

# On unipotent quotients and some $\mathbb{A}^1$ -contractible smooth schemes

Aravind Asok

Department of Mathematics  
University of Washington  
Seattle, WA 98195  
asok@math.washington.edu

Brent Doran

School of Mathematics  
Institute for Advanced Study  
Princeton, NJ 08540  
doranb@ias.edu

November 29, 2006

## Abstract

We study quotients of quasi-affine schemes by unipotent groups over fields of characteristic 0. To do this, we introduce a notion of stability which allows us to characterize exactly when a principal bundle quotient exists and, together with a cohomological vanishing criterion, to characterize whether or not the resulting quasi-affine quotient scheme is affine. We completely analyze the case of  $\mathbb{G}_a$ -invariant hypersurfaces in a linear  $\mathbb{G}_a$ -representation  $W$ ; here the above characterizations admit simple geometric and algebraic interpretations. As an application, we produce arbitrary dimensional families of non-isomorphic smooth quasi-affine but not affine  $n$ -dimensional varieties ( $n \geq 6$ ) that are contractible in the sense of  $\mathbb{A}^1$ -homotopy theory. Indeed, existence follows without any computation; yet explicit defining equations for the varieties depend only on knowing some linear  $\mathbb{G}_a$ - and  $SL_2$ - invariants, which, for a sufficiently large class, we provide. Similarly, we produce infinitely many non-isomorphic examples in dimensions 4 and 5. Over  $\mathbb{C}$ , the analytic spaces underlying these varieties are non-isomorphic, non-Stein, topologically contractible and often diffeomorphic to  $\mathbb{C}^n$ .

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Contractibility in topology and algebraic geometry</b>	<b>7</b>
<b>3</b>	<b>Stability for unipotent groups</b>	<b>10</b>
<b>4</b>	<b>Quotients of invariant hypersurfaces</b>	<b>17</b>
<b>5</b>	<b>Examples: <math>\mathbb{A}^1</math>-contractible strictly quasi-affine varieties</b>	<b>28</b>
<b>6</b>	<b>Consequences, Conjectures, and Comments</b>	<b>34</b>
<b>7</b>	<b>Appendix</b>	<b>37</b>

# 1 Introduction

## History and Motivation

In 1935, J.H.C. Whitehead constructed, as a counterexample to his “proof” of the 3-dimensional Poincaré conjecture, the first example of an open (i.e., without boundary) contractible manifold not homeomorphic to a ball (see [Whi35]). Subsequently, D.R. McMillan produced infinitely many pairwise non-homeomorphic open contractible smooth 3-manifolds (see [McM62]). Slightly earlier, Mazur and Poenaru had provided examples of contractible open 4-manifolds (Mazur’s examples can be constructed as smooth manifolds, see [Maz61, Poe60]). Generalizing these constructions, Curtis and Kwun (see [CK65]) showed that there exist infinitely many pairwise non-homeomorphic, contractible, open  $n$ -manifolds for every  $n \geq 5$ , and Glaser (see [Gla67]) showed that the same result held in dimension 4.

Roughly contemporaneously, geometric topologists began to explore the possibility of “exotic”  $PL$  and smooth structures compatible with the usual topology on  $\mathbb{R}^n$ . Stallings proved (see [Sta62]) that if  $M$  is an open contractible  $n$ -manifold of dimension  $n \geq 5$ , simply connected at infinity<sup>1</sup>, then  $M$  is  $PL$ -homeomorphic to  $\mathbb{R}^n$ ; if further  $M$  is smooth, then  $M$  is in fact diffeomorphic to  $\mathbb{R}^n$  with its usual smooth structure. (This result also follows from the  $h$ -cobordism theorem if  $n \geq 6$ , see [Mil65] §9 Proposition A). In other words, for  $n \geq 5$ ,  $\mathbb{R}^n$  admits, up to the appropriate notion of isomorphism, unique  $PL$  and smooth structures.

Surprisingly, a simple shift in perspective allows one to construct, at least in principle, all contractible manifolds; this will be the motivating theme of this paper. It follows from results of McMillan, Zeeman (see [MZ62]) and Stallings (see [Sta62]) that *all* of the examples just discussed can be realized as quotients of free  $\mathbb{R}^k$  actions on  $\mathbb{R}^{n+k}$  (for appropriate  $n$  and  $k$ ). In fact, Stallings (see *loc. cit.* §4, §5 and Proposition 2.2) shows that any open contractible (PL or smooth) 3-manifold can be constructed as a quotient of  $\mathbb{R}^5$  by a free (PL or smooth)  $\mathbb{R}^2$ -action and, more generally, for any  $n \geq 4$  any open contractible (PL or smooth)  $n$ -manifold can be constructed as a quotient of  $\mathbb{R}^{n+1}$  by a free (PL or smooth)  $\mathbb{R}$ -action.<sup>2</sup>

The naïve algebro-geometric analog of this question is whether all smooth contractible complex algebraic varieties can be constructed as quotients of  $\mathbb{C}^n$  by the free algebraic action of a unipotent group. All of the constructions just mentioned are inherently “topological” so it perhaps came as a shock that a smooth contractible variety, apart from affine space, even exists. The first example was given by Ramanujam in his landmark paper [Ram71].<sup>3</sup> It, together with the fact that Zariski cancellation holds in dimension 2 (see [Fuj79]), provides a counter-example to this analog of the question.

Ramanujam’s example was only the tip of the iceberg. Other authors showed that there

<sup>1</sup>An open manifold  $M$  is said to be simply connected at infinity if for every compact subset  $C \subset M$ , there is a compact subset  $D$  such that  $C \subset D \subset M$  and  $M \setminus D$  is connected and simply connected.

<sup>2</sup>Strictly speaking, Stallings shows that the product of the manifold with  $\mathbb{R}^1$  (or  $\mathbb{R}^2$  when  $n = 3$ ) is PL-equivalent or diffeomorphic to  $\mathbb{R}^n$ ; however, for contractible real manifolds, any  $\mathbb{R}^k$ -bundle is trivial, so our statement is equivalent to his, and generalizes well to our algebraic setting.

<sup>3</sup>Ramanujam proved much more: any smooth, complex algebraic surface whose underlying analytic space is contractible and simply connected at infinity is necessarily *algebraically* isomorphic to  $\mathbb{A}^2$ . Ramanujam’s example, being non-simply connected at infinity, was necessarily not homeomorphic to  $\mathbb{R}^4$ .

exist many examples of contractible smooth algebraic varieties in every (complex) dimension  $\geq 2$  (see the beautiful survey [Zai99] for an overview and many more references). In this paper, we begin a study of contractible algebraic varieties from the standpoint of motivic homotopy theory. Rather, since topological contractibility only makes sense for varieties defined over fields that are embeddable in the complex numbers, we have to reformulate the notion of contractibility appropriately.

Following Morel and Voevodsky (see [MV99]) we view the category of smooth schemes as analogous to the category of topological spaces with the affine line playing the role of the unit interval in ordinary topology. Morel and Voevodsky replace the category of (e.g. locally contractible) topological spaces by the category of (simplicial) Nisnevich sheaves on  $\mathcal{S}m/k$  (the category of smooth manifolds is replaced by the Nisnevich sheaves corresponding to smooth schemes), the notion of homeomorphism is replaced by isomorphism of smooth schemes and finally, the usual topological homotopy category is replaced by the Morel-Voevodsky  $\mathbb{A}^1$ -homotopy or “motivic homotopy” category. These analogies are, of course, not perfect (as we shall explain), but hopefully serve to guide intuition.

Our goal is to study  $\mathbb{A}^1$ -contractible smooth algebraic varieties, i.e. those varieties that are  $\mathbb{A}^1$ -weakly equivalent to  $\text{Spec } k$ . Essentially by construction  $\mathbb{A}^n$  is  $\mathbb{A}^1$ -contractible. However, we will see that there are many examples of smooth algebraic varieties, not isomorphic to affine space, that are  $\mathbb{A}^1$ -contractible. Henceforth, we call such a variety an *exotic  $\mathbb{A}^1$ -contractible variety*. We suggest that this notion gives the “correct” algebro-geometric analog of our thematic question, namely:

**Question 1.1.** *Does every smooth  $\mathbb{A}^1$ -contractible variety arise as a quotient of affine space by the free action of a unipotent group?*

First, however, one needs to produce examples of interesting  $\mathbb{A}^1$ -contractible varieties. We prove, in this spirit, the following results.

**Theorem 1.2** (See Theorem 5.1). *For every  $m \geq 4$ , there exists a denumerably infinite collection of pairwise non-isomorphic exotic  $\mathbb{A}^1$ -contractible varieties, each admitting an embedding into a smooth affine variety with pure codimension 2 smooth boundary.*

**Theorem 1.3** (See Theorem 5.3). *For every  $m \geq 6$  and every  $n \geq 0$ :*

- *there exists a connected  $n$ -dimensional scheme  $S$  and a smooth morphism  $f : X \rightarrow S$  of relative dimension  $m$ , whose fibers over  $k$ -points are  $\mathbb{A}^1$ -contractible and quasi-affine, not affine, and pairwise non-isomorphic.*
- *The morphism  $f : X \rightarrow S$  admits a partial compactification to a flat family  $\bar{f} : \bar{X} \rightarrow S$  whose fibers over  $k$ -points are smooth affine varieties. Furthermore, for any  $k$ -point  $t \in S$ , the map  $X_t \rightarrow \bar{X}_t$  is an open immersion with a smooth complement of codimension  $\geq 2$*

In other words, there exist arbitrary dimensional *moduli* of  $\mathbb{A}^1$ -contractible smooth varieties in dimension  $\geq 6$ . We stress that these examples are completely explicit and non-pathological. The families arise in a simple geometric manner: as  $\mathbb{G}_a$ -quotients of families of  $\mathbb{G}_a$ -invariant

hypersurfaces in a fixed linear  $\mathbb{G}_a$ -representation  $W$ . The resulting quotient varieties are complements of smooth codimension 2 subvarieties in smooth hypersurfaces in  $\mathrm{Spec} k[W]^{\mathbb{G}_a}$ . The simplest case, for example, is the complement in an affine quadric four-fold (defined by the vanishing of  $x_1x_4 - x_2x_3 - x_5(x_5 + 1)$  in  $\mathbb{A}^5$ ) of an explicit embedded copy of  $\mathbb{A}^2$  (defined by  $x_1 = x_2 = 0, x_5 = -1$ ); see Remark 5.2 for details.

At least over  $\mathbb{C}$ , and in fact over fields embeddable in  $\mathbb{C}$ , our method provably does not produce exotic  $\mathbb{A}^1$ -contractibles in dimension  $\leq 2$ . Indeed, there exists a unique up to isomorphism smooth  $\mathbb{A}^1$ -contractible variety of dimension 1, namely  $\mathbb{A}^1$  (see Claim 5.7). It follows from results of several authors, that all the examples of  $\mathbb{A}^1$ -contractible smooth surfaces we produce are necessarily isomorphic to affine space (see also Claim 5.8). Therefore, only dimension 3 seems mysterious. In analogy with the topological setting, one may need to use explicit  $(\mathbb{G}_a)^2$  actions to study dimension 3.

