

Vector bundles on contractible smooth schemes

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Abstract

We discuss algebraic vector bundles on smooth k -schemes X contractible from the standpoint of \mathbb{A}^1 -homotopy theory; when $k = \mathbb{C}$, the smooth manifolds $X(\mathbb{C})$ are contractible as topological spaces. The integral algebraic K-theory and integral motivic cohomology of such schemes are that of $\text{Spec } k$. One might hope that furthermore, and in analogy with the classification of topological vector bundles on manifolds, algebraic vector bundles on such schemes are all isomorphic to trivial bundles; this is almost certainly true when the scheme is affine. However, in the non-affine case this is false: we show that (essentially) every smooth \mathbb{A}^1 -contractible strictly quasi-affine scheme that admits a U -torsor whose total space is affine, for U a unipotent group, possesses a non-trivial vector bundle. Indeed we produce explicit arbitrary dimensional families of non-isomorphic such schemes, with each scheme in the family equipped with “as many” (i.e., arbitrary dimensional moduli of) non-isomorphic vector bundles, of every sufficiently large rank n , as one desires; neither the schemes nor the vector bundles on them are distinguishable by algebraic K-theory. We also discuss the triviality of vector bundles for certain smooth complex affine varieties whose underlying complex manifolds are contractible, but that are not necessarily \mathbb{A}^1 -contractible.

1 Introduction

In this note, we study the set of isomorphism classes of vector bundles on smooth k -schemes that are contractible in the sense of \mathbb{A}^1 -homotopy theory (as introduced in [MV99]); such schemes will be called \mathbb{A}^1 -contractible. We wish to stress three counter-intuitive points. First, as our main results show, there are lots of these, both of the schemes and of the bundles on a typical fixed such scheme (see Theorem 1.2, and Corollaries 3.1 and 3.4). Second, they arise quite naturally and explicitly, so should not be considered pathological. Third, the standard cohomology theories (at least those theories representable on the \mathbb{A}^1 -homotopy category) are completely insensitive to these structures, and so are missing a surprising amount of algebro-geometric data.

Regarding the third point let us be more specific right from the start. Since motivic cohomology is representable in the \mathbb{A}^1 -homotopy category (see [Voe01] Theorem 2.3.1¹), \mathbb{A}^1 -contractible

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¹The proof of this fact requires, at the moment, that k be a perfect field.

schemes have the motivic cohomology of $\mathrm{Spec} k$ and so, for instance, have no non-trivial algebraic cycles. Similarly, and more importantly for our present purposes, since algebraic K-theory is representable in the \mathbb{A}^1 -homotopy category (see [MV99] §4 Theorem 3.13), one knows that the algebraic K-theory of any \mathbb{A}^1 -contractible smooth k -scheme is isomorphic to that of $\mathrm{Spec} k$; already from $K_0(\mathrm{Spec} k) \cong \mathbb{Z}$ this implies that all vector bundles are stably trivial.

Given an \mathbb{A}^1 -contractible smooth scheme X , it is therefore natural to ask whether all the vector bundles on X are in fact trivial, especially given that topological vector bundles on open contractible manifolds are trivial. Indeed, recalling the Quillen-Suslin theorem for affine space (itself the prototypical smooth \mathbb{A}^1 -contractible scheme, and the only one known before [AD07]), one may view this as a generalized Serre problem. We show there is a stark dichotomy between the affine and strictly quasi-affine cases: in the affine case, the answer seems to be yes, whereas in the quasi-affine case we prove that the answer is a resounding no and construct explicit counterexamples in abundance. What is especially interesting is that none of the standard means for distinguishing vector bundles on a scheme (e.g., Chern classes, algebraic K-theory, algebraic cycles) can play any role at all; from their standpoint, all the bundles are indistinguishable from a trivial bundle.

On the other hand, it follows from Corollary 4.2 that there are moduli of strictly quasi-affine surfaces which are not \mathbb{A}^1 -contractible (nor are the complex surfaces even contractible in the sense of manifolds) and yet they admit only trivial vector bundles. Thus having non-trivial vector bundles is by no means a necessary feature of being strictly quasi-affine.

Representability properties of the functor “isomorphism classes of vector bundles”

We put the above discussion in a broader context. Let $\mathcal{S}m_k$ denote the category of separated, finite type, smooth schemes defined over k . The \mathbb{A}^1 -homotopy category is constructed by embedding the category $\mathcal{S}m_k$ in a larger category of *spaces*, equipping that category with the structure of a model category and then forming the associated homotopy category. The category of spaces is taken to be the category of simplicial Nisnevich sheaves on $\mathcal{S}m_k$. The homotopy category can be formed by localizing along two classes of morphisms: first, along the *simplicial weak equivalences* and second along the \mathbb{A}^1 -*weak equivalences*. We refer the reader to ([MV99] §2 Theorem 3.2) for precise details regarding this construction. Here and through the remainder of the paper $[\cdot, \cdot]_s$ and $[\cdot, \cdot]_{\mathbb{A}^1}$ will denote the set of simplicial homotopy classes of maps and \mathbb{A}^1 -homotopy classes of maps between spaces.

Let $\mathcal{V}(\cdot)$ (resp. $\mathcal{V}_n(\cdot)$) denote the functor that assigns to an object $X \in \mathcal{S}m_k$ the set of isomorphism classes of (rank n) locally free sheaves on X . Let BGL_n denote the usual simplicial classifying space defined in [MV99] §4.1, then by *ibid.* §4 Proposition 1.16, one knows that the set of simplicial homotopy classes of maps $[X, BGL_n]_s$ can be identified with $H_{Nis}^1(X, GL_n)$. Using a version of “Hilbert’s Theorem 90” (i.e., that GL_n is a “special group” in the sense of Serre), we know that the last group is isomorphic to $H_{Zar}^1(X, GL_n)$ which is, essentially by construction, isomorphic to $\mathcal{V}_n(X)$. Ideally, one hopes that $\mathcal{V}_n(X)$ descends to a functor on the \mathbb{A}^1 -homotopy category and is representable by the space BGL_n , i.e., that $[X, BGL_n]_{\mathbb{A}^1} = \mathcal{V}_n(X)$.

Positive results

In the case $n = 1$ this ideal scenario is the reality, without restriction on X . Recall that a smooth scheme X is called \mathbb{A}^1 -rigid (see [MV99] §3 Example 2.4), if for any smooth scheme U , the map $\text{Hom}_{\mathcal{S}_{m_k}}(U, X) \rightarrow \text{Hom}_{\mathcal{S}_{m_k}}(U \times \mathbb{A}^1, X)$ induced by pullback along the projection $U \times \mathbb{A}^1 \rightarrow U$ is a bijection. Morel and Voevodsky show that, since the sheaf $\mathbb{G}_{\mathbf{m}}$ is \mathbb{A}^1 -rigid, $B\mathbb{G}_{\mathbf{m}} = BGL_1$ is in fact \mathbb{A}^1 -local (see *ibid.* §3 Definition 2.1). Thus, $[X, B\mathbb{G}_{\mathbf{m}}]_{\mathbb{A}^1} = [X, B\mathbb{G}_{\mathbf{m}}]_s$ and one concludes $\mathcal{V}_1(X) = H_{Zar}^1(X, \mathbb{G}_{\mathbf{m}}) = [X, B\mathbb{G}_{\mathbf{m}}]_{\mathbb{A}^1}$ (see *ibid.* §4 Proposition 3.8).

