Generalized Intersection Bodies and the Low-Dimensional Busemann-Petty Problem

Emanuel Milman
The Weizmann Institute
of Science, Israel

Workshop on
Convexity and Probability

Freudenstadt
September 2006

Busemann-Petty Problem

Notation: $0 \leq m \leq n$
G^n_m - Grassmann manifold of m-dim subspaces of $\mathbb{R}^n$.

Busemann-Petty (1956): K, L convex symmetric in $\mathbb{R}^n$,

Assume $\forall H \in G^n_{n-1}$ $\mathrm{Vol}\,(K \cap H) \leq \mathrm{Vol}\,(L \cap H)$.

Does it follow that $\mathrm{Vol}\,(K) \leq \mathrm{Vol}\,(L)$?

Series of results 1975-1999 (Ball,Bourgain,Gardner, Giannopoulos,Koldobsky,Larman,Lutwak,Papadimitrakis, Rogers,Schlumprecht,Zhang):

Answer: $n \leq 4$ Yes , $n \geq 5$ No!

Lutwak, Gardner:

Answer to BP-problem is positive in $\mathbb{R}^n$ **iff** every symmetric convex body in $\mathbb{R}^n$ is an intersection body.

Intersection Bodies

K star-body: $\forall x \in K$ $[0, x] \in K$ and ρ_K continuous.

Radial function: $\rho_K(\theta) = \max\{r \geq 0; r\theta \in K\}$, $\theta \in S^{n-1}$.

Radial metric: $d_\rho(K_1, K_2) = \max_{\theta \in S^{n-1}} |\rho_{K_1}(\theta) - \rho_{K_2}(\theta)|$.

<u>K int-body of L</u> if $\rho_K(\theta) = \text{Vol}\left(L \cap \theta^\perp\right) \quad \forall \theta \in S^{n-1}$.

<u>K int-body</u> if $\exists \{K_i\}$ int-bodies of $\{L_i\}$, $d_\rho(K_i, K) \to 0$.

Spherical Radon Transform:

$$R : C_e(S^{n-1}) \to C_e(S^{n-1}) \quad R(f)(\theta) = \int_{S^{n-1} \cap \theta^\perp} f(\xi) d\sigma(\xi)$$

$$R^* : \mathcal{M}_e(S^{n-1}) \to \mathcal{M}_e(S^{n-1}) \quad \int_{S^{n-1}} g R^*(d\mu) = \int_{S^{n-1}} R(g) d\mu$$

Easy to see that $R^*(g) = R(g)$, i.e. self-dual.
R is 1-1 and onto a dense subset.

<u>K int-body of L</u> iff $\rho_K = c_n R(\rho_L^{n-1}) = R^*(g)$, $g \geq 0$.

<u>K int-body</u> ($\mathcal{I}^n$) iff $\rho_K = R^*(d\mu)$, $d\mu \geq 0$.

k-Generalized BP Problem

Zhang (1996): K, L convex symmetric in $\mathbb{R}^n$.

Assume $\forall H \in G^n_{n-k}$ Vol $(K \cap H) \leq$ Vol $(L \cap H)$.

Does it follow that Vol $(K) \leq$ Vol (L) ?

Zhang: Answer for k-generalized BP-problem in $\mathbb{R}^n$ is positive **iff** every symmetric convex body in $\mathbb{R}^n$ is a "generalized int-body" called <u>k-BP body</u> (BP^n_k).

Bourgain & Zhang (1998), Koldobsky (2000):
negative for $1 \leq k \leq n-4$.
true for $k = n-1$ (trivially).
open for $1 < k = n-3, n-2$ (all signs are positive).

<u>known positive answers:</u>

Zhang: $n-k = 2, 3$; any L; K body of revolution.

B&Z: $n-k = 2$; L Ball; K convex perturbation.

