Generalized Intersection Bodies and the Low Dimensional Busemann-Petty Problem

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Outline

- The Busemann-Petty Problem
- Intersection Bodies
- The k-Generalized Busemann-Petty Problem
- Low Dimensional Busemann-Petty Problem
- First Generalization of Int-Bodies: k-Busemann-Petty Bodies ($\mathcal{BP}_k^n$), Spherical Radon Transforms.
- Second Generalization of Int-Bodies: k-Intersection Bodies ($\mathcal{I}_k^n$), Fourier Transforms of Homogeneous Distributions.
- Relationship between $\mathcal{BP}_k^n$ and $\mathcal{I}_k^n$. Are these families equivalent?

Notation: $0 \leq m \leq n$
G_m^n - Grassmann manifold of m-dim linear subspaces of $\mathbb{R}^n$.

Busemann-Petty Problem (1956)

Let K, L denote two convex symmetric bodies in $\mathbb{R}^n$.

$$\text{Assume} \quad \forall H \in G_{n-1}^n \quad \text{Vol}_{n-1}(K \cap H) \leq \text{Vol}_{n-1}(L \cap H).$$

$$\text{Does it follow that} \qquad \text{Vol}_n(K) \leq \text{Vol}_n(L) \quad ?$$

Series of results 1975-1999 (Ball,Bourgain,Gardner, Giannopoulos,Koldobsky,Larman,Lutwak,Papadimitrakis, Rogers,Schlumprecht,Zhang):

Answer: $n \leq 4$ Yes , $n \geq 5$ No!

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Intersection Bodies

- Key Observation (Lutwak, Gardner):
  Answer to BP-problem is positive in $\mathbb{R}^n$ **iff** every symmetric convex body in $\mathbb{R}^n$ is an Intersection Body.
- Intersection Bodies were introduced by Lutwak in 1975. They belong to a larger class of bodies:
- K is called a star-body if $\forall x \in K$ $[0, x] \in K$ and its *radial function* ρ_K is continuous (and even).
- $\rho_K(\theta) = \max \{r \geq 0; r\theta \in K\}$, $\theta \in S^{n-1}$. $\rho_K = \|\cdot\|_K^{-1}$, $\|x\|_K = \min \{r \geq 0; x \in rK\}$ is Minkowski's functional.
- Radial metric: $d_\rho(K_1, K_2) = \max_{\theta \in S^{n-1}} \left|\rho_{K_1}(\theta) - \rho_{K_2}(\theta)\right|$.

Definition

K int-body of L if $\rho_K(\theta) = \mathrm{Vol}_{n-1}\left(L \cap \theta^\perp\right) \quad \forall \theta \in S^{n-1}$.

K intersection-body if $\exists \{K_i\}$ int-bodies of $\{L_i\}$, $d_\rho(K_i, K) \to 0$.

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Easy to see that $R^*(g) = R(g)$, i.e. self-adjoint.
R is injective and (by duality) onto a dense subset.

Recall: K int-body of L iff $\rho_K(\theta) = \mathrm{Vol}_{n-1}\left(L \cap \theta^\perp\right) \quad \forall \theta \in S^{n-1}$.

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Bourgain & Zhang (1998), Koldobsky (2000):
negative for $1 \leq k \leq n-4$.
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open for $n \geq 5, k = n-3, n-2$; *low-dim BP problem*.

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Low-Dimensional BP Problem

Denote $d = n - k$. Low-Dim BP Problem: $n \geq 5$, $d = 2, 3$.

$$\text{Assume} \quad \forall E \in G^n_d \quad \text{Vol}_d(K \cap E) \leq \text{Vol}_d(L \cap E)\,.$$

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Known positive answers:

- Zhang (96): $d = 2, 3$; any L; K convex body of revolution, i.e. invariant under $O(n-1)$.
- Generalized by Rubin (07) to K with more general axial symmetries - invariant under $O(m) \times O(n-m)$.
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$\mathcal{BP}_k^n$ - first generalization of $\mathcal{I}^n$ by Zhang

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$$R_m : C_e(S^{n-1}) \to C(G_m^n) \quad R_m(f)(E) = \int_{S^{n-1}\cap E} f(\xi) d\sigma_E(\xi)$$

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More concretely:

$$R_m^*(g)(\theta) = \int_{\theta \in E \in G_m^n} g(E) dE.$$

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Remark

R_m is injective but its image is **not dense** in $C(G_m^n)$ for $1 < m < n-1$, so Ker $R_m^* \neq 0$ in this range.