Furthermore, Morel argues (see [Mor] Theorem 3) that if one restricts $\mathcal{V}_n(\cdot)$ to a functor on the category of smooth *affine* schemes then again this ideal is realized, at least if $n \neq 2$ (it is expected that $n = 2$ works as well, but the details remain to be written out). Consequently, if X is an affine \mathbb{A}^1 -contractible smooth k -scheme, then every vector bundle on X (of rank $n \neq 2$) is isomorphic to a trivial bundle.

Negative results

Unfortunately, for $n \geq 2$, the functors $\mathcal{V}_n(X)$ cannot descend to functors on the homotopy category without a restriction on X : it has long been known (and was pointed out to us by Morel) that even with $X = \mathbb{P}^1$, the canonical map $\mathcal{V}(\mathbb{P}^1) \rightarrow \mathcal{V}(\mathbb{P}^1 \times \mathbb{A}^1)$ induced by pull-back via the projection morphism is *not* a bijection, as we discuss in §2. Observe that this means the space BGL_n is *not* \mathbb{A}^1 -local for $n > 1$ (*cf.*, [MV99] p. 138). Indeed, one can show that \mathbb{A}^1 -locality of BGL_n is equivalent to the assertion that, for any smooth scheme X , and $i = 0, 1$, the canonical map $H_{Nis}^i(X, GL_n) \rightarrow H_{Nis}^i(X \times \mathbb{A}^1, GL_n)$ is a bijection (combine Proposition 1.16 of [MV99] §4 and [Mor04] Lemma 3.2.1); Morel has called this latter cohomological condition on a group “strong \mathbb{A}^1 -invariance.”

Nevertheless, Morel’s results might make one hope that some form of homotopy invariance holds for the functor $\mathcal{V}_n(X)$ for general n beyond the affine case. For instance, perhaps any \mathbb{A}^1 -weak equivalence of smooth schemes $f : X \rightarrow Y$ where X is affine would induce a bijection $f^* : \mathcal{V}(Y) \rightarrow \mathcal{V}(X)$; when, in addition, Y is affine this is true by the discussion above.

Remark 1.1. Indeed, by the Jouanolou-Thomason homotopy lemma (see e.g., [Wei89] Proposition 4.4), given any smooth scheme Y admitting an ample family of line bundles (e.g., a quasi-projective variety), there exists a smooth affine scheme X and a Zariski locally trivial smooth morphism with fibers isomorphic to affine spaces $f : X \rightarrow Y$. In particular, this morphism is an \mathbb{A}^1 -weak equivalence, so the above naïve hope would reduce the study of vector bundles on such schemes to the case of affine varieties! Unfortunately, Theorem 1.2 shows that this is false.

Nevertheless, Morel’s results combined with the Jouanolou-Thomason homotopy lemma give the following general picture. Suppose Y is a smooth scheme over a perfect field k (according to our conventions, this means Y is separated, regular and Noetherian and thus admits an ample family of line bundles). As long as n is a strictly positive integer $\neq 2$, then for any smooth affine scheme X that is \mathbb{A}^1 -weakly equivalent to Y , we have a bijection $[Y, BGL_n]_{\mathbb{A}^1} \cong \mathcal{V}_n(X)$.

Alternatively, homotopy invariance might hold for a slightly broader class of varieties than affine ones, say for quasi-affine schemes with “nice enough” affine closures. In [AD07], using

techniques for studying unipotent group actions developed in [DK07], we constructed many examples in characteristic 0 of non-isomorphic strictly quasi-affine \mathbb{A}^1 -contractible smooth schemes. Using this construction for arbitrary k , we will see that neither of the above generalizations are possible; it seems Morel's results are in fact the strongest one can expect. Furthermore, we will see that any attempt to quantify the lack of homotopy invariance must account for arbitrarily many non-isomorphic vector bundles. Specifically, our main goal in this paper is to prove the following result.

Theorem 1.2. *Let k be a field. Suppose X is a finite type, smooth, affine \mathbb{A}^1 -contractible k -scheme equipped with a free everywhere stable action of a split connected unipotent group U .*

- i) The quotient X/U exists as a smooth \mathbb{A}^1 -contractible quasi-affine scheme.*
- ii) If X/U is affine, then for every positive integer n , the pull-back map $\mathcal{V}_n(X/U) \rightarrow \mathcal{V}_n(X)$ is a bijection.*
- iii) If X is isomorphic to affine space and X/U is affine, then every vector bundle on X/U is isomorphic to a trivial bundle.*
- iv) If X/U is not affine, but admits a smooth quasi-affine closure with at least one codimension ≥ 2 boundary component, then X/U admits non-trivial vector bundles of rank m for all sufficiently large m .*

Remark 1.3. One might suspect that any quasi-affine scheme that is not affine has non-trivial vector bundles, but this is false in general. Indeed, one can show that all vector bundles on the complement of finitely many points in \mathbb{A}^2 are trivial. (We will generalize this fact in Corollary 4.2.)

Remark 1.4. The boundary component condition in (iv) above is imposed for ease of proof; it almost certainly can be removed, and is satisfied for instance when $\mathrm{Spec} k[X]^U$ is smooth, which is true in the generic case of our construction. We have asked (see [AD07, Aso]), in analogy with the structure theory of contractible manifolds, whether any smooth \mathbb{A}^1 -contractible variety can be realized as such a quotient of affine space by the free action of a unipotent group. A positive solution to this question would have the following consequence: removing the boundary component condition in Theorem 1.2 implies the interesting dichotomy that *all* non-affine smooth \mathbb{A}^1 -contractible varieties have a non-trivial vector bundle, whereas *all* affine ones would only have trivial vector bundles.

In §3, we will expand on this theorem by placing a lower bound on “how many” non-trivial vector bundles such a quasi-affine \mathbb{A}^1 -contractible variety can have, and thereby construct large dimensional families of examples with as many non-trivial vector bundles as one likes, all indistinguishable from the trivial bundle from the point of view of algebraic K-theory (or of any invariant representable in the \mathbb{A}^1 -homotopy category).

