = c(\alpha)\lambda^{\frac{1}{3}} n^{\frac{\lambda}{23}} \rho \ ; \varepsilon = C n^{-\frac{\lambda}{23}} \ ; \mu = \exp(-n^{\frac{\lambda}{23}}) \ ,$$

with $\lambda = \lambda(\alpha) > 0$.

<u>Th</u> (M. 06): If K is p-convex (α) with type $s > \frac{2p}{p+2}$ in Löwner's position, (*) holds with:

$$T = c(\alpha, p, s) T_s(X_K)^{-\frac{1}{3}} n^{\frac{v}{23}} \rho \ ; \varepsilon = C n^{-\frac{v}{23}} \ \ \mu = \exp(-n^{\frac{v}{23}}) \ ,$$

with $v = \frac{1}{2} + \frac{1}{p} - \frac{1}{s} > 0$.

<u>Cor</u> (M. 06): If $K \in SQ(L_p)$ $(1 < p < \infty)$ in Löwner's position, (*) holds with:

$$T = c(p) n^{\frac{1}{23r}} \rho \ , \ \ \varepsilon = C n^{-\frac{1}{23r}} \ , \ \ \mu = \exp(-n^{\frac{1}{23r}}) \ ,$$

with $r = \max(p, q)$ and $q = p^*$.

Different Approach

Used Gromov-Milman concentration for $f(x) = |x|/\sqrt{n}$ - global $\|\cdot\|_K$-Lip. constant on K is $R := diam(K)/\sqrt{n}$.

Bobkov-Ledoux (2000): log-Sobolev inequality averages-out the local $\|\cdot\|_K$-Lip. constant of f ($= \|\nabla f\|_K^*$) on K.

Th (BL): If K is p-convex (α), $q = p^*$, then $\forall f$:

$$Ent_{U(K)}(|f|^q) \leq \frac{2^q}{\Gamma(\frac{n}{p}+1)^{q/n}} \left(\frac{q}{\alpha}\right)^{q-1} \int_K (\|\nabla f\|_K^*)^q dx.$$

Bobkov-Zegarlinski 05: q-log-Sobolev implies q-Poincare:

$$\forall f \quad Ent_\mu(|f|^q) \quad \leq \quad C \int \|\nabla f\|^q \, d\mu$$

$$\Downarrow$$

$$\forall f \quad \int \left| f - \int f d\mu \right|^q d\mu \quad \leq \quad C \frac{2^q}{\log 2} \int \|\nabla f\|^q \, d\mu.$$

Apply to $f(x) = |x|^2$:

$$\left(\int_K \left| |x|^2 - \int_K |y|^2 dy \right|^q dx \right)^{\frac{1}{q}} \leq \frac{C}{(\alpha n)^{\frac{1}{p}}} \left(\int_K (\|x\|_K^*)^q dx \right)^{\frac{1}{q}}.$$

Get meaningful concentration of $|x|^2$ in K using Markov's inequality if r.h.s. smaller than $\int_K |y|^2 dy \quad (\geq nL_K^2)$.

Meaningful bound in novel position

K isotropic, T volume-preserving linear transformation:

$$\int_{T(K)} \|x\|^*_{T(K)}\, dx = \int_K \|x\|^*_{T^*T(K)}\, dx = \int_{K\setminus K'} + \int_{K'}$$

$$K' = K \cap C\sqrt{n} L_K D_n.$$

Use Majorizing-Measures Th. (Fernique-Talagrand):

$$\int_{K'} \|x\|^*_{T^*T(K)}\, dx = \int_{K'} \sup_{y\in T^*T(K)} \langle x, y\rangle\, dx \leq$$

$$Cn^{\frac{1}{4}} L_K \int_{\mathbb{R}^n} \sup_{y\in T^*T(K)} \langle x, y\rangle\, d\gamma_n(x) \simeq Cn^{\frac{1}{4}} L_K n^{\frac{1}{2}} M^*(T^*T(K)).$$

$M^*(K) = \int_{S^{n-1}} \|\theta\|^*_K\, d\sigma(\theta)$ - "Mean-Width".

($n^{\frac{1}{4}}$ term is same as in Bourgain's bound $L_K \leq Cn^{\frac{1}{4}} \log n$).

For $T = Id$, no good bound on $M^*(K)$ for isotropic K.

Idea: use T so that $T^*T(K)$ has minimal mean-width.
Figiel & Tomczak-Jaegermann and Pisier:
$M^*(T^*T(K)) \leq n^{1/2} \min(C \log n, c(p, \alpha))$.

Concentration for $T(K)$ half way (geometric mean sense) between isotropic K and minimal mean-width $T^*T(K)$.

Concentration and CLT

Denote: $\rho = \left(\int_{T(K)} |x|^2 dx/n\right)^{\frac{1}{2}}$:

Th (M. 06):

$$\text{Vol}\left\{x \in T(K); \left|\frac{|x|}{\sqrt{n}} - \rho\right| \geq t\rho\right\} \leq C\frac{\min(c(p,\alpha), \log n)^q}{n^{\frac{3}{4}q-1}\alpha^{\frac{q}{p}}L_K^q t^q}.$$

Weak-concentration for $2 \leq p < 4$ (because of $n^{\frac{1}{4}}$ term).

Improvement: no restriction on diameter, no type condition, no dependence on implicit $\lambda = \lambda(\alpha)$.

Control $C_{iso}(T(K))$ to derive CLT with d_{Kol}:

$$(*) \qquad \sigma(\theta \in S^{n-1}; d_{Kol}(g_\theta(T(K)), \phi_\rho) \leq \varepsilon) \geq 1 - \mu.$$

Th (M. 06):
If K is p-convex (α), $2 \leq p < 4$, in above position, (*) holds with:

$$\varepsilon = C\min(c(p,\alpha), \log n)^{\frac{q}{q+1}} n^{-\frac{\frac{3}{4}q-1}{q+1}} \alpha^{-\frac{q}{2p+q}}\ ,\ \mu = \exp(-cn\frac{p-q+2}{4p+2q})\ ,$$

where $q = p^*$.

Cor (M. 06):
If $K \in SQ(L_p)$, $1 < p \leq \frac{4}{3}$, in above position, (*) holds with:

$$\varepsilon = \log^{\frac{2}{3}}(n) n^{-\frac{1}{6}}(p-1)^{-\frac{1}{3}}\ ,\ \mu = \exp(-cn^{\frac{1}{6}})\ .$$

Additional Definitions

$T_p(X)$- type-p ($1 \leq p \leq 2$) constant of Banach space $(X, \|\cdot\|)$, is the minimal $T > 0$ for which:

$$\left(\mathbb{E}\left\|\sum_{i=1}^{m} \varepsilon_i x_i\right\|^2\right)^{1/2} \leq T \left(\sum_{i=1}^{m} \|x_i\|^p\right)^{1/p}$$

for any $m \geq 1$ and any $x_1, \ldots, x_m \in X$, where $\{\varepsilon_i\}$ are i.i.d. r.v.'s uniformly distributed on $\{-1, 1\}$ and $\mathbb{E}$ denotes expectation.

$Ent_\mu(f)$- the entropy of a non-negative function f w.r.t. a probability measure μ:

$$Ent_\mu(f) := \int f \log(f) d\mu - \int f d\mu \log\left(\int f d\mu\right).$$