The motivic homotopy category of schemes over  $\mathrm{Spec} \mathbb{C}$ , admits a “topological realization functor” to the usual homotopy category of topological spaces. This realization functor takes  $\mathbb{A}^1$ -weak equivalences of smooth schemes to ordinary weak equivalences and, in particular, the topological realization of an  $\mathbb{A}^1$ -contractible smooth variety is a contractible smooth manifold. We are unable to produce examples of contractible algebraic varieties that are provably not  $\mathbb{A}^1$ -contractible. Topological intuition encourages us to believe that such varieties exist; however, “motivic” intuition related to the Hodge conjecture imposes very strong topological restrictions on any such examples. Summarizing the above discussion, we make the following conjecture.

**Conjecture 1.4.** *For every  $m \geq 3$ , and every  $n \geq 0$ , there exists a connected  $n$ -dimensional scheme  $S$  and a smooth morphism  $f : X \rightarrow S$  of relative dimension  $m$ , whose fibers are  $\mathbb{A}^1$ -contractible and for a fixed field  $k$ , the fibers of  $f$  over  $k$ -points of  $S$  are all non-isomorphic.*

Finally, we claim that the topological characterization “at infinity” of the  $PL$  or smooth structure of  $\mathbb{R}^n$  for  $n \geq 5$  gives rise to a natural question: can one give a motivic topological characterization of affine space? As a first step in this direction, one can define a notion of homology at infinity. This notion was introduced by Wildeshaus in his paper “Basic properties of the boundary motive” (see [Wil]). All exotic  $\mathbb{A}^1$ -contractibles have, via Poincaré duality, motivic homology at infinity that of a motivic sphere of appropriate dimension (see 6.9). A natural question to ask is whether there exists a good notion of “ $\mathbb{A}^1$ -fundamental group at infinity,” analogous to the fundamental group at infinity, that one might use to characterize when an  $\mathbb{A}^1$ -contractible smooth variety is exotic. In this direction, Morel (see [Mora]) dreams of a “motivic  $s$ -cobordism theorem:” a characterization of affine space as a smooth scheme should be a related consequence.

## Contents

As the techniques used in this paper have been introduced fairly recently, we have endeavored to make the paper as self-contained as possible. We begin, in §2, by making a brief review of  $\mathbb{A}^1$ -contractibility. In particular, we state the main criterion we use to check that a morphism is an  $\mathbb{A}^1$ -weak equivalence (see Lemma 2.4). In addition, we give the simplest examples of (singular)

$\mathbb{A}^1$ -contractible algebraic varieties. In §3, we discuss the relevant elements from geometric invariant theory for non-reductive group actions as developed in [DK]. In particular, after reviewing some basic facts about unipotent groups, we discuss a condition characterizing the existence of principal bundle quotients by unipotent group actions (see Theorems 3.9 and Theorem 3.12). To keep this section self-contained, we have given complete proofs of all the main results; the focus here is on quotients of everywhere stable quasi-affine schemes and the material is essentially orthogonal to that contained in [DK].

In §4, we study the simplest class of unipotent group actions:  $\mathbb{G}_a$ -actions. The Jacobsen-Morozov theorem (see Theorem 4.2) essentially allows us to reduce the study of  $\mathbb{G}_a$ -actions to  $SL_2$ -actions. We completely resolve the question of when the (principal bundle) quotient of a  $\mathbb{G}_a$ -invariant hypersurface in a linear  $\mathbb{G}_a$ -representation is an affine variety, a strictly quasi-affine variety, or not even a scheme. In particular, we give two explicit characterizations (see Theorems 4.11 and 4.20) which are used to produce all the examples discussed in §5. One curious consequence is a natural decomposition of *any*  $\mathbb{G}_a$ -invariant function into a sum of an  $SL_2$ -invariant function and a  $\mathbb{G}_a$ -invariant function of a very particular sort; the authors are not aware of a classical version of this statement in invariant theory (see Theorem 4.20). In §5, we prove the two theorems stated in the introduction by using a very simple class of  $\mathbb{G}_a$ -equivariant linear embeddings of affine space (see Theorems 5.1 and 5.3). We also make a detailed study of strictly quasi-affine quotients in small dimensions. In §6, we discuss various consequences of and conjectures related to the notion of  $\mathbb{A}^1$ -contractibility. We emphasize here that the very existence of the motivic homotopy category allows us to make very strong statements about the motivic topology of exotic  $\mathbb{A}^1$ -contractible varieties. In particular, we discuss briefly the idea of motivic topology at infinity. We close in the Appendix (§7) with a summary of the main tools of the technique of faithfully flat descent, its application to Borel transfer, and the proof of the quite general and formal Theorem 3.12. Consequences of descent and its applications are utilized throughout the paper; rather than interrupt the main discussion with technical sidelights, we have compiled the relevant facts there.

## Conventions

Throughout this paper the word “field” will stand for “field of characteristic zero.” The word “scheme” will mean separated scheme of locally finite type over a field  $k$ , the word “variety” will mean “reduced scheme of finite type,” and all group schemes will be linear algebraic  $k$ -groups. Given a group scheme  $G$  and a scheme  $X$ , we will say  $X$  is a  $G$ -scheme if  $X$  admits an algebraic left  $G$ -action; that being said, the geometric quotient of  $X$  by  $G$ , if it exists as a scheme, will be written  $X/G$  (rather than  $G \backslash X$ ). The word “free” applied to a  $G$ -action on a scheme  $X$  will always mean scheme-theoretically free  $G$ -action, i.e. the action morphism  $G \times X \rightarrow X \times X$  is a closed immersion. Finally, given a closed immersion group homomorphism of linear algebraic groups  $H \hookrightarrow G$  and an  $H$ -scheme  $X$ , we write  $G *_H X$  for the twisted or contracted product; this is the (algebraic space) quotient of  $G \times X$  by the free  $H$ -action defined by  $h \cdot (g, x) = (gh, h \cdot x)$ . If  $X$  is quasi-affine we prove in the appendix (see Corollary 7.2 and Remarks 7.3 and 7.6) that this contracted product exists as a scheme.

### Acknowledgements

The authors owe an intellectual debt to the examples of J. Winkelmann in his paper [Win90], which are the first strictly quasi-affine quotients of a  $\mathbb{C}^n$  to appear in the literature; it was a pleasure, and a wonderful self-check, to see them arise so naturally from another perspective. Our study of  $\mathbb{A}^1$ -contractibility came as a sidenote to discussion and collaboration with Frances Kirwan; this work can be thought of as living in the intersection of two concurrent projects with her, we thank her for various discussions around this subject matter. We would like to thank James Parson, Paul Hacking, Sándor Kovács and Steve Mitchell for helpful discussions and answering various questions we had. Finally, we would like to thank William Stein for providing facilities for computation.

This material is based upon work supported by the National Science Foundation under agreement No. DMS-0111298. Any opinions, findings and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

## 2 Contractibility in topology and algebraic geometry

Let us begin by reviewing some basic notions of  $\mathbb{A}^1$ -homotopy theory. The general references for the material in this section are [MV99] and [Voe98]. Let  $\mathcal{S}m/k$  denote the category of finite type smooth schemes over  $k$  and let  $\mathcal{S}ch/k$  denote the category of all finite type schemes over  $k$ . We will only use the most basic definitions from [MV99]: the main goals of this section are to explain Lemma 2.4, to prove Lemma 2.5 that the topological realization of any  $\mathbb{A}^1$ -weak equivalence of smooth schemes over a field embeddable in  $\mathbb{C}$  is actually a topological weak equivalence, and to give a sense for how smooth  $\mathbb{A}^1$ -contractible schemes are different from the singular case. The reader willing to take these results on faith can proceed directly to the geometric constructions of the next sections.

### Spaces

The category  $\mathcal{S}m/k$  is not suitable for the purposes of homotopy theory: for example, quotients by subspaces do not always exist in this category. Equip  $\mathcal{S}m/k$  with the Nisnevich topology and consider the category  $\mathcal{S}hv_{Nis}(\mathcal{S}m/k)$ ; we refer to objects of this category as *spaces*. As the Nisnevich topology is sub-canonical, every representable presheaf is a sheaf. Therefore, the Yoneda embedding  $\mathcal{S}m/k \rightarrow \mathcal{S}hv_{Nis}(\mathcal{S}m/k)$ , which sends a smooth scheme  $X$  to the representable functor  $U \mapsto X(U)$ , is fully faithful. For a possibly non-smooth scheme  $X$ , the functor  $U \mapsto X(U)$  is also a Nisnevich sheaf and gives a functor  $\mathcal{S}ch/k \rightarrow \mathcal{S}hv_{Nis}(\mathcal{S}m/k)$ ; this extended functor is not fully faithful as the following example shows.

*Example 2.1.* Let  $X_{p,q}$  denote the Nisnevich sheaf attached to the cuspidal curve  $x^p - y^q = 0 \subset \mathbb{A}^2$ . Normalization determines a morphism  $\mathbb{A}^1 \rightarrow X_{p,q}$ ; we now show that the induced morphism of sheaves is an isomorphism. In fact, the presheaves on  $\mathcal{S}m/k$  associated to these schemes are isomorphic.

The map of sheaves  $\mathbb{A}^1 \rightarrow X_{p,q}$  is surjective because any morphism from a connected test scheme  $T$  to  $X_{p,q}$  factors through the normalization. To show the map of sheaves  $\mathbb{A}^1 \rightarrow X_{p,q}$  is injective, we have to show that the diagonal closed immersion  $\Delta : \mathbb{A}^1 \rightarrow \mathbb{A}^1 \times_{X_{p,q}} \mathbb{A}^1$  is surjective and hence identifies  $\mathbb{A}^1$  with the underlying reduced scheme of the product. This follows using the fact that the morphism  $\mathbb{A}^1 \rightarrow X_{p,q}$  is radiciel. We thank James Parson for explaining this example to us.

### $\mathbb{A}^1$ -weak equivalences

As in topology, it is often convenient to consider categories of simplicial spaces; denote by  $\Delta^\circ \mathcal{S}hv_{Nis}(\mathcal{S}m/k)$  the category of simplicial Nisnevich sheaves. The motivic homotopy category can be constructed from either the category  $\mathcal{S}hv_{Nis}(\mathcal{S}m/k)$  or  $\Delta^\circ \mathcal{S}hv_{Nis}(\mathcal{S}m/k)$  by localizing at the class of  $\mathbb{A}^1$ -weak equivalences. For a precise definition of  $\mathbb{A}^1$ -weak equivalence see [MV99] §2.2 Definitions 2.1 and 2.2 and §2.3 Proposition 3.14. Roughly speaking, for every sheaf  $X$ , we invert the projection morphism  $X \times \mathbb{A}^1 \rightarrow X$ . This localization forces many additional morphisms to be weak equivalences. Let us now give some examples of  $\mathbb{A}^1$ -weak equivalences.