Contractible complex affine algebraic varieties

In addition, we will visit the generalized Serre problem as discussed in [Zař99] §8. We will say that a scheme X over \mathbb{C} is topologically contractible if $X(\mathbb{C})$ equipped with its usual structure of a complex manifold is contractible as a topological space. The generalized Serre problem asks: if X is a smooth complex affine algebraic variety which is topologically contractible, then are all algebraic vector bundles on X isomorphic to trivial bundles? Note that if X is an \mathbb{A}^1 -contractible smooth scheme over \mathbb{C} , then X is necessarily topologically contractible (see [AD07] Lemma 2.5). However, not all topologically contractible complex varieties are \mathbb{A}^1 -contractible (see [Aso]); for example any topologically contractible smooth complex surface of log-general type is *not* \mathbb{A}^1 -contractible (in fact such surfaces can be shown to be \mathbb{A}^1 -rigid). We will observe in §4, putting together results of several authors, that the generalized Serre problem is true for all topologically contractible smooth complex varieties of dimension ≤ 2 ; consequently there are positive dimensional moduli of smooth surfaces that each admit only trivial vector bundles. Finally, we will present some examples of topologically contractible smooth complex 3-folds all of whose vector bundles are isomorphic to trivial bundles.

Conventions and Definitions

The word “scheme” will mean separated scheme, locally of finite type over a field k . The word “variety” will mean reduced, finite type scheme. A scheme X is called \mathbb{A}^1 -contractible if the canonical morphism $X \rightarrow \text{Spec } k$ is an \mathbb{A}^1 -weak equivalence in the sense of [MV99] §3 Definition 2.1. A scheme X is called quasi-affine if there exists an affine scheme \bar{X} and an open immersion $X \hookrightarrow \bar{X}$; we will refer to quasi-affine schemes that are not affine as *strictly quasi-affine* schemes. If X is any scheme, we let $\text{Vec}(X)$ denote the category of finite rank locally free \mathcal{O}_X -modules and, as above, $\mathcal{V}(X)$ will denote the set of isomorphism classes of vector bundles on X .

Throughout, U will denote a split *connected* unipotent k -group. Splitness of U implies that U admits an increasing filtration by normal subgroups with sub-quotients isomorphic to \mathbb{G}_a , and in particular that U is isomorphic to affine space as a k -scheme. Observe that if $\text{char}(k) > 0$, then split unipotent groups *can* have non-trivial finite subgroups (e.g., the kernel of the Artin-Schreier morphism $\mathbb{G}_a \rightarrow \mathbb{G}_a$).

Actions of groups on schemes are always assumed to be left actions; actions will be called free if they are scheme-theoretically free, i.e., the action morphism is a closed immersion. If X is a scheme equipped with an action of U , then X/U will denote the geometric quotient of U by X , if it exists as a scheme.

A U -torsor over a scheme X will be a triple (\mathcal{P}, π, U) consisting of a faithfully flat, finite presentation morphism $\pi : \mathcal{P} \rightarrow X$, from a left U -scheme \mathcal{P} , such that the canonical morphism $U \times \mathcal{P} \rightarrow \mathcal{P} \times \mathcal{P}$ is an isomorphism onto $\mathcal{P} \times_X \mathcal{P}$. Observe that in this situation, U acts freely on \mathcal{P} (see [MFK94] Lemma 0.6) and X is a geometric quotient of \mathcal{P} by U .

Our notation and terminology will follow [AD07] unless otherwise mentioned. However, the reader need not be familiar with the results of *ibid.*, as long as she takes on faith Theorems 3.10 and 4.11 therein: in essence, (a) there is a (computable) notion of an everywhere stable U -action on an affine scheme X , (b) it is equivalent to X being endowed with the structure of a U -torsor

over a quasi-affine scheme X/U , and (c) in certain circumstances we can explicitly identify the complement of the open immersion of X/U in $\text{Spec } k[X/U]$ using geometric invariant theory.

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2 Vector bundles and U -torsors

If $q : X \rightarrow X/U$ is a U -torsor, we observe the induced map $q^* : \mathcal{V}(X/U) \rightarrow \mathcal{V}(X)$ can have very different character depending on whether X/U is affine or non-affine. The affine and strictly quasi-affine cases will be used to prove Theorem 1.2.

The affine case: q^* is a bijection

Lemma 2.1. *Suppose $q : X \rightarrow X/U$ is a U -torsor with X/U a smooth, affine scheme. Then q induces a bijection*

$$q^* : \mathcal{V}(X/U) \xrightarrow{\sim} \mathcal{V}(X).$$

If in addition X is isomorphic to affine space, then every vector bundle on X/U is isomorphic to a trivial bundle.

Proof. Lindel proved (see [Lin82]) that if Y is a smooth affine k -scheme then pullback via the projection map $Y \times \mathbb{A}^n \rightarrow Y$ induces a bijection $\mathcal{V}(Y) \xrightarrow{\sim} \mathcal{V}(Y \times \mathbb{A}^n)$.

According to the hypotheses, $q : X \rightarrow X/U$ equips the triple (X, q, U) with the structure of a U -torsor over X/U and X/U is affine. Observe that for any affine scheme Y , $H^1(Y, \mathbb{G}_a) = H^1(Y, \mathcal{O}_Y) = 0$ by [Gro61] Théorème 1.3.1. As U is split, an inductive argument shows that $H^1(Y, U) = 0$ for any such Y . Thus (X, q, U) must be a trivial U -torsor over X/U , whence $X \cong U \times X/U$. Thus, the first result follows from the discussion of the previous paragraph.²

The final statement, where X is assumed to be affine space, now follows from the Quillen-Suslin theorem (see e.g., [Qui76]) that all vector bundles on affine space are isomorphic to trivial bundles. \square

A quasi-projective counterexample: failure of surjectivity

Consider the projection morphism $p_1 : \mathbb{P}^1 \times \mathbb{A}^1 \rightarrow \mathbb{P}^1$. We will show that the pull-back map $p_1^* : \mathcal{V}(\mathbb{P}^1) \rightarrow \mathcal{V}(\mathbb{P}^1 \times \mathbb{A}^1)$ is not a bijection. By Grothendieck's description of the category

²See the proof of Corollary 3.2 in the Appendix to [AD07] for details (note that because of the splitness assumption, there is no restriction on the base field).

of vector bundles on \mathbb{P}^1 , we know that every locally free sheaf on \mathbb{P}^1 is isomorphic to a direct sum of rank 1 locally free sheaves. A vector bundle on $\mathbb{P}^1 \times \mathbb{A}^1$ isomorphic to a pull-back of a vector bundle \mathcal{F} on \mathbb{P}^1 is necessarily isomorphic to the (external) tensor product of \mathcal{F} and $\mathcal{O}_{\mathbb{A}^1}$. There is a rank 2 vector bundle \mathcal{E} on $\mathbb{P}^1 \times \mathbb{A}^1$ whose restriction to $\mathbb{P}^1 \times \{0\}$ is trivial and whose restriction to $\mathbb{P}^1 \times \{1\}$ is isomorphic to $\mathcal{O}(1) \oplus \mathcal{O}(-1)$. This means \mathcal{E} is not isomorphic to the pull-back of any bundle on \mathbb{P}^1 .

