Dual-Mixed-Volumes and the Slicing Problem

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Introduction

$K \subset \mathbb{R}^n$ convex body.
Assume centrally symmetric $K = -K$.

Def: K is *isotropic* if

$$\int_K \langle x, \theta_1 \rangle^2 \, dx = \int_K \langle x, \theta_2 \rangle^2 \, dx \;\; \forall \theta_1, \theta_2 \in S^{n-1}.$$

If in addition $\mathrm{Vol}(K) = 1$, define L_K as:

$$L_K^2 = \int_K \langle x, \theta \rangle^2 \, dx \;\; \forall \theta \in S^{n-1}.$$

Well known: isotropic position always exists.
Therefore we may define L_K for any body K.

The Slicing Problem

Slicing-Problem (Bourgain 84; Milman Pajor 87; Ball):

$$\exists C > 0 \quad \forall K \subset \mathbb{R}^n \quad L_K \leq C \ \ ?$$

Equivalently:

$$\exists c > 0 \quad \forall K \subset \mathbb{R}^n \quad \text{Vol}(K) = 1$$
$$\exists \theta \in S^{n-1} \quad \text{Vol}(K \cap \theta^\perp) \geq c \ \ ?$$

Slicing problem is known to be true for several families of convex bodies, such as unconditional bodies, projection bodies, u.b.'s of subspaces of L_1 (e.g. Bourgain, MP87, Ball 89).

Best general bound (Bourgain 91):

$$L_K \leq C n^{1/4} \log(1+n).$$

Known Bounds

SL_p^n, QL_p^n, SQL_p^n :
classes of unit-balls of n-dimensional subspaces, quotients, subspaces of quotients of L_p.

Outer Volume-Ratio:

$$L_K \leq C \inf \left\{ \left(\frac{\text{Vol}(\mathcal{E})}{\text{Vol}(K)} \right)^{1/n} \middle| \; K \subset \mathcal{E} \; , \; \mathcal{E} \in SL_2^n \right\} .$$

Th (Ball 89):

$$L_K \leq C \inf \left\{ \left(\frac{\text{Vol}(L)}{\text{Vol}(K)} \right)^{1/n} \middle| \; K \subset L \; , \; L \in SL_1^n \right\} .$$

$\big($ In fact, $L_K \leq C_1 wrgl_2(X_K^*) \leq C_2 gl_2(X_K)$ $\big)$.

Th (Junge 94):

$$L_K \leq C \inf \left\{ \sqrt{p}\, q \left(\frac{\text{Vol}(L)}{\text{Vol}(K)} \right)^{1/n} \middle| \; \begin{array}{l} K \subset L \; , \; L \in SQL_p^n \; , \\ 1 < p < \infty \; , \; 1/p + 1/q = 1 \end{array} \right\} .$$

(In fact, for $L \in SQX^n$, if $cotype(X), cotype(X^*), gl_2(X) \leq C$).

As evident from general formulations, Ball and Junge make use of tools from Functional Analysis and Operator Theory.

This is expected if formulations use (variants of) the Gordon-Lewis property.

Perhaps for SL_p^n and QL_q^n the approach could be simplified, bounds improved, and framework unified?

Indeed, Ball and Junge's results are complimentary for SL_p^n and QL_q^n:

$1 \le p \le 2$	$SL_p^n \subset SL_1^n$	use Ball	$\Rightarrow$	$L_K \le C$
$p \ge 2$	SL_p^n	use Junge	$\Rightarrow$	$L_K \le C\sqrt{p}$
$1 \le q \le 2$	QL_q^n	use Junge	$\Rightarrow$	$L_K \le Cp$
$q \ge 2$	$QL_q^n \subset QL_\infty^n$	use outer v.r. or Ball	$\Rightarrow$	$L_K \le C$

Reminder

Th (Ball):

$$L_K \leq C \,\mathsf{inf} \left\{ \left(\frac{\mathsf{Vol}(L)}{\mathsf{Vol}(K)} \right)^{1/n} \middle| \; K \subset L \;,\; L \in SL_1^n \right\} .$$

Th (Junge):

$$L_K \leq C \,\mathsf{inf} \left\{ \sqrt{p} \left(\frac{\mathsf{Vol}(L)}{\mathsf{Vol}(K)} \right)^{1/n} \middle| \; K \subset L \;,\; L \in SL_p^n \;,\; p \geq 2 \right\} .$$

$$L_K \leq C \,\mathsf{inf} \left\{ p \left(\frac{\mathsf{Vol}(L)}{\mathsf{Vol}(K)} \right)^{1/n} \middle| \; \begin{array}{l} K \subset L \;,\; L \in QL_q^n \;, \\ 1 < q \leq 2 \;,\; 1/p + 1/q = 1 \end{array} \right\} .$$

Main Results

Using elementary argument, we unify ranges of $p \geq 0$, and extend to $p < 0$ (using generalized intersection-bodies).