We let  $i_0 : \text{Spec } k \rightarrow \mathbb{A}^1$  and  $i_1 : \text{Spec } k \rightarrow \mathbb{A}^1$  denote the inclusion of  $k$ -rational points 0 and 1. Given a pair of spaces  $X$  and  $Y$  and two morphisms  $f, g : X \rightarrow Y$ , an *elementary  $\mathbb{A}^1$ -homotopy* from  $f$  to  $g$  is a morphism  $H : X \times \mathbb{A}^1 \rightarrow Y$  such that  $H \circ i_0 = f$  and  $H \circ i_1 = g$ . The morphisms  $f$  and  $g$  are said to be  *$\mathbb{A}^1$ -homotopic* if they can be connected by a finite sequence of elementary  $\mathbb{A}^1$ -homotopies. Finally a morphism  $f : X \rightarrow Y$  is said to be a *strict  $\mathbb{A}^1$ -homotopy equivalence* if there exists a morphism  $g : Y \rightarrow X$  such that  $f \circ g$  and  $g \circ f$  are  $\mathbb{A}^1$ -homotopic to  $\text{Id}_X$  and  $\text{Id}_Y$ .

**Lemma 2.2** ([MV99] §2.3 Lemma 3.6). *Any strict  $\mathbb{A}^1$ -homotopy equivalence is an  $\mathbb{A}^1$ -weak equivalence.*

**Definition 2.3.** A space  $X$  is said to be  *$\mathbb{A}^1$ -contractible* if the structure morphism  $X \rightarrow \text{Spec } k$  is an  $\mathbb{A}^1$ -weak equivalence. A smooth  $\mathbb{A}^1$ -contractible scheme not isomorphic to affine space will be called an *exotic  $\mathbb{A}^1$ -contractible* scheme.

**Lemma 2.4** ([MV99] §3.2 Example 2.3). *Let  $\pi : Y \rightarrow X$  be a Zariski locally trivial, smooth morphism of smooth schemes with  $\mathbb{A}^1$ -contractible fibers. Then  $\pi$  is an  $\mathbb{A}^1$ -weak equivalence.*

### Comparison with topological contractibility

We now explain the relation between  $\mathbb{A}^1$ -contractibility and the usual topological weak equivalences when  $k$  is a field that admits an embedding into  $\mathbb{C}$ . We follow the discussion of [DI04]; let  $\mathcal{Top}$  denote the category of *all* topological spaces with continuous maps as morphisms. The usual notion of open set endows  $\mathcal{Top}$  with the structure of a Grothendieck site. We let  $\mathcal{H}$  denote the usual homotopy category of topological spaces.

Consider the site  $\mathcal{Sm}/\mathbb{C}_{\text{Nis}}$ . Dugger shows (see [Dug01] Proposition 8.1) how to construct a model category  $U(\mathcal{Sm}/\mathbb{C}_{\text{Nis}})_{\mathbb{A}^1}$  that is Quillen equivalent to the Morel-Voevodsky category (and hence resulting homotopy categories are isomorphic). Given a smooth  $\mathbb{C}$ -scheme  $X$ , the assignment  $X \rightarrow X(\mathbb{C})$  sending  $X$  to its set of complex points equipped with the usual topology extends to an adjoint pair of Quillen functors from  $U(\mathcal{Sm}/\mathbb{C}_{\text{Nis}})_{\mathbb{A}^1}$  to  $\mathcal{Top}$ . In particular, any such functor preserves weak equivalences of cofibrant objects. We let  $t^{\mathbb{C}}$  denote the induced functor of homotopy categories. The following result then follows from the fact that representable sheaves are cofibrant objects in  $U(\mathcal{Sm}/\mathbb{C}_{\text{Nis}})_{\mathbb{A}^1}$ .

**Lemma 2.5.** *If  $f : X \rightarrow Y$  is a morphism of smooth schemes which is an  $\mathbb{A}^1$ -weak equivalence, then the induced map  $t^{\mathbb{C}}(f) : X(\mathbb{C}) \rightarrow Y(\mathbb{C})$  is a topological weak equivalence. In particular, if  $X$  is any  $\mathbb{A}^1$ -contractible smooth scheme, then  $X(\mathbb{C})$  is a contractible topological space.*

### Contracting $\mathbb{G}_m$ -actions and singular $\mathbb{A}^1$ -contractible varieties

A natural way to produce explicit strict  $\mathbb{A}^1$ -homotopy equivalences is to consider  $\mathbb{G}_m$ -actions. If  $X$  is a variety equipped with an algebraic  $\mathbb{G}_m$ -action such that there is a unique closed  $\mathbb{G}_m$ -orbit that is a fixed-point, then one expects  $X$  to be contractible since, from the standpoint of Morse theory, all points “flow toward the fixed points.” More generally, suppose  $T$  is a  $k$ -torus acting on a scheme  $X$  with a unique closed  $T$ -orbit that is furthermore a fixed point (which is necessarily a  $k$ -rational point). In this case, we will say that  $X$  admits a *contracting  $T$ -action*.

**Lemma 2.6.** *Suppose  $T$  is a  $k$ -split torus. If  $X$  is an affine  $T$ -scheme equipped with a contracting  $T$ -action, then  $X$  is an  $\mathbb{A}^1$ -contractible scheme.*

*Proof.* The inclusion of the  $T$ -fixed point determines a  $T$ -equivariant morphism  $\iota : \operatorname{Spec} k \rightarrow X$  and the structure morphism  $X \rightarrow \operatorname{Spec} k$  (equivalently the categorical quotient morphism  $X \rightarrow \operatorname{Spec} k[X]^T$ ) is a  $T$ -equivariant morphism as well.

Choose a “generic” one-parameter subgroup  $\mu : \mathbb{G}_m \rightarrow T$  that has the same fixed-point locus as  $T$  (we can do this because  $T$  is split). The action morphism  $\mu : \mathbb{G}_m \times X \rightarrow X$  extends to a morphism  $\bar{\mu} : \mathbb{A}^1 \times X \rightarrow X$ . We claim that  $\bar{\mu}$  defines an elementary  $\mathbb{A}^1$ -homotopy between the identity map  $Id : X \rightarrow X$  and the composite map  $X \rightarrow \operatorname{Spec} k \xrightarrow{\iota} X$ . The structure morphism  $X \rightarrow \operatorname{Spec} k$  is an  $\mathbb{A}^1$ -weak equivalence by Lemma 2.2, and therefore by Definition 2.3  $X$  is  $\mathbb{A}^1$ -contractible.  $\square$

*Example 2.7.* Consider the linear action of  $\mathbb{G}_m$  on  $\mathbb{A}^n$  with weights  $a_1, \dots, a_n$ . Assume further that each of the  $a_i$  is a strictly positive integer. Then any closed  $\mathbb{G}_m$ -stable subvariety necessarily has the origin as the unique  $\mathbb{G}_m$ -fixed point and hence satisfies the hypotheses of Lemma 2.6. It follows from Example 2.1 that the case  $n = 2$  only produces examples isomorphic as spaces to affine space.

*Example 2.8.* Let  $X$  be the affine nodal cubic curve defined to be the vanishing locus of  $y^2 = x^2(x+1)$  in  $\mathbb{A}^2$ . The curve  $X$  admits a contracting  $\mathbb{G}_m$ -action, and one can show, using the same ideas as Example 2.1, that the sheaf  $X$  is not isomorphic to the sheaf  $\mathbb{A}^1$ .

*Remark 2.9.* Note that the assumption that  $X$  is affine is necessary in the statement of Lemma 2.6 as the following example shows. Take  $\mathbb{P}^1$  with the usual  $\mathbb{G}_m$ -action and consider the quotient that identifies the two  $\mathbb{G}_m$ -fixed points. The resulting variety is non-normal, and admits a  $\mathbb{G}_m$ -action that has a unique closed orbit that is a  $\mathbb{G}_m$ -fixed point. However, since it is a strictly positive dimensional projective variety, it can not be  $\mathbb{A}^1$ -contractible.

By Luna’s slice theorem (see [Lun73] III.1 Corollaire 2), every smooth affine  $\mathbb{G}_m$ -variety with a contracting  $\mathbb{G}_m$ -action is necessarily  $\mathbb{G}_m$ -equivariantly isomorphic to a vector space endowed with a linear  $\mathbb{G}_m$ -action. Therefore,  $\mathbb{A}^1$ -contractible smooth varieties constructed by means of Lemma 2.6 are scheme-theoretically isomorphic to affine space. It seems difficult (to us) to sort out when a sheaf  $X$  attached to an arbitrary scheme is actually isomorphic to the sheaf attached to a smooth scheme.

### 3 Stability for unipotent groups

In this section, we investigate a general technique for studying quasi-affine schemes that have the structure of a  $U$ -torsor over some base (where  $U$  is a unipotent  $k$ -group). We use a notion of stability for  $U$ -actions in the spirit of the geometric invariant theory for reductive groups (see 3.4). We begin by recalling some general facts about unipotent groups. Then we prove the basic results about stability and in particular about “everywhere stable actions” on quasi-affine schemes (see Definition 3.8 and Theorem 3.9). We present, following Greuel and Pfister, a characterization of when the quotient of an affine  $U$ -scheme is affine (see Theorem 3.12) but defer the proof to the Appendix, as it uses different language than the body of the paper (see §7). Taken together, these results give a way to characterize when quotients of an affine variety are affine, quasi-affine, or not even a scheme; Corollary 3.16 summarizes the results and suggests how they apply to constructing  $\mathbb{A}^1$ -contractible schemes.

#### Unipotent groups and a key lemma

Let  $U$  be a connected unipotent group over  $k$ . Recall the following structure theorem; this is the essential ingredient in understanding the structure of  $U$ -torsors.