M. (05): $n-k = 2, 3$; any L; K s.t. ρ^k_K is convex.
In particular if K is a C^2 perturbation of Ball.

BP_k^n

Spherical m-dim Radon Transform:

$$R_m : C_e(S^{n-1}) \to C(G_m^n) \quad R(f)(E) = \int_{S^{n-1} \cap E} f(\xi) d\sigma(\xi)$$

$$R_m^* : \mathcal{M}(G_m^n) \to \mathcal{M}_e(S^{n-1}) \quad \int_{S^{n-1}} g R_m^*(d\mu) = \int_{G_m^n} R_m(g) d\mu$$

More concretely:

$$R_m^*(f)(\theta) = \int_{\theta \in E \in G_m^n} f(E) dE.$$

R_m is 1-1 but its image $R_m(C(S^{n-1}))$ not dense in $C(G_m^n)$ for $1 < m < n-1$, so $Ker R_m^* \neq 0$ in this range.

$$\begin{aligned} K \in \mathcal{I}^n \Longleftrightarrow \rho_K &= R^*(d\mu') \;\; d\mu' \in \mathcal{M}_+(S^{n-1}) \\ &= R_{n-1}^*(d\mu) \;\; d\mu \in \mathcal{M}_+(G_{n-1}^n) \end{aligned}$$

$$\underline{K \in BP_k^n} \Longleftrightarrow \rho_K^k = R_{n-k}^*(d\mu) \;\; d\mu \in \mathcal{M}_+(G_{n-k}^n)$$

$\mathcal{I}_k^n$

Second generalization of $\mathcal{I}^n$ proposed by Koldobsky.

$$K \text{ int-body of } L \iff \rho_K(\theta) = \text{Vol}\left(L \cap \theta^{\perp}\right) \quad \forall \theta \in S^{n-1}$$
$$\iff \frac{1}{2}\text{Vol}\left(K \cap E^{\perp}\right) = \text{Vol}\left(L \cap E\right) \quad \forall E \in G_{n-1}^n$$

$$\underline{K\ k\text{-int-body of } L} \iff \text{Vol}\left(K \cap E^{\perp}\right) = \text{Vol}\left(L \cap E\right) \quad \forall E \in G_{n-k}^n$$

$\underline{K\ k\text{-int-body}}$ $(\mathcal{I}_k^n)$ if limit in the radial-metric.

Natural to describe using Fourier Transforms of homogeneous distributions (Koldobsky):

$$K \in \mathcal{I}_k^n \iff (\|\cdot\|_K^{-k})^{\wedge} \geq 0$$

In some sense an extension of L_p to L_{-k}.

$\mathcal{I}_k^n$ played important role in unified solution to BP-problem (Gardner Koldobsky Schlumprecht 99).

$BP_k^n = \mathcal{I}_k^n$?

Koldobsky 00, M. 05: $BP_k^n \subset \mathcal{I}_k^n$.

Koldobsky 00: $BP_k^n = \mathcal{I}_k^n$ implies positive answer to unresolved cases ($n-k=2,3$) in k-gen BP-problem .

<u>Reason:</u>

Koldobsky: $\{$Convex bodies in $\mathbb{R}^n\} \subset \mathcal{I}_k^n$ iff $k \geq n-3$.

if $BP_k^n = \mathcal{I}_k^n$ for $k = n-3, n-2$:

$$\{\text{Convex bodies in } \mathbb{R}^n\} \subset BP_{n-3}^n, BP_{n-2}^n.$$

Zhang: this would imply positive answer to k-gen BP-problem for $k = n-3, n-2$.

<u>Conclusion:</u>

$BP_k^n = \mathcal{I}_k^n$ is an important question.
Need to understand structures of $BP_k^n, \mathcal{I}_k^n$.

Identical Structures of $BP_k^n, \mathcal{I}_k^n$

Th. (M. 05) For $\mathcal{C} = BP, \mathcal{I}$ (using different methods):

1. $\mathcal{C}_k^n$ closed under full-rank linear transformations, k-radial sums ($\rho_L^k = \rho_{K_1}^k + \rho_{K_2}^k$), limit in radial metric.

2. $\mathcal{C}_1^n = \mathcal{I}^n$, $\mathcal{C}_{n-1}^n = \{$symmetric star-bodies in $\mathbb{R}^n\}$.