Th 1. K isotropic, $\text{Vol}(K) = \text{Vol}(D_n)$:

$$L_K \leq C \inf \left\{ \frac{\sqrt{p_0}}{M_p(L)} \,\middle|\, K \subset L \,,\ L \in SL_p^n \,,\ p \geq 0 \right\},$$

where:

$p_0 = \max(1, \min(p, n))$, $M_p(L) = \left(\int_{S^{n-1}} \|\theta\|_L^p \, d\sigma(\theta)\right)^{1/p}$.

Generalization because:

$$\frac{1}{M_p(L)} \leq \left(\frac{\text{Vol}(L)}{\text{Vol}(D_n)}\right)^{1/n}.$$

Example:

Improvement for $K = [-1, 1]^n$, $K \simeq K' \in SL_{\log n}^n$.
Junge $\Rightarrow L_K \leq C\sqrt{\log n}$. Th1 $\Rightarrow L_K \leq C$.

Extension to $p < 0$

Th 2. K isotropic, $\text{Vol}(K) = \text{Vol}(D_n)$:

$$L_K \le C \ \inf \left\{ \mathcal{L}_k \widetilde{M}_k(L) \middle| \ K \subset L \ , \ L \in \mathcal{I}_k^n \ , \ k = 1, \dots, n \right\},$$

where:

$\mathcal{I}_k^n =$ Zhang's n-dim. generalized k-intersection bodies.
In some sense $\mathcal{I}_k^n \subset SL_p^n$ for $p = -k$.

$\mathcal{L}_k = \sup \left\{ L_G | G \subset \mathbb{R}^k \right\}$, $\widetilde{M}_k(L) = \left(\int_{S^{n-1}} \rho_L(\theta)^k d\sigma(\theta) \right)^{1/k}$.

($\rho_L =$ radial function, $\rho_L(\theta) = 1 / \|\theta\|_L \ \forall \theta \in S^{n-1}$).

Generalization because:

$$\left(\frac{1}{M_p(L)} \le \right) \ \widetilde{M}_k(L) \le \left(\frac{\text{Vol}(L)}{\text{Vol}(D_n)} \right)^{1/n},$$

and for $1 \le p \le 2$, $k = 1, \dots, n$:

$$SL_p^n \underset{\text{(Koldobsky)}}{\subset} \mathcal{I}_1^n \underset{\text{(Grinberg Zhang)}}{\subset} \mathcal{I}_k^n.$$

Again: norm-moments and not volume-ratio.

QL_q^n

Th 3. K isotropic, $\mathrm{Vol}(K) = \mathrm{Vol}(D_n)$:

$$L_K \leq C \inf \left\{ \sqrt{p_0}\, M_p^*(T(L)) \;\middle|\; \begin{array}{l} K \subset L \,,\; L \in QL_q^n \,,\; T \in SL(n) \\ 1 \leq q \leq \infty \,,\; 1/p + 1/q = 1 \end{array} \right\},$$

where $M_p^*(G) = M_p(G^\circ)$, $p_0 = \max(1, \min(p, n))$.

Why is this a generalization?

Lemma A.

$$\forall L \in QL_q^n \;\; \exists T \in SL(n) \;\text{ s.t. } M_p^*(T(L)) \leq C\sqrt{p} \left(\frac{\mathrm{Vol}(L)}{\mathrm{Vol}(D_n)} \right)^{1/n}.$$

In fact, $T(L)$ may be chosen as:

1. John's maximal volume ellipsoid position (actually gives stronger result).

2. Dual to isotropic position (follows from Th 1).

Cor.

$$L_K \leq C \inf \left\{ p \left(\frac{\mathrm{Vol}(L)}{\mathrm{Vol}(K)} \right)^{1/n} \;\middle|\; \begin{array}{l} K \subset L \,,\; L \in QL_q^n \,, \\ 1 < q \leq \infty \,,\; 1/p + 1/q = 1 \end{array} \right\}.$$

Dual counterpart to $\mathcal{I}_k^n$

Th 4. K isotropic, $\mathrm{Vol}(K) = \mathrm{Vol}(D_n)$:

$$L_K \leq C \inf \left\{ \frac{\mathcal{L}_{2k}^2}{\widetilde{M}_k(T(L))} \;\middle|\; \begin{array}{c} L \subset K^\circ \,,\, L \in \mathcal{I}_k^n \,,\, T \in SL(n) \\ k = 1, \ldots, \lfloor n/2 \rfloor \end{array} \right\}.$$

Why is this a generalization? $L \in \mathcal{I}_k^n$ need not be convex so no generic analogue of Lemma A. Denote $\mathcal{CI}_k^n = \mathcal{I}_k^n \bigcap \{\text{convex}\}$.