**Theorem 3.1** (Lazard (see [KMT74] Theorem 8.0)). *Let  $U$  be a connected unipotent  $k$ -group. Then  $U$  admits an increasing filtration  $F_i(U)$  with successive quotients  $F_i(U)/F_{i-1}(U) \cong \mathbb{G}_a$ .*

**Corollary 3.2** (Grothendieck). *Let  $X$  be a  $k$ -scheme and let  $U$  be a connected unipotent  $k$ -group. Then all  $U$ -torsors on  $X$  are Zariski locally trivial.*

**Key Lemma 3.3.** *Let  $U$  be a connected unipotent group. Suppose  $U$  acts freely on an  $\mathbb{A}^1$ -contractible smooth scheme  $X$  such that a principal bundle quotient  $\pi : X \rightarrow X/U$  exists as a scheme. Then  $X/U$  is smooth and  $\mathbb{A}^1$ -contractible.*

*Proof.* In this situation, we know that  $\pi$  is a Zariski locally trivial morphism by Corollary 3.2. As unipotent groups are isomorphic to affine spaces, the fibers of  $\pi$  are isomorphic to affine spaces and are hence  $\mathbb{A}^1$ -contractible. If  $X$  and  $X/U$  are smooth, then  $\pi$  is an  $\mathbb{A}^1$ -weak equivalence by Lemma 2.4. Since  $X$  is  $\mathbb{A}^1$ -contractible by assumption, the result would follow. One need only observe that if a principal bundle quotient  $X \rightarrow X/U$  exists as a scheme, that  $X/U$  is smooth if and only if  $X$  is smooth. □

#### Stability and $U$ -torsors

Next, we discuss how to construct principal bundle quotients of actions of unipotent groups using elements of the geometric invariant theory for non-reductive groups studied by the second author and F. Kirwan in [DK]. In particular, we will show that quotients satisfying the hypotheses of Lemma 3.3 abound. Suppose  $U$  is a connected unipotent group. Any such group can be realized as a closed subgroup of a reductive group  $G$ ; in fact every unipotent group is isomorphic to a closed subgroup of  $GL_n$  for  $n$  sufficiently large. Let  $i : U \hookrightarrow G$  denote the corresponding group

homomorphism. We can now define a notion of stability for actions of unipotent groups using the corresponding notion for reductive groups.

Let  $X$  be a quasi-affine  $U$ -scheme. Consider the contracted product scheme  $G *_U X$  (which exists by Corollary 7.2 and Remark 7.3) with structure sheaf  $\mathcal{O}_{G *_U X}$ . In this scenario, there is a canonical closed immersion  $\iota : X \hookrightarrow G *_U X$  which sends a point  $x \in X$  to the point  $[e, x]$ . If the character group  $X^*(G)$  is trivial (e.g.,  $G$  is semi-simple), then  $\mathcal{O}_{G *_U X}$  actually admits a canonical  $G$ -linearization. For us it is convenient to always work with the  $G$ -linearization associated to the trivial character. We can then make the following definition.

**Definition 3.4.** A geometric point  $x \in X$  is *stable*, denoted  $x \in X^s$ , if for every reductive group  $G$ , and every embedding  $i : U \hookrightarrow G$ , the geometric point  $[e, x] = \iota(x)$  is a (properly) stable point in the sense of Mumford (see [MFK94]) of  $G *_U X$  with respect to the  $G$ -linearized sheaf  $\mathcal{O}_{G *_U X}$ , where the linearization is given by the trivial character.

*Remark 3.5.* In the above definition, we can replace the unipotent group  $U$  by any linear algebraic group. In addition, the set of stable points is the set of geometric points of an open subscheme of  $X$ ; the notation  $X^s$  refers to this open subscheme. As noted in [MFK94] Proposition 1.14, if  $K/k$  is a field extension, then we have an identification  $X_K^s \cong X_k^s \times_k K$  so that stability is preserved under field extensions. We will therefore often make arguments by passing to an algebraic closure and then applying Galois descent; note that this is a special case of the descent theory described in Theorem 7.1.

Our definition of stability is intrinsic to the  $U$ -action on  $X$ , but it is not *a priori* obvious that the set of stable points on  $X$  is ever non-empty. We now show that to check whether a point is stable, it suffices to check stability for a *single* reductive group.

**Lemma 3.6** (Doran-Kirwan (compare [DK] Lem 5.1.6)). *Given a reductive group  $G$ , a closed embedding group homomorphism  $i : U \hookrightarrow G$  and a geometric point  $x \in X$ , the point  $\iota(x) = [e, x]$  in  $G *_U X$  is stable with respect to the linearization corresponding to the trivial character, if and only if it is semi-stable in the sense of Mumford with respect to the same linearization; i.e.  $(G *_U X)^s = (G *_U X)^{ss}$ .*

*Proof.* Let  $O$  be a  $U$ -orbit of geometric points in  $X$  such that  $G *_U O$  is strictly semi-stable, i.e., lies in the locally closed subscheme  $X^{ss} \setminus X^s$ . Consequently  $G *_U O$  is a  $G$ -orbit in the complement of a  $G$ -stable affine hypersurface defined by a  $G$ -invariant polynomial  $F$  (i.e.,  $G *_U O \subset (G *_U X)_F$ ), and furthermore is either closed of non-maximal dimension or not closed in  $(G *_U X)_F$ . In either case, there is a unique closed orbit  $(G *_U O')$  in the closure of  $G *_U O$  in  $(G *_U X)_F$  that is of non maximal dimension. Now, a closed sub-scheme of an affine scheme is affine, and hence the  $G$ -orbit  $(G *_U O')$  is necessarily affine; this means any point  $y \in G *_U O'$  necessarily has strictly positive dimensional reductive stabilizer group. However, the stabilizer group of  $y$  must be conjugate to a subgroup of  $U$  and is therefore unipotent as well; it follows that the stabilizer group is in fact trivial, which is a contradiction. Thus such an  $O$  does not exist.  $\square$

**Proposition 3.7** (Doran-Kirwan (compare [DK] Prop. 5.1.8)). *A geometric point  $x \in X$  is stable if and only if, for any fixed reductive group  $G$ , together with a fixed closed embedding*

group homomorphism  $i : U \hookrightarrow G$ , the point  $\iota(x) = [e, x]$  in  $G *_U X$  is (properly) stable in the sense of Mumford with respect to the linearization on  $\mathcal{O}_{G *_U X}$  corresponding to the trivial character of  $G$ .

*Proof.* By Lemma 3.6 it suffices to show that if  $G_1$  and  $G_2$  are two reductive groups both containing  $U$  that the intersections  $\iota_1(X) \cap (G_1 *_U X)^{ss}$  and  $\iota_2(X) \cap (G_2 *_U X)^{ss}$  co-incide. By definition, a point  $y$  in  $G_i *_U X$  is semi-stable if and only if it is contained in a  $G_i$ -invariant open affine  $(G_i *_U X)_F$ , for  $F$  a  $G$ -invariant function in  $k[G_i *_U X]$ .

Fix a reductive group  $G$  containing both  $G_1$  and  $G_2$  (e.g.  $G_1 \times G_2$ ). Using the Borel transfer (see Section 7), we have isomorphisms:

$$k[G *_U X]^G \xrightarrow{\sim} k[G_i *_U X]^{G_i} \xrightarrow{\sim} k[X]^U.$$

If  $f$  is a  $U$ -invariant, we let  $F_i$  be the corresponding  $G_i$ -invariants in  $k[G_i *_U X]$  and  $F$  the corresponding  $G$ -invariant in  $k[G *_U X]$  obtained by the transfer isomorphisms. It follows that  $\iota_1(x) \in (G_1 *_U X)_{F_1}$  if and only if  $\iota_2(x) \in (G_2 *_U X)_{F_2}$ . It suffices then to check that the hypersurface complement  $(G_1 *_U X)_{F_1}$  is affine if and only if  $(G_2 *_U X)_{F_2}$ . To do this, one need only check that  $(G_i *_U X)_{F_i}$  is affine if and only if  $(G *_U X)_F$  is affine.

Now, since  $G *_U X = G *__{G_i} (G_i *_U X)$  we know that  $G *_U X$  has the structure of an étale locally trivial fiber space over  $G/G_i$ . Therefore, using the transfer isomorphism we have an identification  $(G *_U X)_F = G *__{G_i} (G_i *_U X)_{F_i}$ . As  $G_i$  is reductive,  $G/G_i$  is affine by Matsushima's theorem. The projection onto the first factor makes  $G *__{G_i} (G_i *_U X)_{F_i}$  an étale locally trivial fiber bundle over the affine variety  $G/G_i$  with fibers isomorphic to  $(G_i *_U X)_{F_i}$ . By Corollary 7.5, the result follows. □

**Definition 3.8.** Suppose a linear algebraic group  $U$  acts on a quasi-affine scheme  $X$ . We say that the action is *everywhere stable* if every geometric point of  $X$  is stable, i.e. that  $X = X^s$ .

**Theorem 3.9** (Doran-Kirwan (compare [DK] 5.3.1)). *Let  $U$  be a connected unipotent group and suppose  $X$  is a finite type quasi-affine  $U$ -scheme. The action of  $U$  on  $X$  is everywhere stable if and only if the quotient morphism  $\pi : X \rightarrow X/U$  is a  $U$ -torsor with  $X/U$  a quasi-affine scheme. In this case, the scheme  $X/U$  can be realized as an open subscheme of the scheme  $\text{Spec } k[X]^U$ . If furthermore  $X$  is a variety, then  $X/U$  is a variety as well.*

*Proof.* First, note that any finite subgroup of a unipotent group over a characteristic zero field is trivial. Since the  $U$ -action on  $X$  is everywhere stable, it is proper. Since  $U$  has no finite subgroups, the  $U$ -action on  $X$  is set-theoretically free as well; therefore the  $U$ -action on  $X$  is free.

Note that since  $U$  acts everywhere stably on  $X$ ,  $G$  acts everywhere stably on  $G *_U X$ ; by the discussion of the previous paragraph,  $G$  must actually act freely on  $G *_U X$ . Since  $G$  acts everywhere stably on  $G *_U X$ , we know by [MFK94] Chapter 1 Theorem 1.10 that a geometric quotient  $G *_U X/G$  exists as a scheme and because the  $G$ -action is free, it follows that the quotient morphism  $q : G *_U X \rightarrow G *_U X/G$  is a  $G$ -torsor.

We begin with the forward implication. *A priori* the quotient  $X/U$  is just an algebraic space. We claim that  $G *_U X/G$  is in fact a geometric quotient of  $X$  by  $U$ , so that we have  $G *_U X/G \xrightarrow{\sim} X/U$  as schemes.

Since  $G *_U X/G$  is a geometric quotient by  $G$ , it follows that  $k$ -points in  $G *_U X/G$  correspond bijectively to  $G$ -orbits in  $G *_U X$ . By faithfully flat descent (see 7.5, closed  $G$ -stable subschemes of  $G *_U X$  are in bijection with closed  $U$ -stable subschemes of  $X$ ). Therefore,  $k$ -points of  $G *_U X/G$  are in bijection with  $U$ -orbits in  $X$ . Next,  $q : G *_U X \rightarrow G *_U X/G$  is a submersive morphism hence a subscheme  $V \subset G *_U X/G$  is open if and only if  $q^{-1}(V)$  is open in  $G *_U X$ . Now,  $q^{-1}(V)$  is a  $G$ -stable open and hence corresponds to a  $U$ -stable open of  $X$  (again by Corollary 7.5), and agrees with  $\pi^{-1}(V)$  by definition. Finally if  $\iota : X \rightarrow G *_U X$ , we note by Corollary 7.2 combined with [SGA71] Expose VIII Cor. 1.9, that  $q_*(\mathcal{O}_{G *_U X})^G \cong \pi_*(\mathcal{O}_X)^U$ . Therefore,  $G *_U X/G$  is in fact a geometric quotient of  $X$  by  $U$ , and we will write  $X/U$  and  $G *_U X/G$  interchangeably henceforth.