The strictly quasi-affine case: failure of injectivity

Now assume $q : X \rightarrow X/U$ is a U -torsor with X/U *not* affine. The existence of \mathbb{A}^1 -contractible strictly quasi-affine X/U will be proved in §3. More generally, for the rest of this subsection, and in particular for the statements of Lemmas 2.2, 2.3 and 2.4, we assume we are in the following situation:

- i) X/U is an open dense subscheme of a finite type smooth scheme $\overline{X/U}$, with the inclusion denoted $j : X/U \hookrightarrow \overline{X/U}$,
- ii) we denote by Z the closed complement of X/U in $\overline{X/U}$ equipped with the reduced induced scheme structure and assume it is *non-empty*.

In this situation, we have a localization sequence in G -theory (see [Sri96] Proposition 5.15):

$$\cdots \rightarrow G_1(X/U) \rightarrow G_0(Z) \rightarrow G_0(\overline{X/U}) \rightarrow G_0(X/U) \rightarrow 0.$$

As both X/U and $\overline{X/U}$ are finite type smooth schemes, we know by Poincaré duality (see [Sri96] §5.6) that $G_i(X/U) \cong K_i(X/U)$ and $G_i(\overline{X/U}) \cong K_i(\overline{X/U})$. Since X/U is smooth and \mathbb{A}^1 -contractible, it follows that $K_i(\text{Spec } k) \rightarrow K_i(X/U)$ is an isomorphism.

Lemma 2.2. *If X/U is \mathbb{A}^1 -contractible, the localization sequence gives a short exact sequence*

$$0 \rightarrow G_0(Z) \rightarrow G_0(\overline{X/U}) \rightarrow \mathbb{Z} \rightarrow 0.$$

Proof. Since X/U is \mathbb{A}^1 -contractible, it follows that $G_1(X/U) \cong G_1(\text{Spec } k) \cong k^*$, $G_0(X/U) \cong G_0(\text{Spec } k) \cong \mathbb{Z}$. We just need to show that the boundary map $G_1(X/U) \rightarrow G_0(Z)$ is trivial, or equivalently, that the morphism $G_1(\overline{X/U}) \rightarrow G_1(X/U)$ is surjective. To see this, observe that each pair (\mathcal{V}, α) consisting of a vector bundle on $\overline{X/U}$ and an automorphism α of \mathcal{V} represents an element of $G_1(\overline{X/U})$. Now, the map $G_1(\overline{X/U}) \rightarrow G_1(X/U)$ is induced by restriction. Since $G_1(X/U) \cong k^*$, we can represent any class in this group by a pair consisting of a trivial bundle and an automorphism corresponding to multiplication by an element of k^* . Such a pair can be extended to give a class in $G_1(\overline{X/U})$. (This produces a splitting of the map $G_1(\overline{X/U}) \rightarrow G_1(X/U)$ by the canonical morphism $G_1(\text{Spec } k) \rightarrow G_1(\overline{X/U})$). \square

Lemma 2.3. *If X/U is a smooth \mathbb{A}^1 -contractible open dense subscheme of a finite type smooth scheme $\overline{X/U}$, then there exists a non-trivial vector bundle on $\overline{X/U}$.*

Proof. First, observe that $G_0(Z)$ is always non-trivial. Thus, using Lemma 2.2, the map $j^* : K_0(\overline{X/U}) \rightarrow K_0(X/U)$ always has a kernel. In particular, $K_0(\overline{X/U})$ has a generator which is not isomorphic to $[\mathcal{O}_{\overline{X/U}}]$. Choosing any vector bundle representing the isomorphism class of this non-trivial generator gives the result. \square

Now under the hypothesis that the boundary component is of codimension at least two, non-isomorphic bundles on $\overline{X/U}$ will restrict to non-isomorphic bundles on X/U . Indeed, this follows by the following “well-known” result about restrictions of vector bundles on normal varieties.

Lemma 2.4. *Assume that the complement of X/U in $\overline{X/U}$ is of codimension at least two. Then the restriction functor $j^* : \text{Vec}(\overline{X/U}) \rightarrow \text{Vec}(X/U)$ is fully-faithful. Furthermore, the Picard groups of X/U and $\overline{X/U}$ are isomorphic.*

Proof. Since X/U is dense in $\overline{X/U}$, the restriction functor is faithful (any morphism is uniquely determined by restriction to the generic point). To check that the functor is full, it suffices to show that given any pair of locally free sheaves \mathcal{V}_1 and \mathcal{V}_2 on $\overline{X/U}$, any morphism $\varphi|_{X/U} : \mathcal{V}_1|_{X/U} \rightarrow \mathcal{V}_2|_{X/U}$ extends to a morphism $\varphi : \mathcal{V}_1 \rightarrow \mathcal{V}_2$.

By assumption, X/U has complement of codimension ≥ 2 in $\overline{X/U}$, and $\overline{X/U}$ is smooth and hence normal. Observe that the canonical morphism $\mathcal{O}_{\overline{X/U}} \rightarrow j_*\mathcal{O}_{X/U}$ is an isomorphism (since regular functions on X/U extend to regular functions on $\overline{X/U}$ by normality). Given \mathcal{V}_i as above, we can choose an open cover U_i of $\overline{X/U}$ on which \mathcal{V}_i trivialize. Consider the induced open cover $X/U \cap U_i$ of X/U . Any morphism $\varphi_{X/U} : \mathcal{V}_1|_{X/U} \rightarrow \mathcal{V}_2|_{X/U}$ is specified by a matrix of regular functions on the $X/U \cap U_i$. By the extension property of regular functions mentioned above, this matrix extends uniquely to give a morphism $\varphi : \mathcal{V}_1|_{U_i} \rightarrow \mathcal{V}_2|_{U_i}$; furthermore, these morphisms glue to give the required extension.

Interpreting line bundles in terms of Čech cocycles, the extension property of regular functions on smooth schemes shows that line bundles on X/U extend to line bundles on $\overline{X/U}$. \square

Remark 2.5. Note that any quasi-affine variety admits a canonical open immersion into the spectrum of its ring of regular functions, which (by definition of a geometric quotient) here is isomorphic to $\text{Spec}(k[X]^U)$. Although this last scheme is affine by definition, it is well-known (Hilbert’s 14th Problem) that it need not be Noetherian, though it is known to be locally of finite type (whence our conventions). We established in [AD07] Corollary 3.18 (iii) that in fact the complement of X/U in $\text{Spec}(k[X]^U)$ consists of codimension ≥ 2 affine subschemes. In particular, whenever $k[X]^U$ is finitely generated, then X/U admits codimension ≥ 2 affine “closures”.