Lemma B (Cor. of Th 2).

$$\forall L \in \mathcal{CI}_k^n \quad \underset{\text{(isotropic)}}{\exists T \in SL(n)} \quad \text{s.t. } \widetilde{M}_k(T(L)) \geq \frac{C}{\mathcal{L}_k} \left(\frac{\mathrm{Vol}(L)}{\mathrm{Vol}(D_n)} \right)^{1/n}.$$

Cor.

$$L_K \leq C \inf \left\{ \mathcal{L}_{2k}^2 \mathcal{L}_k \left(\frac{\mathrm{Vol}(L^\circ)}{\mathrm{Vol}(K)} \right)^{1/n} \;\middle|\; \begin{array}{c} K \subset L^\circ \,,\, L \in \mathcal{CI}_k^n \\ k = 1, \ldots, \lfloor n/2 \rfloor \end{array} \right\}.$$

Ingredients of Proof

Use dual mixed-volumes (Lutwak 75).

$$\widetilde{V}_p(L_1, L_2) = \frac{1}{n}\int_{S^{n-1}} \rho_{L_1}(x)^p \rho_{L_2}(x)^{n-p}\, dx \quad p \in \mathbb{R},$$

$$\widetilde{V}_p(L, L) = \mathrm{Vol}(L) \ , \ \widetilde{V}_p(L, D_n) = \mathrm{Vol}(D_n)\widetilde{M}_p(L)^p.$$

Use the representations / definitions:

$$L \in SL_p^n \Leftrightarrow \|x\|_L^p = \int_{S^{n-1}} |\langle x, \theta\rangle|^p\, d\mu(x) \ , \ \mu \in \mathcal{M}_+(S^{n-1}).$$

$$L \in \mathcal{I}_k^n \Leftrightarrow \rho_L(x)^k = R^*_{n-k}(d\mu) \ , \quad \mu \in \mathcal{M}_+(G(n, n-k)).$$

$$R_{n-k} : C(S^{n-1}) \longrightarrow C(G(n, n-k))$$
$$R_{n-k}(f)(E) = \int_{S^{n-1}\cap E} f(\theta) d\sigma_{n-k-1}(\theta).$$

Idea of Proof

Actually prove (a little stronger):

Basic Th. K isotropic $\mathrm{Vol}(K) = \mathrm{Vol}(D_n)$:

1. If $L \in SL_p^n$, $p \geq 0$, then:

$$C_1/\sqrt{p_0} \underset{\text{Th 3}}{\leq} L_K \Big/ \left(\frac{\widetilde{V}_{-p}(L,K)}{\widetilde{V}_{-p}(L,D_n)} \right)^{1/p} \underset{\text{Th 1}}{\leq} C_2\sqrt{p_0}.$$

2. If $L \in \mathcal{I}_k^n$, $k = 1, \ldots, n$, then:

$$C_3 \underset{\text{Th 4}}{\leq} L_K \Big/ \left(\frac{\widetilde{V}_k(L,D_n)}{\widetilde{V}_k(L,K)} \right)^{1/k} \underset{\text{Th 2}}{\leq} C_4\mathcal{L}_k.$$

Idea: Plug in representations inside dual mixed-volumes, and use Fubini (1) or Duality (2). Use $\mu \in \mathcal{M}_+$ to preserve directions of inequalities.

Remark: Proofs of dual counterparts resemble Bourgain's proof of $L_K \leq Cn^{1/4}\log(1+n)$, in the sense that we get an inequality $L_K \geq \mathrm{expr}(T(K))L_K^2$.

The class $\mathcal{I}_k^n$ (propaganda)

May be useful for bounding L_K:

$$\mathcal{L}_n \leq C \text{ inf} \left\{ \mathcal{L}_k \left(\frac{\text{Vol}(L)}{\text{Vol}(K)} \right)^{1/n} \middle| \; K \subset L \; , \; L \in \mathcal{I}_k^n \; , \; k = 1, \ldots, n \right\}.$$

Tempting to try iterating this inequality:

If $\forall n, K \; \exists L \in \mathcal{I}_k^n$ s.t. $K \subset L, k = n^{1-\epsilon}$ and $\left(\frac{\text{Vol}(L)}{\text{Vol}(K)}\right)^{1/n} \leq C$,
Then $\mathcal{L}_n \leq C_1 \log(1+n)^{C_2/\log(1+\epsilon)}$.

Known: $\mathcal{I}_{n-4}^n$ does not contain all convex bodies, $n \geq 5$ (Bourgain Zhang 98, Koldobsky 2000).
$\mathcal{I}_{n-1}^n$ contains all star-bodies.

The advantage of $L \in \mathcal{I}_k^n$ over SL_1^n, is that L is no longer required to be convex, so we might try **constructing** tight $L \supset K$.
Tempting because of characterization of $\mathcal{I}_k^n$ (Grinberg Zhang 99, extending Goodey Weil 95 for $k = 1$):

$$L \in \mathcal{I}_k^n \Leftrightarrow L \text{ radial-limit of } L_i \; , \; \rho_{L_i}^k = \rho_{\mathcal{E}_1}^k + \ldots + \rho_{\mathcal{E}_m}^k.$$