Now we observe that  $X \rightarrow X/U$  is in fact a  $U$ -torsor. Indeed,  $G *_U X \rightarrow X/U$  is a  $G$ -torsor over  $X/U$ . By pull-back the scheme  $G \times X$  is a  $G \times U$ -torsor over  $X/U$ . Therefore, by faithfully flat descent,  $X$  is a  $U$ -torsor over  $X/U$ . (In fact, the same argument shows that  $X \rightarrow X/U$  is a  $U$ -torsor if and only if  $G *_U X \rightarrow G *_U X/G$  is a  $G$ -torsor.) Since  $X \rightarrow X/U$  is a  $U$ -torsor it is necessarily an affine morphism.

Let us now check that  $X/U$  is in fact a quasi-affine scheme. To do this we check that  $G *_U X$  embeds  $G$ -equivariantly in an affine scheme  $\overline{G *_U X}$  such that  $G *_U X/G \hookrightarrow \overline{G *_U X}/G$ . We can pick a finite set  $S$  of  $G$ -invariant functions in  $k[G *_U X]$  such that given geometric points  $x, y \in G *_U X$  if there exists some  $G$ -invariant function  $f$  such  $f(x) \neq f(y)$  then in fact we can find  $f' \in S$  such that  $f'(x) \neq f'(y)$ ; this follows from the fact that  $G *_U X$  is of finite type (see Corollary 7.5). We then consider the morphism  $G *_U X \rightarrow \mathbb{A}^k$  defined by the functions  $f_i \in S$ . By definition, this gives an embedding  $G *_U X/G$  into  $\text{Spec } k[f_1, \dots, f_n]$ . We denote this closure by  $\overline{G *_U X}/G$ , and this identifies  $X/U$  as a quasi-affine scheme.

Now, for the reverse direction, suppose  $X \rightarrow X/U$  is a principal bundle quotient. For any  $G$ -invariant function  $F$ , the induced morphism  $(G *_U X)_F \rightarrow (G *_U X)_F/G$  is necessarily a principal  $G$ -bundle, and if  $f$  denotes the  $U$ -invariant in  $k[X]$  corresponding to  $F$ ,  $X_f \rightarrow X_f/U$  is a  $U$ -torsor. If  $X_f/U$  is affine, then  $(G *_U X)_F/G$  is necessarily affine. Therefore, if  $x$  is a geometric point of  $X_f$ ,  $\iota(x) = [e, x]$  is a geometric point of  $(G *_U X)_F$ . Therefore,  $\iota(x) = [e, x]$  is semi-stable in the sense of Mumford. The result follows then by Lemma 3.6.

Zariski's main theorem shows that we have an open immersion  $i_G : G *_U X/G \hookrightarrow \text{Spec } k[G *_U X/G]$ . By properties of a geometric quotient,  $\text{Spec } k[G *_U X/G] \xrightarrow{\sim} \text{Spec } k[G *_U X]^G$ ; we thus obtain a morphism  $X/U \hookrightarrow \text{Spec } k[X]^U$ , which is *a fortiori* an open immersion because  $i_G$  has the same property. It follows that the scheme theoretic image of  $X$  under  $\pi$  is a variety which is an open subscheme of  $\text{Spec } k[X]^U$ .  $\square$

*Remark 3.10.* Observe that Theorem 3.9 implies that the quotient of an everywhere stable action of a unipotent group  $U$  on a strictly quasi-affine scheme  $X$  (i.e. quasi-affine but not affine) is necessarily strictly quasi-affine. Indeed, the morphism  $X \rightarrow X/U$  is affine and so if  $X/U$  were affine, since the composite of two affine morphisms is affine, this would mean that  $X$  was affine.

*Remark 3.11.* Assume that  $X$  is an  $\mathbb{A}^1$ -contractible smooth affine scheme (e.g.  $\mathbb{A}^n$ ). By Lemma 3.3, if  $X$  admits an everywhere stable action of a unipotent group, the quotient  $X/U$  exists as a scheme and is smooth and  $\mathbb{A}^1$ -contractible. If the quotient  $X/U$  is *not* affine, then it cannot be isomorphic to affine space, and so is an exotic  $\mathbb{A}^1$ -contractible scheme.

Combining the above remarks, if  $X$  is an affine scheme equipped with an everywhere stable  $U$ -action, we would like a criterion which characterizes when the quotient is an affine scheme. There is an effective cohomological criterion for this, which we adapt from a theorem of Greuel and Pfister (see [GP93] Theorem 3.10). To state the result, we need some notation.

Let  $\mathfrak{u}$  denote the Lie algebra corresponding to  $U$ ; it is necessarily nilpotent. The exponential map then defines an *algebraic* isomorphism  $\exp : \mathfrak{u} \xrightarrow{\sim} U$ . Now, specifying a  $U$ -action on an affine  $k$ -scheme  $X$  is equivalent to giving a map  $U \hookrightarrow \text{Aut}_k(X)$ . Such an action determines a Lie algebra homomorphism  $\mathfrak{u} \rightarrow \text{Der}_k(k[X])$ . The image of this map necessarily consists of locally nilpotent derivations. Conversely, a unipotent group action can be specified completely by giving a set of locally nilpotent derivations generating the  $\mathfrak{u}$ -action. The following theorem is extremely useful, but we defer the proof to the appendix (see Section 7).

**Theorem 3.12.** *Suppose  $X$  is an affine scheme equipped with an action of a unipotent group  $U$ . The following conditions are equivalent:*

- i) *The quotient map  $\pi : X \rightarrow X/U$  is a  $U$ -torsor and  $\pi$  induces an isomorphism  $X/U \cong \text{Spec}(k[X]^U)$ , i.e.  $X/U$  is an affine variety.*
- ii) *The Lie algebra cohomology  $H^1(\mathfrak{u}, \Gamma(X, \mathcal{O}_X)) = 0$ .*
- iii) *The quotient map  $X \rightarrow X/U$  is a trivial  $U$ -torsor over  $X/U$ .*

*Example 3.13.* Consider the situation where  $X$  is an affine  $\mathbb{G}_a$ -scheme. Then the condition  $H^1(\mathfrak{g}_a, k[X]) = 0$  can be made more explicit. Let  $D$  be the locally nilpotent derivation defining the action of  $\mathfrak{g}_a$  on  $k[X]$ . Then, the  $\mathfrak{g}_a$ -cohomology of  $k[X]$  can be computed as the cohomology of the Chevalley complex:

$$\text{Hom}(k, k[X]) \xrightarrow{\delta} \text{Hom}(\mathfrak{g}_a, k[X])$$

where the differential  $\delta$  is defined by  $\delta(\omega)(f) = \omega(Df)$ . Because  $H^1(\mathfrak{g}_a, k[X]) = \text{Coker}(\delta)$ , we know that  $H^1(\mathfrak{g}_a, k[X]) = 0$  only if  $\delta$  is surjective. Therefore, viewing  $D$  as a map  $k[X] \rightarrow k[X]$ ,  $H^1(\mathfrak{g}_a, k[X])$  is zero if and only if  $1 \in \text{Im}(D)$ . If  $1 \in \text{Im}(D)$ , an element  $s \in k[X]$  such that  $D(s) = 1$  is sometimes called a *slice*. Assuming  $k[X]^{\mathbb{G}_a}$  is finitely generated, one can give an algorithm to determine whether or not  $1 \in \text{Im}(D)$ .

### Spaces of actions

Before we proceed, let us summarize the state of unipotent group quotients as it now stands for us. Fix an affine scheme  $X$  and a unipotent group  $U$  whose associated Lie algebra is  $\mathfrak{u}$ . Consider the set of locally nilpotent derivations  $\text{Der}_k^{\text{ln}}(k[X])$ ; this set is not in general a  $k$ -vector space. Specifying a  $U$ -action on  $X$  is equivalent to specifying a Lie sub-algebra of  $\text{Der}_k^{\text{ln}}(k[X])$

isomorphic to  $\mathfrak{u}$ . We will pay special attention to the case  $U = \mathbb{G}_{\mathbf{a}}$ , where the set of actions of  $\mathbb{G}_{\mathbf{a}}$  on  $X$  is equivalent to the set  $\text{Der}_k^{ln}(k[X])$ . We have the following schematic.

$$H^1(\mathfrak{u}, k[X]) = 0 \text{ actions} \subset \text{everywhere stable actions} \subset \text{free actions} \subset \text{actions}$$

These inclusions correspond to quotients which are:

$$\begin{aligned} &\text{affine varieties (geometric quotient)} \subset \text{quasi-affine varieties (geometric quotient)} \\ &\subset \text{algebraic spaces (geometric quotient)} \subset \text{categorical quotient needn't exist} \end{aligned}$$

What is more, each inclusion above is strict, in that we can give examples of each. Indeed, in principle there are “algorithms” to detect which case holds. The rest of the paper will be devoted to studying the first two inclusions, so let us give examples of the other two; indeed, to show how widespread these phenomena are, we give examples for  $\mathbb{G}_{\mathbf{a}}$ -actions on affine space  $\mathbb{A}^5$ .

*Example 3.14.* We now construct a free action of  $\mathbb{G}_{\mathbf{a}}$  on an affine variety, in fact  $\mathbb{A}^5$ , for which a quotient exists only as an algebraic space. The rough idea, which we do not justify here, is to find a  $\mathbb{G}_{\mathbf{a}}$ -invariant subvariety  $X$  of a linear  $\mathbb{G}_{\mathbf{a}}$ -representation  $W$  such that (i) the action is free, (ii) it has an open subset of stable points and (iii) it has a positive dimensional subset of unstable points (i.e., points where all homogeneous invariants vanish). All of these can be explicitly checked. Here is one such construction, which recovers the example due to Deveney and Finston [DF95].

Let  $V$  be the 2-dimensional representation of  $\mathbb{G}_{\mathbf{a}}$  defined by the usual embedding of  $\mathbb{G}_{\mathbf{a}}$  into  $SL_2$  as strictly lower triangular matrices with ones along the diagonal. Let  $W = V \oplus V \oplus \text{Sym}^3 V \cong \mathbb{A}^8$  with coordinates  $w_1, \dots, w_8$ . Let  $X \cong \mathbb{A}^5$  be presented as a codimension 3 closed subvariety with defining equations  $w_5 = 2w_1w_3^2, w_6 = 2w_1w_3w_4$ , and  $w_7 = 1 + w_1w_4^2$ . One can check the action on  $X$  is proper and set-theoretically free; hence, it is free. It has a 3-dimensional subspace, determined by  $w_1 = w_3 = 0$ , in the “unstable” locus.