The smoothness hypothesis on the partial compactification we impose is for the technical convenience of identifying K-theory with G -theory (via Lemma 2.3), and almost certainly could be removed. For the general case, (when $\text{Spec } k[X]^U$ is neither smooth nor finitely generated), one would have to replace G -theory by Thomason’s K-theory (see [TT90]). We believe all the lemmas, with the exception of Lemma 2.3, go through in this setting: one needs a more subtle argument to extract a vector bundle from a class in Thomason K-theory. In any case, the above results hold for any \mathbb{A}^1 -contractible smooth variety Y that admits a smooth partial compactification \overline{Y} (we assume \overline{Y} is a variety) such that $\overline{Y} \setminus Y$ has codimension ≥ 2 in \overline{Y} .

Proof of Theorem 1.2

Proof of Theorem 1.2 (i). Suppose X is a smooth affine \mathbb{A}^1 -contractible scheme admitting a free everywhere stable action of a unipotent group U . Since unipotent groups in positive characteristic can have non-trivial finite subgroups, everywhere stability implies properness of the action of U on X but not necessarily that the action is free. Imposing this additional condition, the same proof as that of Theorem 3.10 of [AD07] shows that in this situation a quotient X/U exists as a quasi-affine smooth scheme (indeed, the proof of *loc. cit.* shows existence of such a quotient is equivalent to the action being free and everywhere stable).

Now, using §3 Example 2.3 of [MV99], together with fact that U is a special group (i.e., all U -torsors are Zariski locally trivial) we conclude that furthermore X/U is an \mathbb{A}^1 -contractible smooth scheme (see also [AD07] Key Lemma 3.3). \square

Proof of Theorem 1.2 (ii), (iii). Statements (ii) and (iii) follow immediately from Lemma 2.1. \square

Proof of Theorem 1.2 (iv). Combining Lemmas 2.2, 2.3, and 2.4, we obtain the required non-trivial vector bundle on X/U . We can refine this statement however. Note that $\text{Pic}(\overline{X/U})$ is necessarily trivial by Lemma 2.4, thus the non-trivial generator corresponds to a vector bundle of rank $m \geq 2$. Furthermore, the vector bundle representing the non-trivial class on $\overline{X/U}$ is not stably trivial either so taking direct sums with the trivial bundle produces non-trivial vector bundles on $\overline{X/U}$ of any rank $\geq m$. Restricting these bundles to X/U produces non-trivial vector bundles of that same rank. \square

Remark 2.6. Take $X = \mathbb{A}^n$. If a unipotent group U acts freely and everywhere stably on X with a strictly quasi-affine quotient \mathbb{A}^n/U satisfying the hypotheses Theorem 1.2, we know there is a non-trivial vector bundle on \mathbb{A}^n/U . The pull-back of this vector bundle to \mathbb{A}^n is necessarily trivial, thus we see that pull-back by the quotient morphism does not induce an injection on isomorphism classes of vector bundles, at least once we have shown such an example exists.

3 “Lower bounds” on the failure of \mathbb{A}^1 -invariance

We must now show that Theorem 1.2 describes a large class of schemes. We are not trying to present a classification, so we simply supply a class of examples that work over an arbitrary field.

The basic idea is to rewrite the problem of finding U -torsors with \mathbb{A}^n as total space by “linearizing”, i.e., by restricting from a linear U -representation W to a U -invariant subvariety isomorphic to \mathbb{A}^n . The simplest cases to consider are \mathbb{G}_a -equivariant closed immersions of \mathbb{A}^n as hypersurfaces in W , chosen so that \mathbb{A}^n inherits the structure of a \mathbb{G}_a -torsor from a larger \mathbb{G}_a -torsor – namely an appropriate open subscheme $W' \subset W$. More specifically, the geometric points of W' will be “stable” points of W with trivial isotropy; such a set can be explicitly identified with the help of a modified Hilbert-Mumford numerical criterion from geometric invariant theory (GIT). Furthermore, given such a \mathbb{G}_a -equivariant closed immersion, we can identify the “boundary” locus (i.e., complement) of $\mathbb{A}^n/\mathbb{G}_a$ in $\text{Spec}(k[\mathbb{A}^n]^{\mathbb{G}_a})$ again using geometric invariant theory. (For

more details on this point of view, we refer the reader to [AD07] and [DK07].) With some care one can thereby arrange that the conditions of Theorem 1.2 are satisfied.

Let V denote the standard 2-dimensional representation of SL_2 . By abuse of notation, we will write V instead of $\mathbb{A}(V)$, and furthermore, we choose coordinates u, v on V throughout and write 0 for the origin of V . We embed $\mathbb{G}_{\mathbf{a}} \hookrightarrow SL_2$ as the subgroup of *lower* triangular matrices. Recall that $SL_2/\mathbb{G}_{\mathbf{a}} \cong V \setminus 0$; thus V is an SL_2 -equivariant completion of $SL_2/\mathbb{G}_{\mathbf{a}}$, and we can identify the identity coset $[e]$ in $SL_2/\mathbb{G}_{\mathbf{a}}$ with $\{(0, 1)\} \in V$.

Via this embedding $\mathbb{G}_{\mathbf{a}} \hookrightarrow SL_2$, an arbitrary SL_2 -representation W can be considered as a $\mathbb{G}_{\mathbf{a}}$ -representation by restriction. Any given SL_2 -orbit in $V \times W$ is contained either in $0 \times W$ or the complement; if it is in the complement, then it restricts to a $\mathbb{G}_{\mathbf{a}}$ -orbit in $[e] \times W$. Similarly any SL_2 -invariant subscheme of $(V \setminus 0) \times W$ restricts to a $\mathbb{G}_{\mathbf{a}}$ -invariant subscheme of $[e] \times W$. We argued in [AD07] (Theorem 3.10, Lemma 4.5, and Theorem 4.11), assuming k was of characteristic 0, using faithfully flat descent and the functoriality for quasi-affine maps in GIT, that if the geometric points of an SL_2 -invariant subscheme Y in $(V \setminus 0) \times W$ are stable for the SL_2 -action on $V \times W$, then the corresponding $\mathbb{G}_{\mathbf{a}}$ -invariant subscheme X in W is a $\mathbb{G}_{\mathbf{a}}$ -torsor over a quasi-affine variety. Furthermore we showed the complement of $X/\mathbb{G}_{\mathbf{a}}$ in $\text{Spec}(k[X]^{\mathbb{G}_{\mathbf{a}}})$ is the GIT SL_2 -quotient of the boundary of the closure \bar{Y} of Y in $V \times W$ (i.e., the complement of the quotient is the quotient of the complement). Note that $\mathbb{G}_{\mathbf{a}}$ acts freely on X if and only if SL_2 acts freely on Y . As we explained in the proof of Theorem 1.2 (i), if furthermore Y was contained in the open subscheme of $V \times W$ where SL_2 acts freely, the same result holds in arbitrary characteristic.