3. Let $K_1 \in \mathcal{C}_{k_1}^n$, $K_2 \in \mathcal{C}_{k_2}^n$ and $l = k_1 + k_2 \leq n - 1$. If $\rho_L^l = \rho_{K_1}^{k_1} \rho_{K_2}^{k_2}$ then $L \in \mathcal{C}_l^n$. As corollaries:
   (a) $\mathcal{C}_{k_1}^n \cap \mathcal{C}_{k_2}^n \subset \mathcal{C}_{k_1+k_2}^n$ if $k_1 + k_2 \leq n - 1$.
   (b) $\mathcal{C}_k^n \subset \mathcal{C}_l^n$ if k divides l (*open: $k < l$?*)
   (c) If $K \in \mathcal{C}_k^n$ and $\rho_L = \rho_K^{k/l}$ then $L \in \mathcal{C}_l^n$ for $l \geq k$.

4. If $K \in \mathcal{C}_k^n$ then $K \cap E \in \mathcal{C}_k^m$ for $E \in G_m^n$ and $m > k$.

(1) and (2) well-known and basically follow from defs.

For $\mathcal{C} = \mathcal{I}$, (3) independently noticed by Koldobsky.

For $\mathcal{C} = BP$, (4) and (3b) for $k = 1$ were proved by Grinberg and Zhang. To deduce (3), need generalization of Blaschke-Petkantschin formula.

Gen. Blaschke-Petkantschin formula

$k_1, \ldots, k_r \geq 1$, $l = \sum k_i \leq n-1$.
Denote $G^a = G^n_{n-a}$, $G^a_F = \{E \in G^a; F \subset E\}$,
$d\mu^Y_X$ normalized Haar measures on corresponding spaces.
$\bar{E} = (E_1, \ldots, E_r)$ with $E_i \in G^{k_i}$.

Th. (M. 05):

For any $f(\bar{E}) = f(E_1, \ldots, E_r)$ in $C(G^{k_1} \times \ldots \times G^{k_r})$:

$$\int_{E_1 \in G^{k_1}} \cdots \int_{E_r \in G^{k_r}} f(\bar{E}) d\mu^{k_1}(E_1) \cdots d\mu^{k_r}(E_r) =$$
$$\int_{F \in G^l} \int_{E_1 \in G^{k_1}_F} \cdots \int_{E_r \in G^{k_r}_F} f(\bar{E}) \Delta(\bar{E}) d\mu^{k_1}_F(E_1) \cdots d\mu^{k_r}_F(E_r) d\mu^l(F),$$

where $\Delta(\bar{E}) = C_{n,\{k_i\},l} \Omega(\bar{E})^{n-l}$, and $\Omega(\bar{E})$ denotes the l-dimensional volume of the parallelepiped spanned by unit volume elements of $E_1^\perp, \ldots, E_r^\perp$.

Original Blaschke-Petkantschin formula is some equivalent version of above for $k_i = 1$.

Symmetry between k and $n-k$

Th. (M. 05):

$$BP_k^n = \mathcal{I}_k^n \Longleftrightarrow BP_{n-k}^n = \mathcal{I}_{n-k}^n$$

Motivation:

$$I : G_m^n \to G_{n-m}^n \quad I(E) = E^\perp$$

We show:

$$KerR_{n-k}^* = KerR_k^* \circ I,$$

or equivalently:

$$\overline{ImR_{n-k}} = \overline{ImI \circ R_k}.$$

Equivalent formulations of $BP_k^n = \mathcal{I}_k^n$

Notation:
G class of functions $\Longrightarrow$ G_+ non-negative members.

Th. (M. 05): The following are equivalent:

1.
$$BP_k^n = \mathcal{I}_k^n$$

2.
$$\overline{R_{n-k}(C(S^{n-1}))_+} = \overline{R_{n-k}(C_+(S^{n-1})) + I \circ R_k(C_+(S^{n-1}))}.$$

3. Other formulations...

$$\overline{R_{n-k}(C(S^{n-1}))_+} \supset \overline{R_{n-k}(C_+(S^{n-1})) + I \circ R_k(C_+(S^{n-1}))}.$$

because $\overline{ImI \circ R_k} = \overline{ImR_{n-k}}$, explaining why $BP_k^n \subset \mathcal{I}_k^n$.