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- If $K \in \mathcal{BP}_k^n$ then **positive** answer to k-generalized BP-problem in $\mathbb{R}^n$ for any star-body L.
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Proof of Positive Part

We are given $\mathrm{Vol}_{n-k}(K \cap E) \leq \mathrm{Vol}_{n-k}(L \cap E)$ $\forall E \in G^n_{n-k}$ and:

$$K \in \mathcal{BP}^n_k \iff \rho^k_K = R^*_{n-k}(d\mu) \quad \mu \in \mathcal{M}_+(G^n_{n-k})$$

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Recall:

$$K \text{ int-body of } L \iff \rho_K(\theta) = \mathrm{Vol}_{n-1}\left(L \cap \theta^\perp\right) \quad \forall \theta \in S^{n-1}$$

$$\iff \frac{1}{2}\mathrm{Vol}_1\left(K \cap E^\perp\right) = \mathrm{Vol}_{n-1}\left(L \cap E\right) \quad \forall E \in G^n_{n-1}$$

Definition of $\mathcal{I}_k^n$ (Koldobsky)

K k-int-body of L $\iff$

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$\mathcal{I}_k^n$ - second generalization of $\mathcal{I}^n$ by Koldobsky

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- $\mathcal{I}_k^n$ played important role in unified solution to BP-problem (Gardner Koldobsky Schlumprecht 99).
- In some sense an extension of L_p^n to L_{-k}^n (Koldobsky).
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Given $f \in C(S^{n-1})$, $p > -n$, denote the locally integrable:

$$E_p(f)(r\theta) = f(\theta)r^p \quad r > 0, \theta \in S^{n-1},$$

$$E_p^\wedge(f) = \text{Fourier Transform of } E_p(f) \text{ as distribution,}$$

i.e. for any test function ϕ:

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Facts:

- $E_p^\wedge(f)$ is homogeneous distribution of degree $-n-p$.
- In general, $E_p^\wedge(f)$ is not a locally integrable function nor even a measure on $\mathbb{R}^n$.
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Some Properties of the Fourier Transform

Thm (Koldobsky)

- Parseval Formula: for any nice f, g, $0 < q < n$,

$$\int_{S^{n-1}} E^{\wedge}_{-q}(f)(\theta) g(\theta) d\sigma(\theta) = \int_{S^{n-1}} f(\theta) E^{\wedge}_{-q}(g)(\theta) d\sigma(\theta),$$

so $E^{\wedge}_{-q} = (E^{\wedge}_{-q})^*$ is “self-adjoint".

- Integration on Perpendicular subspaces: For any nice f,

$$R_k(f)(H^{\perp}) = d_{n,k} R_{n-k}(E^{\wedge}_{-k}(f))(H) \;\; \forall H \in G^n_{n-k},$$

so:

$$I \circ R_k = d_{n,k} R_{n-k} \circ E^{\wedge}_{-k}$$

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Connection of $\mathcal{I}_k^n$ to Fourier Transforms

Thm (Koldobsky)

$$K \in \mathcal{I}_k^n \iff (\|\cdot\|_K^{-k})^\wedge \geq 0$$

Remark: R.H.S. makes sense for non-integer $0 < k < n$.

Idea of Proof:

$$K \in \mathcal{I}_k^n \;\; \text{“}\!\iff\!\text{”} \;\; \mathrm{Vol}_k\left(K \cap E^\perp\right) = \mathrm{Vol}_{n-k}\left(L \cap E\right) \;\; \forall E \in G_{n-k}^n$$

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Relationship between $\mathcal{BP}^n_k, \mathcal{I}^n_k$

Two generalizations of $\mathcal{I}^n$:

- $K \in \mathcal{BP}^n_k \iff \|\cdot\|_K^{-k} = \rho_K^k = R^*_{n-k}(d\mu) \quad \mu \in \mathcal{M}_+(G^n_{n-k})$
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Thm (Koldobsky 00): $\mathcal{BP}^n_k \subset \mathcal{I}^n_k$.

Proof (M. 05):

$$I \circ R_k = d_{n,k} R_{n-k} \circ E^\wedge_{-k}$$

Dualizing:

$$(I \circ R_k)^* = d_{n,k} (E^\wedge_{-k})^* \circ R^*_{n-k} = d_{n,k} E^\wedge_{-k} \circ R^*_{n-k}$$

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$$I \circ R_k = d_{n,k} R_{n-k} \circ E_{-k}^\wedge$$

Dualizing:

$$(I \circ R_k)^* = d_{n,k}(E_{-k}^\wedge)^* \circ R_{n-k}^* = d_{n,k} E_{-k}^\wedge \circ R_{n-k}^*$$

So if $K \in \mathcal{BP}_k^n$, i.e. $\|\cdot\|_K^{-k} = R_{n-k}^*(d\mu)$, $\mu \geq 0$,

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hence $K \in \mathcal{I}_k^n$.