Corollary 3.1. *For any integers $n \geq 4$, and any $m \geq 1$, there exists a strictly quasi-affine \mathbb{A}^1 -contractible smooth scheme of dimension n with at least m non-isomorphic, stably trivial, non-trivial vector bundles of every rank l for sufficiently large l .*

Proof. Let $W = V^{\oplus 3}$ with coordinates $\{w_1, \dots, w_6\}$. The Hilbert-Mumford numerical criterion applied to SL_2 -orbits in the SL_2 -representation $V \times W$ and then restricted to $\mathbb{G}_{\mathbf{a}}$ -orbits in $[e] \times W$, implies, as justified above, that all geometric points in the complement of the subscheme defined by $\{w_1 = 0, w_3 = 0, w_5 = 0\}$ are stable for the $\mathbb{G}_{\mathbf{a}}$ -action; in particular, $V \times W \setminus \{w_1 = 0, w_3 = 0, w_5 = 0\}/\mathbb{G}_{\mathbf{a}}$ is a quasi-affine geometric quotient.

For $f + 1$ a 1-variable polynomial with no repeated roots and constant term 1, consider a $\mathbb{G}_{\mathbf{a}}$ -invariant hypersurface X given by the $\mathbb{G}_{\mathbf{a}}$ -invariant equation $w_1 = 1 + f(w_3w_6 - w_4w_5)$. Observe first that by the preceding paragraph all geometric points of X are stable, since any non-stable point must satisfy $w_1 = w_3 = w_5 = 0$ which violates the hypersurface equation because $0 \neq 1$. It follows that the geometric quasi-affine quotient $X/\mathbb{G}_{\mathbf{a}}$ exists. Second, X is isomorphic to \mathbb{A}^5 via the closed immersion $w_2 = z_1, \dots, w_6 = z_5$, where $\{z_1, \dots, z_5\}$ are coordinates on \mathbb{A}^5 . Third, note that $f(w_3w_6 - w_4w_5)$ is SL_2 -invariant; it follows easily that the associated hypersurface equation defining \bar{Y} in $V \times W$ is $uw_2 - vw_1 = 1 + f(w_3w_6 - w_4w_5)$, where (u, v) are the coordinates on the first factor of V . Fourth, again by the Hilbert-Mumford criterion, all geometric points of \bar{Y} are stable with respect to the SL_2 -action on $V \times W$, so in particular the SL_2 action on \bar{Y} is proper, and the affine geometric quotient \bar{Y}/SL_2 exists.

The boundary $B = \bar{Y} \setminus Y$ of \bar{Y} , equivalently the closed subscheme of \bar{Y} defined by the simultaneous vanishing of u and v , is explicitly given by $f(w_3w_6 - w_4w_5) + 1 = 0$ in $0 \times W$; it is SL_2 -

invariant, codimension 2 in \overline{Y} , and its geometric points are all SL_2 -stable. As per the discussion preceding this Corollary, this means the complement of $X/\mathbb{G}_a = \mathbb{A}^5/\mathbb{G}_a$ in $\text{Spec}(k[\mathbb{A}^5]^{\mathbb{G}_a})$ is given by B/SL_2 , which is necessarily again codimension 2.

Observe that by the Jacobian criterion, the fact that $f+1$ has no repeated roots implies that both \overline{Y} and B are smooth schemes for any k . Also, B has m disjoint components, where m is the degree of f . Recall that an action is called set-theoretically free if its stabilizers at k -points are trivial. Furthermore, proper, set-theoretically free actions are free (*cf.* [AD07] Lemma 3.11). A direct computation shows the only geometric points having non-trivial stabilizers for the \mathbb{G}_a -action on W lie in the non-stable locus, so \mathbb{G}_a acts freely on X and hence SL_2 acts freely on Y . Also note that SL_2 acts freely on B in $0 \times W$; indeed B is a finite disjoint union of rank 2 vector bundles over SL_2 . Consequently SL_2 acts not only properly but also set-theoretically freely, hence freely, on all of \overline{Y} , so \overline{Y}/SL_2 is smooth. Denote the SL_2 -quotient of the boundary B by Z ; then Z is smooth and codimension 2 in $\overline{Y}/SL_2 \cong \text{Spec}(k[\mathbb{A}^5]^{\mathbb{G}_a})$.

Indeed, Z in these examples is isomorphic to a disjoint union of m copies of the affine plane. It follows that for any $m \geq 1$, we can choose f and hence X so that $K_0(Z)$ is isomorphic to $\mathbb{Z}^{\oplus m}$. Thus Lemma 2.2 shows that $K_0(\text{Spec } k[\mathbb{A}^5]^{\mathbb{G}_a}) \cong \mathbb{Z}^{\oplus m+1}$. As vector bundles representing different classes in K_0 are not stably equivalent, by taking direct sums with trivial bundles we get m non-trivial, non-isomorphic bundles in every sufficiently large rank. Then restriction, by Lemma 2.4, gives the desired bundles on $\mathbb{A}^5/\mathbb{G}_a$.

Higher dimensional examples immediately follow by taking other representations; for example, $W = V^{\oplus 3} \oplus k^r$, where k denotes the trivial representation and \mathbb{A}^{5+r} is presented as a hypersurface with the same equation as above. \square

Remark 3.2. We expect it is possible to construct a smooth quasi-affine 3-dimensional variety with the desired properties via unipotent quotients of an affine space. However, we do not believe there are any smooth \mathbb{A}^1 -contractible surfaces other than \mathbb{A}^2 (see also Remark 4.4); this is known to be true over \mathbb{C} (see [Aso]).

Remark 3.3. The quasi-affine quotient scheme in the simplest case of the construction from Corollary 3.1 (where f is the identity so that \mathbb{A}^5 is defined by $w_1 = 1 + (w_3w_6 - w_4w_5)$) is very pleasant to visualize. An easy computation with invariants presents $\text{Spec}(k[X]^{\mathbb{G}_a})$ as a quadric hypersurface in \mathbb{A}^5 . When $k = \mathbb{C}$ this may be thought of as the complexification of a sphere, that is, $T^*(S^4)$. Here B is a single affine plane: over \mathbb{C} , B is the cotangent plane at a point, so the complement of B is clearly contractible as a complex manifold. This particular example of a quasi-affine contractible complex variety, with a different presentation, was known to Winkelmann [Win90].

In the other direction, given a desired boundary we can often pick the defining hypersurface equation for X so as to yield a quotient with that specified boundary. Varying the boundary in a family may be realized by varying the defining hypersurface equation in the fixed \mathbb{G}_a -representation W . By arranging for a boundary Z with a large $K_0(Z)$, we can then by the above process get smooth \mathbb{A}^1 -contractible schemes with arbitrarily many non-isomorphic vector bundles, and indeed find arbitrary dimensional families of such schemes.