*Example 3.15.* Let  $V$  be the 2-dimensional representation of  $\mathbb{G}_{\mathbf{a}}$  defined by the usual embedding of  $\mathbb{G}_{\mathbf{a}}$  into  $SL_2$  as strictly lower triangular matrices with ones along the diagonal. Consider the  $\mathbb{G}_{\mathbf{a}}$ -action on  $\text{Sym}^4 V \cong \mathbb{A}^5$ . Then a categorical quotient of  $\text{Sym}^4 V$  by the  $\mathbb{G}_{\mathbf{a}}$ -action does not exist. In particular, there are not enough invariants to separate any closed orbits occurring in certain families. So the “quotient” with respect to invariants turns out to just be a constructible set, albeit sitting in a slightly larger canonical quasi-affine variety. If a categorical quotient  $Y$  of  $X$  by a group  $U$  exists, then the quotient morphism  $X \rightarrow Y$  must be surjective; thus here a categorical quotient cannot exist. See [DK] Section 6 for details (in the projective case, although the arguments are the same). Note that if  $G$  is a reductive group and  $X$  is an affine  $G$ -scheme, then while similar “dimensional collapsing” phenomena still occur, the morphism  $X \rightarrow \text{Spec } k[X]^G$  is always surjective, so this constructible set problem never occurs.

In summary, the following corollary of the above theorems is computationally effective:

**Corollary 3.16.** *Suppose a unipotent group  $U$  acts freely on a finite type smooth affine scheme  $X$ .*

- i) The algebraic space quotient  $X/U$  is a smooth quasi-affine scheme if and only if the action of  $U$  on  $X$  is everywhere stable.
- ii) The algebraic space  $X/U$  is an affine scheme if and only if  $H^1(\mathfrak{u}, k[X]) = 0$  in which case  $k[X/U] = \text{Spec } k[X]^U$ , i.e.  $X \rightarrow \text{Spec } k[X]^U$  is surjective; and  $k[X]^U$  is necessarily finitely generated.
- iii) Assume further that the action of  $U$  on  $X$  is everywhere stable. If  $H^1(\mathfrak{u}, k[X])$  is non-zero, then the smooth scheme  $X/U$  has complement of codimension  $\geq 2$  in  $\text{Spec } k[X]^U$ .
- iv) Given a unipotent group  $U$  and two everywhere stable actions of  $U$  on  $X$ , the resulting quotients  $X/U$  are  $\mathbb{A}^1$ -weakly equivalent. If  $X$  is an  $\mathbb{A}^1$ -contractible variety, then  $X/U$  is an  $\mathbb{A}^1$ -contractible variety. If  $H^1(\mathfrak{u}, k[X])$  is non-trivial, then  $X/U$  is not isomorphic to affine space.

*Proof.* The only statement that doesn't follow immediately from Theorems 3.9, 3.12 and Lemmas 3.3 and 2.4 is (iii). Again by Theorem 3.12 (i) and 3.9 (i) we can assume that the morphism  $j : X/U \rightarrow \text{Spec } k[X]^U$  is an open embedding and not an isomorphism.

Note that since  $X$  is smooth and finite type we know that  $k[X]$  is normal. Whether or not  $k[X]^U$  is finitely generated, it is equal to the intersection of  $k[X]$  and  $k(X)^U$  in  $k(X)$  and is hence by [Bou98] Example VII.1.3 (4) a Krull domain. By [Bou98] VII.1.6 Theorem 4,  $k[X]^U$  is therefore equal to the intersection of its localizations at height 1 prime ideals.

We will show that pull-back  $j^* : k[X]^U \rightarrow \Gamma(X/U, \mathcal{O}_{X/U})$  is a  $k$ -algebra isomorphism if and only if  $X/U$  has complement of codimension  $\geq 2$  in  $\text{Spec } k[X]^U$ . If  $X/U$  has complement of codimension  $\geq 2$  in  $\text{Spec } k[X]^U$ , any element  $f \in \Gamma(X/U, \mathcal{O}_{X/U})$  uniquely to  $k[X]^U$ . Indeed, this follows because any such  $f$  can be viewed as an element of every localization of  $k[X]^U$  at a height 1 prime ideal. Conversely, assume  $X/U$  has complement of codimension 1. This means that the complement of  $X/U$  in  $\text{Spec } k[X]^U$  contains a non-trivial height 1 prime ideal and hence, there exists an element  $f \in \Gamma(X/U, \mathcal{O}_{X/U})$  not an element of  $k[X]^U$ .

Now, just as in the proof of Theorem 3.12, since the quotient morphism is a faithfully flat and affine morphism, and  $X$  is affine, we can identify  $H^i(X/U, \mathcal{O}_{X/U})$  with  $H^i(U, \Gamma(X, \mathcal{O}_X))$  (again use the Čech resolution just prior to the proof of Theorem 3.12). By taking  $i = 0$ , this identification gives an isomorphism  $\Gamma(X/U, \mathcal{O}_{X/U}) \cong k[X]^U$ . Combining this with the discussion of the previous paragraph we see that  $X/U$  always has complement of codimension  $\geq 2$  in  $\text{Spec } k[X]^U$ .

□

## 4 Quotients of invariant hypersurfaces

In this section, we focus on the study of quotients of  $\mathbb{G}_a$ -invariant hypersurfaces in affine spaces of the form  $\mathbb{A}(W)$  where  $W$  carries a  $k$ -rational representation of  $\mathbb{G}_a$ ; we will say that the corresponding  $\mathbb{G}_a$ -action on  $\mathbb{A}(W)$  is *linear* (and similarly for more general groups). Henceforth, we abuse notation and write  $W$  for the affine space  $\mathbb{A}(W)$ . In this section we give a geometric characterization (see Theorem 4.11) of when a  $\mathbb{G}_a$ -invariant hypersurface  $X$  in  $W$  admits the structure of a  $\mathbb{G}_a$ -torsor over an affine or strictly quasi-affine base  $X/\mathbb{G}_a$ . Furthermore, we characterize the corresponding form of the polynomial  $f$  defining  $X$ ; all of the information is encoded in invariant and covariant theory for  $SL_2$  (see Theorem 4.20). As a simple consequence we will see that  $k[X]^{\mathbb{G}_a}$  is finitely generated (see Corollary 4.13) in this situation; however, for  $X$  of higher codimension, we note this finite generation statement fails (see Remark 4.14). In the case of strictly quasi-affine  $X/\mathbb{G}_a$  we determine the boundary locus in  $\text{Spec } k[X]^{\mathbb{G}_a}$ .

### Constructing everywhere stable actions

Let us first describe the method by which we shall produce varieties with everywhere stable actions of unipotent groups. Suppose  $U$  is a unipotent group and  $G$  is any reductive group for which  $U$  is a closed subgroup. Note that any affine  $U$ -variety  $X$  embeds as a closed subscheme of a finite dimensional  $k$ -rational  $U$ -representation  $W$ . As the embedding  $f : X \rightarrow W$  is  $U$ -equivariant and closed, it follows from Corollary 7.5 that the induced morphism

$$G *_U f : G *_U X \rightarrow G *_U W$$

is a closed immersion as well.

It can be shown that every everywhere stable affine  $U$ -variety can be  $U$ -equivariantly embedded as a closed subscheme of the scheme  $W^s$  for some finite dimensional  $U$ -representation  $W$ , but as we will not use this fact, we do not prove it here. Instead, the following lemma is our main tool in construction of everywhere stable actions.

**Lemma 4.1.** *Suppose  $W$  is a linear  $U$ -representation and  $X$  is a quasi-affine  $U$ -stable subscheme of  $W$  contained in  $W^s$ . Then the  $U$ -action on  $X$  is everywhere stable.*

*Proof.* By Corollary 7.5, the  $U$ -equivariant quasi-affine morphism  $f : X \hookrightarrow W$  induces a quasi-affine morphism  $G *_U f : G *_U X \hookrightarrow G *_U W$  (which is also locally closed). By the theorem hypothesis together with definition 3.4, we see that  $G *_U X \hookrightarrow (G *_U W)^s$  where  $(G *_U W)^s$  is the set of stable points for the  $G$ -action on  $G *_U W$ . We may then apply Proposition 1.18 of [MFK94] to conclude that the intersection  $G *_U X \cap (G *_U W)^s$  is contained in the locus  $(G *_U X)^s$ . We can identify the first intersection with  $G *_U (W^s \cap X)$  by the definition of stability for  $U$ . By assumption,  $W^s \cap X = X$  and hence  $X = X^s$  so that the  $U$ -action on  $X$  is everywhere stable by Definition 3.8.  $\square$

We will henceforth abuse terminology and refer to any  $U$ -stable quasi-affine subvariety  $X$  of  $W^s$  for a linear  $U$ -representation  $W$  as an *everywhere stable subvariety of  $W$* . In the sequel, we will be interested in *everywhere stable hypersurfaces*, i.e.,  $U$ -stable codimension 1 closed subvarieties of  $W$  contained in  $W^s$ .

## Linear Representations

Consider  $\mathbb{G}_{\mathbf{a}}$  as a closed subgroup of  $SL_2$  via the homomorphism defined by sending  $x \mapsto \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$ . Given a  $k$ -rational representation  $W$  of  $\mathbb{G}_{\mathbf{a}}$ , linearity has the following extremely useful consequence.

**Theorem 4.2** (Jacobsen-Morozov (see e.g. [AKO02] Thm. 19.5.1)). *Given a linear  $\mathbb{G}_{\mathbf{a}}$ -representation on a vector space  $W$ , i.e. a morphism  $\rho : \mathbb{G}_{\mathbf{a}} \rightarrow GL(W)$ , there exists a factorization of  $\rho$  as the composite  $\mathbb{G}_{\mathbf{a}} \hookrightarrow SL_2 \xrightarrow{\rho'} GL(W)$ . Briefly, we will say that any linear  $\mathbb{G}_{\mathbf{a}}$ -representation  $\rho$  extends to a linear representation of  $SL_2$ .*

*Remark 4.3.* If the  $\mathbb{G}_{\mathbf{a}}$ -action on  $W$  admits, in Jordan canonical form, a single Jordan block, then this extension is canonical (once given the closed embedding  $\mathbb{G}_{\mathbf{a}} \hookrightarrow SL_2$  as above). If there are multiple blocks, then there is a choice of relative scaling amongst the blocks for the action of the maximal torus  $\mathbb{G}_{\mathbf{m}} \subset SL_2$  of diagonal matrices. It is important to note that the above extension induces a bijection between the isomorphism classes of indecomposable representations of  $\mathbb{G}_{\mathbf{a}}$  and  $SL_2$ . Fix a  $\mathbb{G}_{\mathbf{a}}$ -representation  $W$  and any pair of extensions to  $SL_2$  representations. The resulting  $SL_2$ -representations are necessarily isomorphic, as they have the same underlying sets of irreducibles, but this isomorphism is non-canonical. For a much more complete discussion of functoriality in the Jacobsen-Morozov theorem, see [AKO02] §19 and especially Thm. 19.5.1.