Two generalizations of $\mathcal{I}^n$:

- $K \in \mathcal{BP}_k^n \iff \|\cdot\|_K^{-k} = \rho_K^k = R_{n-k}^*(d\mu) \quad \mu \in \mathcal{M}_+(G_{n-k}^n)$
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Relationship between $\mathcal{BP}_k^n, \mathcal{I}_k^n$

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Convex Bodies and $\mathcal{I}_k^n$

Thm (Gardner-Koldobsky-Schlumprecht 99)

{Convex symmetric bodies in $\mathbb{R}^n$} $\subset \mathcal{I}_p^n$ iff $n-3 \leq p < n$.

Cor (Bourgain-Zhang 98, Koldobsky 00)

Negative answer to k-Generalized BP problem for $k < n-3$, in particular ($k = 1$) negative answer to BP problem for $n > 4$.

Proof: By GKS there exists a convex symmetric $K \notin \mathcal{I}_k^n$, and since $\mathcal{BP}_k^n \subset \mathcal{I}_k^n$, $K \notin \mathcal{BP}_k^n$, and conclude by Zhang's Thm.

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$$(\|\cdot\|_K^{-n+1+q})^\wedge(\xi) = \frac{A_{K,\xi}^{(q)}(0)}{a_{n,q}}$$

$A_{K,\xi} = \mathrm{Vol}_{n-1}\left(K \cap \{t\xi + \xi^\perp\}\right)$, $A_{K,\xi}^{(q)}$ is its fractional derivative.

- When $q \leq 2$, this depends only on the usual first two derivatives of $A_{K,\xi}$.
- When K is (symmetric) convex, by Brunn's Thm $A_{K,\xi}$ is (even) concave function, so $A_{K,\xi}^{(2)}(0) \leq 0$, $A_{K,\xi}^{(1)}(0) = 0$, $A_{K,\xi}^{(0)}(0) \geq 0$, hence $(\|\cdot\|_K^{-p})^\wedge \geq 0$ for $p \geq n-3$.
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Equivalence of $\mathcal{BP}_k^n$ and $\mathcal{I}_k^n$

- Koldobsky 00: Question - $\mathcal{BP}_k^n = \mathcal{I}_k^n$?
  Positive answer would imply positive answer to Low-Dim BP problem ($n - k = 2, 3$) (but not conversely!)

  Reason:
  GKS: {Convex symmetric bodies in $\mathbb{R}^n$} $\subset \mathcal{I}_k^n$ iff $k \geq n - 3$.
  if $\mathcal{BP}_k^n = \mathcal{I}_k^n$ for $k = n - 3, n - 2$:
  {Convex symmetric bodies in $\mathbb{R}^n$} $\subset \mathcal{BP}_{n-3}^n, \mathcal{BP}_{n-2}^n$.
  Zhang: implies positive answer to Low-Dim BP problem.

Conclusion

$\mathcal{BP}_k^n = \mathcal{I}_k^n$ is an interesting question, with potential Geometric consequences.

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Plan for the rest of the talk

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- Motivation why $\mathcal{BP}_k^n = \mathcal{I}_k^n$ (already know $\mathcal{BP}_k^n \subset \mathcal{I}_k^n$).
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Identical Structures of $\mathcal{BP}^n_k, \mathcal{I}^n_k$

Th. (M. 05): for $\mathcal{C} = \mathcal{BP}, \mathcal{I}$ (using different methods):

1. $\mathcal{C}^n_k$ closed under full-rank linear transformations, k-radial sums ($\rho^k_L = \rho^k_{K_1} + \rho^k_{K_2}$), limit in radial metric.
2. $\mathcal{C}^n_1 = \mathcal{I}^n$, $\mathcal{C}^n_{n-1} = \{$symmetric star-bodies in $\mathbb{R}^n\}$.
3. Let $K_1 \in \mathcal{C}^n_{k_1}$, $K_2 \in \mathcal{C}^n_{k_2}$ and $l = k_1 + k_2 \leq n - 1$. If $\rho^l_L = \rho^{k_1}_{K_1} \rho^{k_2}_{K_2}$ then $L \in \mathcal{C}^n_l$. As corollaries:
   1. $\mathcal{C}^n_{k_1} \cap \mathcal{C}^n_{k_2} \subset \mathcal{C}^n_{k_1+k_2}$ if $k_1 + k_2 \leq n - 1$.
   2. $\mathcal{C}^n_k \subset \mathcal{C}^n_l$ if k divides l (*open: $k < l$?*)
   3. If $K \in \mathcal{C}^n_k$ and $\rho_L = \rho^{k/l}_K$ then $L \in \mathcal{C}^n_l$ for $l \geq k$.
4. If $K \in \mathcal{C}^n_k$ then $K \cap E \in \mathcal{C}^m_k$ for $E \in G^n_m$ and $m > k$.