Corollary 3.4. *For any integers $n \geq 6$, $m \geq 1$, and $l \geq 1$ there exists an m -dimensional smooth scheme S and a smooth morphism $f : X \rightarrow S$ of relative dimension n whose fibers are strictly quasi-affine \mathbb{A}^1 -contractible smooth schemes, pair-wise non-isomorphic, each of which possesses at least l -dimensional moduli of stably trivial, non-trivial vector bundles in every suitably large rank.*

Proof. We use notation and terminology as in the proof of Corollary 3.1. Consider $W = V^{\oplus 4}$, with coordinates $\{w_1, \dots, w_8\}$, as an SL_2 -representation and hence a \mathbb{G}_a -representation (where $\mathbb{G}_a \hookrightarrow SL_2$ as lower triangular matrices, as before). Then the hypersurface $w_1 = 1 + f(w_3w_6 - w_4w_5, w_3w_8 - w_4w_7, w_5w_8 - w_6w_7)$ in W is isomorphic to \mathbb{A}^7 ; the closed immersion is determined by function $w_2 = z_1, \dots, w_8 = z_7$, where $\{z_1, \dots, z_7\}$ are the coordinates on \mathbb{A}^7 . It is easily checked that the restriction of the linear \mathbb{G}_a -action on W to this \mathbb{A}^7 hypersurface is everywhere stable, by using SL_2 -stability for $V \times W$ as before.

Note that $f(w_3w_6 - w_4w_5, w_3w_8 - w_4w_7, w_5w_8 - w_6w_7)$ is an SL_2 -invariant, so the associated hypersurface equation defining \bar{Y} in $V \times W$ is $uw_2 - vw_1 = 1 + f(w_3w_6 - w_4w_5, w_3w_8 - w_4w_7, w_5w_8 - w_6w_7)$. Over any field k , for generic f this describes a smooth hypersurface in the SL_2 -stable locus of $V \times W$, all of whose points have trivial isotropy in SL_2 ; we leave the details to the reader. In particular for generic f the SL_2 action on \bar{Y} is free, and the quotient \bar{Y}/SL_2 is smooth. Since the boundary B is defined by the simultaneous vanishing of u and v , it is a hypersurface in $0 \times W$ and so is codimension 2 in \bar{Y} . Indeed, B consists of a rank 2 vector bundle over a principal SL_2 -bundle over a smooth affine surface. The quotient $Z = B/SL_2$ is thus codimension 2 and a smooth subvariety of the smooth \bar{Y}/SL_2 . Consequently if Y_1 and Y_2 (respectively, Z_1 and Z_2) are the SL_2 -invariant varieties (respectively, boundaries of the quotients) associated with two different choices of f , say f_1 and f_2 , then any morphism from Y_1/SL_2 to Y_2/SL_2 extends to a morphism from \bar{Y}_1/SL_2 to \bar{Y}_2/SL_2 and vice-versa; so $Y_1/SL_2 \cong Y_2/SL_2 \Rightarrow \bar{Y}_1/SL_2 \cong \bar{Y}_2/SL_2 \Rightarrow Z_1 \cong Z_2$. In particular, if $Z_1 \not\cong Z_2$ then $Y_1/SL_2 \not\cong Y_2/SL_2$. Thus the fact that there are arbitrary dimensional moduli of the surfaces $1 + f(x, y, z) = 0$, and hence of the boundaries Z , means there are arbitrary dimensional moduli of $Y/SL_2 \cong \mathbb{A}^7/\mathbb{G}_a$ associated with varying the \mathbb{G}_a -action (cf. [AD07] Lemma 5.5).

Since $Z \cong B/SL_2$ is a vector bundle over a smooth affine surface S , the map $K_0(Z) \rightarrow K_0(S)$ is an isomorphism. Furthermore, the smooth affine surface is defined as a hypersurface in \mathbb{A}^3 . So for example, if we take a hypersurface isomorphic to a product of a smooth affine curve and the affine line (the reader may check that a family of examples in any genus may be chosen so that \bar{Y} is smooth and so that B is contained in the open subscheme of $0 \times W$ on which SL_2 acts freely, thus guaranteeing Z is smooth), we see that $K_0(Z)$ can be made arbitrarily large by making the genus of the curve high: specifically, line bundles are cancellation stable so there is an injection from $Pic(Z)$ into $K_0(Z)$, and affine curves have moduli of line bundles of dimension increasing with the genus. Because everything is smooth, the same argument as in the previous Corollary now implies the desired statement for \mathbb{G}_a -quotients of \mathbb{A}^7 ; quotients for larger dimensional \mathbb{A}^n may be achieved by taking other representations, e.g., $W \oplus k^r$ for k the trivial representation and the defining equation the same as above. \square

4 Some comments on the generalized Serre problem

Proposition 4.1. *Suppose X is a topologically contractible smooth complex variety of dimension ≤ 2 , then every vector bundle on X is isomorphic to a trivial bundle.³*

Proof. If X is a topologically contractible smooth complex curve, then X is isomorphic to the affine line and the result follows from the Quillen-Suslin theorem. Therefore, we can assume that X has dimension 2. Suppose therefore that X is a topologically contractible smooth complex surface.

By a Lemma of Fujita (see e.g., [Zai99] Lemma 2.1), we know that any such surface is affine. By a Theorem of Gurjar-Shastri, (see [Zai99] Theorem 2.1) we know that any topologically contractible smooth complex surface is rational. In particular, X admits a smooth projective compactification \bar{X} which is a smooth projective rational surface. By the classification of surfaces \bar{X} is birationally equivalent to a ruled surface. Murthy (see [Mur69] Theorem 3.2) has shown that every vector bundle on any affine surface birationally equivalent to a ruled surface is necessarily isomorphic to the direct sum of a trivial bundle and a line bundle. Thus, if $Pic(X)$ is trivial, it follows that every vector bundle on X is isomorphic to a trivial bundle.

