Generalized Intersection Bodies are Not Equivalent

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Busemann-Petty Problem

Notation: $0 \le m \le n$
G^n_m - Grassmann manifold of m-dim subspaces of $\mathbb{R}^n$.

Busemann-Petty (1956): K, L convex symmetric in $\mathbb{R}^n$,

$$\text{Assume} \quad \forall H \in G^n_{n-1} \quad \mathrm{Vol}\,(K \cap H) \le \mathrm{Vol}\,(L \cap H)\,.$$

$$\text{Does it follow that} \qquad \mathrm{Vol}\,(K) \le \mathrm{Vol}\,(L) \quad ?$$

Series of results 1975-1999 (Ball,Bourgain,Gardner, Giannopoulos,Koldobsky,Larman,Lutwak,Papadimitrakis, Rogers,Schlumprecht,Zhang):

Answer: $n \le 4$ Yes , $n \ge 5$ No!

Lutwak, Gardner:

Answer to BP-problem is positive in $\mathbb{R}^n$ **iff** every symmetric convex body in $\mathbb{R}^n$ is an intersection body.

Intersection Bodies

K star-body: $\forall x \in K$ $[0,x] \in K$ and ρ_K continuous.

Radial function: $\rho_K(\theta) = \max\{r \geq 0; r\theta \in K\}$, $\theta \in S^{n-1}$.

Radial metric: $d_\rho(K_1, K_2) = \max_{\theta \in S^{n-1}} |\rho_{K_1}(\theta) - \rho_{K_2}(\theta)|$.

$\underline{K \text{ int-body of } L}$ if $\rho_K(\theta) = \text{Vol}\left(L \cap \theta^\perp\right)$ $\forall \theta \in S^{n-1}$.

$\underline{K \text{ int-body}}$ if $\exists \{K_i\}$ int-bodies of $\{L_i\}$, $d_\rho(K_i, K) \to 0$.

Spherical Radon Transform:

$$R : C_e(S^{n-1}) \to C_e(S^{n-1}) \quad R(f)(\theta) = \int_{S^{n-1} \cap \theta^\perp} f(\xi) d\sigma_{\theta^\perp}(\xi)$$

$$R^* : \mathcal{M}_e(S^{n-1}) \to \mathcal{M}_e(S^{n-1}) \quad \int_{S^{n-1}} g R^*(d\mu) = \int_{S^{n-1}} R(g) d\mu$$

Easy to see that $R^*(g) = R(g)$, i.e. self-adjoint.

$\underline{K \text{ int-body of } L}$ iff $\rho_K = c_n R(\rho_L^{n-1}) = R^*(g)$, $g \geq 0$.

$\underline{K \text{ int-body}}$ ($\mathcal{I}^n$) iff $\rho_K = R^*(d\mu)$, $\mu \geq 0$.

k-Generalized BP Problem

Zhang (1996): K, L convex symmetric in $\mathbb{R}^n$,
Fix $1 \leq k \leq n-1$.

Assume $\forall H \in G^n_{n-k}$ Vol $(K \cap H) \leq$ Vol $(L \cap H)$.

Does it follow that Vol $(K) \leq$ Vol (L) ?

Zhang: Answer for k-generalized BP-problem in $\mathbb{R}^n$ is positive **iff** every symmetric convex body in $\mathbb{R}^n$ is a "generalized int-body" called k-BP body (BP^n_k).

Bourgain & Zhang (1998), Koldobsky (2000):
negative for $1 \leq k \leq n-4$.
true for $k = n-1$ (trivially).
open for $n \geq 5, k = n-3, n-2$; *low-dim BP problem*.

Known positive answers:

Zhang: $n-k = 2, 3$; any L; K body of revolution.
Generalized by Rubin (07) to K with axial symmetries.

B&Z: $n-k = 2$; L Ball; K convex perturbation.
Generalized by M. (05) to $n-k = 2, 3$; any L; K convex perturbation of Ball.

BP_k^n

Spherical m-dim Radon Transform:

$$R_m : C_e(S^{n-1}) \to C(G_m^n) \quad R_m(f)(E) = \int_{S^{n-1}\cap E} f(\xi)\,d\sigma_E(\xi)$$

$$R_m^* : \mathcal{M}(G_m^n) \to \mathcal{M}_e(S^{n-1}) \quad \int_{S^{n-1}} g R_m^*(d\mu) = \int_{G_m^n} R_m(g)\,d\mu$$

$$\begin{aligned} K \in \mathcal{I}^n \iff \rho_K &= R^*(d\mu') & \mu' \in \mathcal{M}_+(S^{n-1}) \\ &= R_{n-1}^*(d\mu) & \mu \in \mathcal{M}_+(G_{n-1}^n) \end{aligned}$$

$$\underline{K \in BP_k^n} \iff \rho_K^k = R_{n-k}^*(d\mu) \quad \mu \in \mathcal{M}_+(G_{n-k}^n)$$

$\mathcal{I}_k^n$

Second generalization of $\mathcal{I}^n$ proposed by Koldobsky.

$$K \text{ int-body of } L \iff \rho_K(\theta) = \text{Vol}\left(L \cap \theta^\perp\right) \quad \forall \theta \in S^{n-1}$$

$$\iff \frac{1}{2}\text{Vol}\left(K \cap E^\perp\right) = \text{Vol}\left(L \cap E\right) \quad \forall E \in G^n_{n-1}$$

$$\underline{K \ k\text{-int-body of } L} \iff \text{Vol}\left(K \cap E^\perp\right) = \text{Vol}\left(L \cap E\right) \quad \forall E \in G^n_{n-k}$$

$\underline{K \ k\text{-int-body}}$ ($\mathcal{I}_k^n$) if limit in the radial-metric.