(1) and (2) well-known and basically follow from defs.
For $\mathcal{C} = \mathcal{I}$, (3) independently noticed by Koldobsky.
For $\mathcal{C} = \mathcal{BP}$, (4) and (3-2) for $k = 1$ were proved by Grinberg and Zhang.

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In fact, we construct C^∞ *body of revolution* in $\mathcal{I}_k^n \setminus \mathcal{BP}_k^n$.

Proof relies on:

Th. (M. 05): The following are equivalent:

1.
$$\mathcal{BP}_k^n \subsetneq \mathcal{I}_k^n$$

2. $\exists g \in C^\infty(G_{n-k}^n)$, $R_{n-k}^*(g) \geq 1$ and $(I \circ R_k)^*(g) \geq 1$, but g is not non-negative functional on $R_{n-k}(C(G_{n-k}^n))$:
$\exists h \in R_{n-k}(C(G_{n-k}^n))_+$ s.t. $\int_{G_{n-k}^n} g(E)h(E)d\eta(E) < 0$.
(where $I : C(G_k^n) \to C(G_{n-k}^n)$ $\quad I(f)(E) = f(E^\perp)$)

<u>Idea of Proof:</u> construct $g \in C^\infty(G_{n-k}^n)$ in (2) invariant under natural action of $O(n-1) < O(n)$, by analyzing the action of R_{n-k} and R_{n-k}^* on functions of revolution.

3. Dual statement (non-constructive!)

$$\overline{R_{n-k}(C(S^{n-1}))_+} \supsetneq \overline{R_{n-k}(C_+(S^{n-1})) + I \circ R_k(C_+(S^{n-1}))}.$$

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Additional formulations using Fourier Transforms

Thm (Grinberg-Zhang 99, generalizing $k = 1$ Goodey-Weil 95)

$K \in \mathcal{BP}_k^n$ iff K can be approximated in radial-metric by K_i: $\rho_{K_i}^k = \rho_{\mathcal{E}_{i,1}}^k + \ldots + \rho_{\mathcal{E}_{i,m_i}}^k$, where $\mathcal{E}_{i,j}$ are ellipsoids.

Equivalently

$\mathcal{BP}_k^n$ is the smallest family (containing the Euclidean Ball) which is closed under full-rank linear transformations, k-radial sums ($\rho_L^k = \rho_{K_1}^k + \rho_{K_2}^k$), limit in radial metric.

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Equivalent formulation to $\mathcal{BP}^n_k = \mathcal{I}^n_k$:

If $f \in C^\infty_e(S^{n-1})$ satisfies $f \geq 0$ and $E^\wedge_{-k}(f) \geq 0$, is f the limit of $\sum_{i=1}^m T_i(E_{-k}(1))$, $T_i \in PD(n)$? *(write $f = \rho^k_K = \|\cdot\|^{-k}_K$)*

Answer: No, for $n \geq 4$ and $2 \leq k \leq n-2$.
(Yes for $k = 1$ (Goodey & Weil) and $k = n-1$).

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$K \in \mathcal{BP}^n_k$ iff K can be approximated in radial-metric by K_i: $\rho^k_{K_i} = \rho^k_{\mathcal{E}_{i,1}} + \ldots + \rho^k_{\mathcal{E}_{i,m_i}}$, where $\mathcal{E}_{i,j}$ are ellipsoids.

- Note: $E_{-k}(1) = \|\cdot\|^{-k}_{D_n}$, $T \in PD(n)$ $T(E_{-k}(1)) = \|\cdot\|^{-k}_{\mathcal{E}_T}$.
- For $1 \leq k \leq n-1$, $E^{\wedge}_{-k}(1) = b_{n,k} E_{-n+k}(1) \geq 0$.
- If $T \in PD(n)$, $T(E_{-k}(1))^{\wedge} = det(T) T^{-1}(E^{\wedge}_{-k}(1)) \geq 0$.

Equivalent formulation to $\mathcal{BP}^n_k = \mathcal{I}^n_k$:

If $f \in C^{\infty}_e(S^{n-1})$ satisfies $f \geq 0$ and $E^{\wedge}_{-k}(f) \geq 0$, is f the limit of $\sum_{i=1}^m T_i(E_{-k}(1))$, $T_i \in PD(n)$? *(write $f = \rho^k_K = \|\cdot\|^{-k}_K$)*

Answer: No, for $n \geq 4$ and $2 \leq k \leq n-2$.
(Yes for $k = 1$ (Goodey & Weil) and $k = n-1$).

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