To see that $Pic(X)$ is trivial for a smooth contractible surface, choose a compactification of \bar{X} whose boundary is a simple normal crossings divisor D . We have an exact sequence for Chow groups

$$CH_i(D) \longrightarrow CH_i(\bar{X}) \longrightarrow CH_i(X) \longrightarrow 0.$$

In particular, taking $i = 1$ and using the fact that \bar{X} and X are smooth, we see that $Pic(\bar{X}) \longrightarrow Pic(X)$ is surjective. By Corollary 2.2 of [Zai99], we know that $Pic(\bar{X})$ is freely generated by the irreducible components of D and thus $Pic(X)$ is necessarily trivial. \square

Corollary 4.2. *If X is any topologically contractible smooth complex algebraic surface, and p_1, \dots, p_n are finitely many points on X , then all vector bundles on $X \setminus \{p_1, \dots, p_n\}$ are trivial.*

Proof. Indeed, if \mathcal{F} is a locally free sheaf on $X \setminus \{p_1, \dots, p_n\}$, then there always exists a coherent extension $\bar{\mathcal{F}}$ of \mathcal{F} to X . The double dual $\bar{\mathcal{F}}^{\vee\vee}$ is a reflexive sheaf on X , which must be locally free since X is a smooth surface; this provides a locally free extension of \mathcal{F} . We have just shown that all vector bundles on such an X are in fact trivial, and thus \mathcal{F} must be a trivial bundle as well. \square

Corollary 4.3. *There are positive dimensional moduli of smooth algebraic surfaces which admit only trivial vector bundles. These can be chosen so that they are affine and, as complex manifolds, contractible or quasi-affine and non-contractible.*

Proof. There are contractible smooth affine algebraic surfaces of log Kodaira dimension 1 that admit deformations (see [FZ94] Example 6.9). Upon removing finitely many points, Corollary 4.2 finishes the result. \square

³Added in proof: Proposition 4.1 can be found in Corollary 2 of [GS89] with a similar proof.

Remark 4.4. None of the examples mentioned in the proof of the Corollary are \mathbb{A}^1 -contractible. In fact, contractible smooth surfaces of positive log Kodaira dimension are known not to be \mathbb{A}^1 -contractible (see [Aso]). Roughly speaking, this is the case because positive log Kodaira dimension surfaces do not have “many” rational curves; in order for a variety to even be \mathbb{A}^1 -connected, one expects that it should be covered by chains of \mathbb{A}^1 s.

Remark 4.5. For topologically contractible smooth complex affine varieties of dimension $n \geq 3$, the functor $X \mapsto \mathcal{V}(X)$ becomes even more subtle. Results of Suslin imply that for projective modules of rank $\geq n$ stable isomorphism implies isomorphism. In this direction, results of Murthy imply (see [Mur02] Corollary 2.11) that if f, g are elements of the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$ with $g \neq 0$, then all stably free modules over $\mathbb{C}[x_1, \dots, x_n, f/g]$ of rank $\geq n - 1$ are free. If X is a topologically contractible smooth complex 3-fold, of this form, then all vector bundles on X are trivial if and only if $\text{Pic}(X)$ is trivial. In particular, Murthy ([Mur02] Theorem 3.6) uses this to deduce that all the Koras-Russell threefolds, in particular the famous Russell cubic surface $x + x^2y + z^2 + t^3 = 0$, satisfy the generalized Serre problem. At the moment, it is not known whether or not the Russell cubic is \mathbb{A}^1 -contractible.

References

- [AD07] A. Asok and B. Doran. On unipotent quotients and some \mathbb{A}^1 -contractible smooth schemes. *Int. Math. Res. Pap.*, 5, 2007. art. id. rpm005. 2, 3, 4, 5, 6, 8, 9, 10, 11, 12
- [Aso] A. Asok. \mathbb{A}^1 -homotopy types and contractible smooth schemes. *Preprint*, available at <http://www.math.washington.edu/~asok/>. 4, 5, 11, 14
- [DK07] B. Doran and F. Kirwan. Towards non-reductive geometric invariant theory. *Quart. J. of Pure and App. Math*, 3(1), 2007. 4, 10
- [FZ94] H. Flenner and M. Zaidenberg. \mathbb{Q} -acyclic surfaces and their deformations. In *Classification of algebraic varieties (L'Aquila, 1992)*, volume 162 of *Contemp. Math.*, pages 143–208. Amer. Math. Soc., Providence, RI, 1994. 13
- [Gro61] A. Grothendieck. Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I. *Inst. Hautes Études Sci. Publ. Math.*, (11):167, 1961. 6
- [GS89] R. V. Gurjar and A. R. Shastri. On the rationality of complex homology 2-cells. I. *J. Math. Soc. Japan*, 41(1):37–56, 1989. 13
- [Lin82] H. Lindel. On the Bass-Quillen conjecture concerning projective modules over polynomial rings. *Invent. Math.*, 65(2):319–323, 1981/82. 6
- [MFK94] D. Mumford, J. Fogarty, and F. C. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 2. Folge [Results in Mathematics and Related Areas (2)]*. Springer-Verlag, Berlin, third edition, 1994. 5
- [Mor] F. Morel. \mathbb{A}^1 -homotopy classification of vector bundles over smooth affine schemes. *Preprint*, available at <http://www.mathematik.uni-muenchen.de/~morel/preprint.html>. 3
- [Mor04] F. Morel. An introduction to \mathbb{A}^1 -homotopy theory. In *Contemporary developments in algebraic K-theory*, ICTP Lect. Notes, XV, pages 357–441 (electronic). Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004. 3
- [Mur69] M. P. Murthy. Vector bundles over affine surfaces birationally equivalent to a ruled surface. *Ann. of Math. (2)*, 89:242–253, 1969. 13

- [Mur02] M. P. Murthy. Cancellation problem for projective modules over certain affine algebras. In *Algebra, arithmetic and geometry, Part I, II (Mumbai, 2000)*, volume 16 of *Tata Inst. Fund. Res. Stud. Math.*, pages 493–507. Tata Inst. Fund. Res., Bombay, 2002. 14
- [MV99] F. Morel and V. Voevodsky. \mathbb{A}^1 -homotopy theory of schemes. *Inst. Hautes Études Sci. Publ. Math.*, (90):45–143 (2001), 1999. 1, 2, 3, 5, 9
- [Qui76] D. G. Quillen. Projective modules over polynomial rings. *Invent. Math.*, 36:167–171, 1976. 6
- [Sri96] V. Srinivas. *Algebraic K-theory*, volume 90 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, second edition, 1996. 7
- [TT90] R. W. Thomason and T. Trobaugh. Higher algebraic K -theory of schemes and of derived categories. In *The Grothendieck Festschrift, Vol. III*, volume 88 of *Progr. Math.*, pages 247–435. Birkhäuser Boston, Boston, MA, 1990. 8
- [Voe01] V. Voevodsky. Lectures on Motivic Cohomology (written by P. Deligne), 2000/2001. Available at <http://www.math.ias.edu/~vladimir/rear.html>. 1
- [Wei89] C. A. Weibel. Homotopy algebraic K -theory. In *Algebraic K-theory and algebraic number theory (Honolulu, HI, 1987)*, volume 83 of *Contemp. Math.*, pages 461–488. Amer. Math. Soc., Providence, RI, 1989. 3
- [Win90] J. Winkelmann. On free holomorphic \mathbb{C} -actions on \mathbb{C}^n and homogeneous Stein manifolds. *Math. Ann.*, 286(1-3):593–612, 1990. 11
- [Zai99] M. Zaidenberg. Exotic algebraic structures on affine spaces. *Algebra i Analiz*, 11(5):3–73, 1999. 5, 13