Natural to describe using Fourier Transforms of homogeneous distributions (Koldobsky):

$$K \in \mathcal{I}_k^n \iff (\|\cdot\|_K^{-k})^\wedge \geq 0$$

In some sense an extension of L_p to L_{-k}.

$\mathcal{I}_k^n$ played important role in unified solution to BP-problem (Gardner Koldobsky Schlumprecht 99).

$BP_k^n = \mathcal{I}_k^n$?

Koldobsky 00, M. 05: $BP_k^n \subset \mathcal{I}_k^n$.

Koldobsky 00: Question - $BP_k^n = \mathcal{I}_k^n$?
This would imply positive answer to unresolved cases ($n - k = 2, 3$) in k-gen BP-problem.

Conclusion: $BP_k^n = \mathcal{I}_k^n$ is an interesting question.

Plan:

1. Motivation why $BP_k^n = \mathcal{I}_k^n$.

2. Th. (M. 07): $BP_k^n \neq \mathcal{I}_k^n$ for $n \geq 4, 2 \leq k \leq n - 2$.

Identical Structures of $BP_k^n, \mathcal{I}_k^n$

Th. (M. 05) For $\mathcal{C} = BP, \mathcal{I}$ (using different methods):

1. $\mathcal{C}_k^n$ closed under full-rank linear transformations, k-radial sums ($\rho_L^k = \rho_{K_1}^k + \rho_{K_2}^k$), limit in radial metric.

2. $\mathcal{C}_1^n = \mathcal{I}^n$, $\mathcal{C}_{n-1}^n = \{$symmetric star-bodies in $\mathbb{R}^n\}$.

3. Let $K_1 \in \mathcal{C}_{k_1}^n$, $K_2 \in \mathcal{C}_{k_2}^n$ and $l = k_1 + k_2 \leq n - 1$.
   If $\rho_L^l = \rho_{K_1}^{k_1} \rho_{K_2}^{k_2}$ then $L \in \mathcal{C}_l^n$. As corollaries:
   - (a) $\mathcal{C}_{k_1}^n \cap \mathcal{C}_{k_2}^n \subset \mathcal{C}_{k_1+k_2}^n$ if $k_1 + k_2 \leq n - 1$.
   - (b) $\mathcal{C}_k^n \subset \mathcal{C}_l^n$ if k divides l (*open:* $k < l$?)
   - (c) If $K \in \mathcal{C}_k^n$ and $\rho_L = \rho_K^{k/l}$ then $L \in \mathcal{C}_l^n$ for $l \geq k$.

4. If $K \in \mathcal{C}_k^n$ then $K \cap E \in \mathcal{C}_k^m$ for $E \in G_m^n$ and $m > k$.

(1) and (2) well-known and basically follow from defs.

For $\mathcal{C} = \mathcal{I}$, (3) independently noticed by Koldobsky.

For $\mathcal{C} = BP$, (4) and (3b) for $k = 1$ were proved by Grinberg and Zhang.

$BP_k^n \neq \mathcal{I}_k^n$

Th. (M. 07): $BP_k^n \neq \mathcal{I}_k^n$ for $n \geq 4, 2 \leq k \leq n-2$.
Construct C^∞ *body of revolution* in $\mathcal{I}_k^n \setminus BP_k^n$.

Proof relies on:

Th. (M. 05): The following are equivalent:

1. $$BP_k^n \subsetneq \mathcal{I}_k^n$$

2. $\exists g \in C^\infty(G_{n-k}^n)$, $R_{n-k}^*(g) \geq 1$ and $(I \circ R_k)^*(g) \geq 1$, but g is not non-negative functional on $R_{n-k}(C(G_{n-k}^n))$: $\exists h \in R_{n-k}(C(G_{n-k}^n))_+$ s.t. $\int_{G_{n-k}^n} g(E)h(E)d\eta(E) < 0$.

   (where $I : C(G_k^n) \to C(G_{n-k}^n)$ $\quad I(f)(E) = f(E^\perp)$)

3. Dual statement:

$$\overline{R_{n-k}(C(S^{n-1}))_+} \supsetneq \overline{R_{n-k}(C_+(S^{n-1})) + I \circ R_k(C_+(S^{n-1}))}.$$

<u>Idea of Proof:</u> construct $g \in C^\infty(G_{n-k}^n)$ in (2) invariant under natural action of $O(n-1) < O(n)$, by analyzing the action of R_{n-k} and R_{n-k}^* on functions of revolution.

Additional Equivalent Formulations using Fourier Transforms

Given $f \in C(S^{n-1}), p \in \mathbb{R}$ denote:

$$E_p(f)(r\theta) = f(\theta)r^p \quad r > 0, \theta \in S^{n-1},$$

$$E_p^\wedge(f) = \text{Fourier Transform of } E_p(f) \text{ as distribution.}$$

For $1 \leq k \leq n-1$, $E_{-k}^\wedge(1) = c_{n,k}E_{-n+k}(1) \geq 0$.

If $T \in PD(n)$, $T(E_{-k}(1))^\wedge = det(T)T^{-1}(E_{-k}^\wedge(1)) \geq 0$.

Equivalent formulation to $BP_k^n = \mathcal{I}_k^n$:

If $f \geq 0$ and $E_{-k}^\wedge(f) \geq 0$, is f the limit of $\sum_{i=1}^m T_i(E_{-k}(1))$, $T_i \in PD(n)$?

Answer: No, for $n \geq 4$ and $2 \leq k \leq n-2$.
(Yes for $k = 1$ (Goodey & Weil) and $k = n-1$).