As in Section 3, consider the inclusion  $\iota : W \hookrightarrow SL_2 *_{\mathbb{G}_{\mathbf{a}}} W$ . Since the  $\mathbb{G}_{\mathbf{a}}$ -action on  $W$  extends to a  $SL_2$ -action, the product of the projection  $SL_2 *_{\mathbb{G}_{\mathbf{a}}} W \rightarrow SL_2/\mathbb{G}_{\mathbf{a}}$  and the action map  $SL_2 *_{\mathbb{G}_{\mathbf{a}}} W \rightarrow W$  defines a canonical isomorphism  $SL_2 *_{\mathbb{G}_{\mathbf{a}}} W \cong SL_2/\mathbb{G}_{\mathbf{a}} \times W$ . Furthermore,  $SL_2/\mathbb{G}_{\mathbf{a}}$  is the complement of 0 in the affine space  $V$  where  $V$  is the standard representation of  $SL_2$ . The point  $e \in SL_2/\mathbb{G}_{\mathbf{a}}$  then corresponds to the vector  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$  in  $V$ .

**Lemma 4.4.** *The composite map*

$$W \hookrightarrow SL_2/\mathbb{G}_{\mathbf{a}} \times W \hookrightarrow V \times W.$$

*determines (by pull-back) an isomorphism  $k[V \times W]^{SL_2} \xrightarrow{\sim} k[W]^{\mathbb{G}_{\mathbf{a}}}$ . In addition,  $k[W]^{\mathbb{G}_{\mathbf{a}}}$  is a finitely generated  $k$ -algebra.*

*Proof.* The inclusion  $SL_2/\mathbb{G}_{\mathbf{a}} \hookrightarrow V$  has complement of codimension 2, hence  $SL_2/\mathbb{G}_{\mathbf{a}} \times W$  has complement of codimension 2 in  $V \times W$ . By normality of  $V \times W$ , restriction determines an isomorphism of  $k$ -algebra  $k[V \times W] \xrightarrow{\sim} k[SL_2/\mathbb{G}_{\mathbf{a}} \times W]$ , and hence gives an isomorphism after taking  $SL_2$ -invariants. Finally, we get an isomorphism  $k[W]^{\mathbb{G}_{\mathbf{a}}} \cong k[SL_2/\mathbb{G}_{\mathbf{a}} \times W]^{SL_2}$  by transfer (see §7). To conclude we note that  $k[V \times W]^{SL_2}$  is finitely generated by Nagata's theorem on finite generation of rings of invariants under reductive group actions.  $\square$

We now discuss stability in  $W$ . By Proposition 3.7, stability for the  $\mathbb{G}_{\mathbf{a}}$ -action on  $W$  is independent of the factorization  $\rho'$  of  $\rho : \mathbb{G}_{\mathbf{a}} \rightarrow GL(W)$  through  $SL_2$ . Therefore, we may extend the  $\mathbb{G}_{\mathbf{a}}$ -representation on  $W$  to an  $SL_2$ -representation in such a way that the maximal

torus of diagonal matrices  $\mathbb{G}_m \subset SL_2$  acts with equal weights on each indecomposable summand of the representation  $W$ .

Note that the stable set for the  $SL_2$ -action on  $V \times W$  can then be determined with the Hilbert-Mumford numerical criterion (see [MFK94] Theorem 2.1). In fact, because the inclusion  $SL_2/\mathbb{G}_a \times W \hookrightarrow V \times W$  is a quasi-affine morphism, it follows from Proposition 1.18 of [MFK94] that the geometric points of  $(SL_2/\mathbb{G}_a \times W) \cap (V \times W)^s$  are a subset of the geometric points of  $(SL_2/\mathbb{G}_a \times W)^s$ . This distinguishes a subset of stable points on  $W$  via restriction from the closed immersion  $\iota : W \rightarrow SL_2/\mathbb{G}_a \times W$ . Again, this set of geometric points necessarily underlies an open subscheme of  $W$  which we denote by  $W^{\bar{s}}$ . This definition, coupled with a Hilbert-Mumford criterion computation, yields the following Lemma, which is sufficient for the uses of stability of this paper.

**Lemma 4.5.** *The set  $W^{\bar{s}}$  is an open subscheme of  $W^s$ . It is the complement in  $W$  of the union of linear coordinate subspaces defined by the vanishing of any coordinate with positive weight relative to the action of the central torus  $\mathbb{G}_m$  of  $SL_2$ .*

Suppose that  $X$  is a closed  $\mathbb{G}_a$ -stable subvariety of  $W$ . Then we obtain an induced morphism  $SL_2 *_{\mathbb{G}_a} X \hookrightarrow SL_2 *_{\mathbb{G}_a} W$  such that the following diagram commutes:

$$\begin{array}{ccc} X & \longrightarrow & SL_2 *_{\mathbb{G}_a} X \\ \downarrow & & \downarrow \\ W & \longrightarrow & SL_2 *_{\mathbb{G}_a} W \xrightarrow{\sim} SL_2/\mathbb{G}_a \times W. \end{array}$$

In particular, we can view  $SL_2 *_{\mathbb{G}_a} X$  as a locally closed subvariety of  $V \times W$ .

**Lemma 4.6.** *Suppose  $G$  is any linear algebraic group with trivial character group (e.g.  $G = \mathbb{G}_a$  or  $SL_2$ ). Any  $G$ -invariant hypersurface  $X \subset W$  is actually defined by a  $G$ -invariant polynomial.*

*Proof.* Let  $G$  be any linear algebraic group whose character group is trivial. We know that any height 1 prime ideal in  $k[W]$  is principal, so choosing a generator  $f$  gives a defining equation for  $X$ . Now,  $G$  acts on  $X$  and hence the ideal defining  $X$  is actually  $G$ -stable. Furthermore, the ideal sheaf of a codimension 1 scheme is actually an invertible sheaf which we denote  $\mathcal{O}(f)$ . Since  $Pic(W)$  is trivial, we note that  $\mathcal{O}(f)$  is isomorphic to  $\mathcal{O}_W$ . Since the character group of  $G$  is trivial, the line bundle  $\mathcal{O}(f)$  admits a unique  $G$ -equivariant structure. This means that the module of sections  $f \cdot k[W]$  admits a unique  $G$ -invariant structure. It follows that the action of  $G$  on  $f$  is trivial and hence  $f$  is actually invariant.  $\square$

By the transfer isomorphism of Lemma 4.4, any  $\mathbb{G}_a$ -invariant polynomial  $f$  on  $W$  determines an  $SL_2$ -invariant polynomial  $F$  on  $V \times W$ . Henceforth, we will use lower case Roman letters to denote  $\mathbb{G}_a$ -invariants in  $W$  and capital Roman letters to denote the corresponding  $SL_2$ -invariants in  $V \times W$ . We now discuss the relation between the geometry of the hypersurfaces defined by the vanishing of each of these polynomials.

**Lemma 4.7.** *The  $\mathbb{G}_a$ -invariant  $f$  is irreducible if and only if the associated  $SL_2$ -invariant  $F$  is irreducible. If  $X$  is irreducible and everywhere stable,<sup>4</sup> then the closure  $\overline{SL_2 *_{\mathbb{G}_a} X} \subset V \times W$  is given by  $F = 0$ .*

*Proof.* Note that if the vanishing locus of  $f$  is a reduced scheme, then the vanishing locus of  $F$  has the same property. Now, for the forward implication, suppose  $f$  is irreducible and  $F$  is not, and let  $F_1$  and  $F_2$  be distinct irreducible factors of  $F$ . Then the vanishing locus of each  $F_i$  defines an  $SL_2$ -invariant hypersurface  $X_i$  in  $V \times W$ . The hypersurface  $X_i$  is smooth and so all Weil divisors are Cartier; because the character group of  $SL_2$  is trivial, the  $F_i$  must be  $SL_2$ -invariant, as per the argument in Lemma 4.6. By the transfer isomorphism we then have  $f = f_1 f_2$ , which is a contradiction.

Similarly, for the reverse direction, if  $F$  is irreducible but we can write  $f = f_1 f_2$ , for non-constant  $f_i$ , then by the proof of Lemma 4.6 both  $f_1$  and  $f_2$  are  $\mathbb{G}_a$ -invariants. Again by the transfer isomorphism this means  $F = F_1 F_2$ , which is a contradiction.

If  $X$  is irreducible and stable then  $X/\mathbb{G}_a$  is irreducible, since  $X/\mathbb{G}_a$  is a categorical quotient and a categorical quotient of an irreducible variety is irreducible. The total space of the  $SL_2$ -principal bundle  $SL_2 *_{\mathbb{G}_a} X \rightarrow X/\mathbb{G}_a$  is irreducible as it contains a dense, Zariski open subset isomorphic to  $V \times SL_2$  where  $V$  is an open subset of  $X/\mathbb{G}_a$  and hence irreducible by irreducibility of  $X/\mathbb{G}_a$ .

Then  $\overline{SL_2 *_{\mathbb{G}_a} X}$  is the closure of an irreducible variety, and must be a codimension 0 closed subvariety of  $F = 0$ , i.e., a maximal dimensional irreducible component of  $F = 0$ . But by the first part of this lemma,  $F = 0$  is itself irreducible.  $\square$

Choose coordinates  $u, v$  on  $V$ , and let  $w_1, \dots, w_n$  be coordinates on  $W$ . Then  $F = \sum_{i,j} F_{i,j} u^i v^j$  where the  $F_{i,j}$  are regular functions of  $\{w_1, \dots, w_n\}$ . Note that  $f$  can be recovered by restricting  $F$  to the subvariety defined by setting  $u = 1$  and  $v = 0$ .

**Definition 4.8.** We will say that  $X, f$ , or  $F$  *misses the boundary* if  $F_{0,0}$  is a non-zero constant. We will say  $X, f$ , or  $F$  *contains the boundary* if  $F_{0,0} = 0$ . Otherwise we will say  $X, f$ , or  $F$  *intersects the boundary*.

*Remark 4.9.* Of course, this terminology is shorthand for how the geometric points of  $\overline{SL_2 *_{\mathbb{G}_a} X}$  relate to the geometric points of the “boundary”  $\{0\} \times W$  in  $V \times W$ . We can use geometric statements about intersections with the boundary over  $\bar{k}$  together with Galois descent to deduce which one of the three cases in the definition holds. In the sequel we will do this without further formal justification.

By Lemma 4.7,  $\overline{SL_2 *_{\mathbb{G}_a} X} \subset V \times W$  is the hypersurface with defining equation  $F = 0$ . Note that because  $F$  is  $SL_2$ -invariant and the  $SL_2$  action restricts to  $\{0\} \times W$ , it follows that  $F_{0,0}$  is  $SL_2$ -invariant. Thus there is a natural way to decompose  $f$ , namely  $f = F_{0,0} + g$ , a sum of an  $SL_2$ -invariant function (*a priori* possibly zero) and a  $\mathbb{G}_a$ -invariant function that contains the boundary. Everywhere stability of  $X$  constrains  $f$  further.

<sup>4</sup>The stability of  $X$  may not be necessary here.

**Lemma 4.10.** *Let  $X$  be everywhere stable. Then  $f = F_{0,0} + g$ , where  $F_{0,0}$  is an  $SL_2$ -invariant not vanishing at the origin (i.e., with a nonzero constant term), and  $g$  is a  $\mathbb{G}_a$ -invariant that contains the boundary. In particular  $f$  does not contain the boundary.*

*Proof.* By the preceding discussion, all that need be shown is that  $F_{0,0}$  has a non-zero constant term. But if it did not, then the origin in  $W$ , which is not stable for any representation (indeed, all homogeneous invariants automatically vanish there), would be a solution to  $f = 0$  and hence would be a point of  $X$ .  $\square$

**Theorem 4.11** (Geometric characterization I). *Let  $X$  be an everywhere stable hypersurface in a linear representation  $W$ . Then the quotient  $X/\mathbb{G}_a$  is affine if and only if  $X$  misses the boundary as per Definition 4.8. Otherwise,*

- i)  $X$  intersects the boundary and,
- ii)  $X/\mathbb{G}_a$  is strictly quasi-affine and can be realized as an open subset of an affine variety with complement of codimension  $\geq 2$ .

*Proof.* Note that it suffices to prove the result under the following two assumptions: 1)  $X$  is irreducible and 2)  $k$  is algebraically closed. For the second statement, we note that stability is preserved under base-change to the algebraic closure (see [MFK94] Proposition 1.14; also see Remark 4.9).

Now, if  $F$  misses the boundary, then  $SL_2 *_{\mathbb{G}_a} X$  is a closed affine subvariety of  $V \times W$ . Because  $SL_2$  is reductive, the quotient  $SL_2 *_{\mathbb{G}_a} X / SL_2 = X/\mathbb{G}_a$  is affine.

Conversely, assume  $X/\mathbb{G}_a$  is affine yet  $F$  intersects the boundary. Since  $X$  is everywhere stable  $F_{0,0}$  is non-constant. In particular, the intersection  $F_{0,0} \cap \{0\} \times W$  has codimension two in  $\{F = 0\} = \overline{SL_2 *_{\mathbb{G}_a} X}$ . Let

$$\pi : \overline{SL_2 *_{\mathbb{G}_a} X} \longrightarrow \overline{SL_2 *_{\mathbb{G}_a} X} // SL_2$$

denote the categorical quotient map. Then, again because  $SL_2$  is reductive, the image of  $\pi$  is an affine variety. Let  $B$  denote the complement of  $SL_2 *_{\mathbb{G}_a} X$  in  $\overline{SL_2 *_{\mathbb{G}_a} X}$ . By upper-semi-continuity of the dimension of the fibers of  $\pi$  (see e.g. [Bor91] Corollary 10.3), the scheme theoretic image of  $B$  under  $\pi$  must have codimension at least two in the quotient  $\overline{SL_2 *_{\mathbb{G}_a} X} // SL_2$ . Furthermore,  $\pi(B)$  is disjoint from  $\pi(SL_2 *_{\mathbb{G}_a} X)$  because  $SL_2 *_{\mathbb{G}_a} X \subset \overline{SL_2 *_{\mathbb{G}_a} X}^s$ . Consequently,  $SL_2 *_{\mathbb{G}_a} X / SL_2 = X/\mathbb{G}_a$  is an open subset of  $\overline{SL_2 *_{\mathbb{G}_a} X} // SL_2$  with complement codimension at least two. Hence  $X/\mathbb{G}_a$  is a strictly quasi-affine subvariety of an affine variety with complement of codimension  $\geq 2$ .  $\square$

*Remark 4.12.* We know by Corollary 3.16 that  $X/\mathbb{G}_a$  is affine if and only if  $H^1(\mathfrak{g}_a, k[X]) = 0$ . Equivalently, by Example 3.13, if  $D$  denotes the locally nilpotent derivation defining the  $\mathfrak{g}_a$ -action on  $k[X]$ , we know that  $1 \in \text{Im}(D)$ . In practice, to determine whether  $1 \in \text{Im}(D)$ , one first needs a generating set for  $k[X]^{\mathbb{G}_a}$ . If we know in advance that  $X/\mathbb{G}_a$  is affine, then in principle one can compute such a generating set (see Example 3.13 and references therein).

Although all  $\mathbb{G}_a$ -invariants of  $W$  restrict to  $\mathbb{G}_a$ -invariants of  $X$ , not all  $\mathbb{G}_a$ -invariants of  $X$  need extend to invariants of  $W$ . So one cannot *a priori* inherit the generating set of invariants

from  $W$ . Nevertheless the next result, Corollary 4.13, shows that at least assuming a normality hypothesis, the invariants do extend.

**Corollary 4.13.** *Suppose  $X$  is a normal, everywhere stable, hypersurface in  $W$ . Then the ring of invariants  $k[X]^{\mathbb{G}_a}$  is finitely generated, and in fact, all elements of  $k[X]^{\mathbb{G}_a}$  arise as restrictions of elements of  $k[V \times W]$ .*

*Proof.* First, we show that if  $X$  is normal, then so is  $SL_2 *_{\mathbb{G}_a} X$ . To see this, note that the morphism  $\pi : SL_2 *_{\mathbb{G}_a} X \rightarrow SL_2/\mathbb{G}_a$  is a Zariski locally trivial fiber bundle with fibers isomorphic to  $X$ . Now, normality is a local property for the Zariski topology, so picking an affine open cover of  $\{V_i\}_{i \in I}$  of  $X$  over which  $\pi$  trivializes, we are reduced to showing that  $V_i \times X$  is normal, but this follows because each  $V_i$  is smooth.

Now, consider the inclusion  $SL_2 *_{\mathbb{G}_a} X \hookrightarrow \overline{SL_2 *_{\mathbb{G}_a} X}$ . By Lemma 4.7,  $\overline{SL_2 *_{\mathbb{G}_a} X}$  is defined as the vanishing locus of  $F$  in  $V \times W$ . Denote by  $\widehat{SL_2 *_{\mathbb{G}_a} X}$  the normalization of  $\overline{SL_2 *_{\mathbb{G}_a} X}$ . Since  $\widehat{SL_2 *_{\mathbb{G}_a} X}$  is affine and normal, the GIT quotient  $\psi : \widehat{SL_2 *_{\mathbb{G}_a} X} \rightarrow \widehat{SL_2 *_{\mathbb{G}_a} X} // SL_2$  is affine and normal. We now have the commutative diagram

$$\begin{array}{ccccc}
 & & SL_2 *_{\mathbb{G}_a} X // G & \longrightarrow & \widehat{SL_2 *_{\mathbb{G}_a} X} // G \\
 & \nearrow q & & \searrow \hat{q} & \downarrow \\
 SL_2 *_{\mathbb{G}_a} X & \longrightarrow & \widehat{SL_2 *_{\mathbb{G}_a} X} & & \overline{SL_2 *_{\mathbb{G}_a} X} // G \\
 & \searrow & \downarrow & \nearrow \bar{q} & \\
 & & \overline{SL_2 *_{\mathbb{G}_a} X} & & 
 \end{array}$$

Because  $X$  is everywhere stable  $X/\mathbb{G}_a$  is isomorphic to an open subscheme of  $\overline{SL_2 *_{\mathbb{G}_a} X} // SL_2$  with complement of codimension at least 2, as argued in the proof of Theorem 4.11. Furthermore  $X/\mathbb{G}_a$ , being a geometric and hence categorical quotient of a normal variety, is itself normal. Now, normalization is the identity on normal varieties hence  $X/\mathbb{G}_a$  is an open subset of  $\widehat{SL_2 *_{\mathbb{G}_a} X} // SL_2$ . Furthermore, normalization is a finite map, so the inverse image of a codimension  $\geq 2$  subvariety has again codimension  $\geq 2$ ; consequently,  $X/\mathbb{G}_a$  has complement codimension  $\geq 2$  in  $\widehat{SL_2 *_{\mathbb{G}_a} X} // SL_2$ . By normality, all functions on  $X/\mathbb{G}_a$  extend uniquely to functions on  $\widehat{SL_2 *_{\mathbb{G}_a} X} // SL_2$ , which being affine implies  $k[X/\mathbb{G}_a]$  is finitely generated. By the properties of a geometric (indeed, categorical) quotient,  $k[X/\mathbb{G}_a] = k[X]^{\mathbb{G}_a}$ , so  $k[X]^{\mathbb{G}_a}$  is finitely generated.

Note that if  $F = 0$  is already a normal variety, then all of the  $SL_2$ -invariant functions on  $SL_2 *_{\mathbb{G}_a} X$  extend to  $SL_2$ -invariant functions on  $\overline{SL_2 *_{\mathbb{G}_a} X}$  which then extend, by reductivity, to  $SL_2$ -invariant functions on  $V \times W$ . Hence, in this case,  $k[X]^{\mathbb{G}_a} \cong k[W]^{\mathbb{G}_a} / (I(X) \cap k[W]^{\mathbb{G}_a})$ .  $\square$

*Remark 4.14.* If  $X$  were of higher codimension in  $W$  then the conclusion of Corollary 4.13, namely that  $k[X]^{\mathbb{G}_a}$  need be finitely generated, does not necessarily hold. We now produce an explicit counter-example in the situation where  $X$  is of codimension  $\geq 3$  in  $W$ . Take an action of  $\mathbb{G}_a$  on an affine variety  $X$  with non-finitely generated ring of invariants. Pick a  $\mathbb{G}_a$ -equivariant

embedding  $X \hookrightarrow W$ . Consider  $SL_2$  as a hypersurface, necessarily  $\mathbb{G}_a$ -invariant, given by the defining equation  $\det(x) = 1$  in  $End(V)$  (where  $V$  is the 2-dimensional representation of  $SL_2$ ). Note that this hypersurface is everywhere stable. Then  $SL_2 \times X$  viewed as a closed subvariety of  $End(V) \times W$  is everywhere stable as well and of codimension at least 2.

By transfer,  $k[X]^{\mathbb{G}_a} = k[SL_2 *_{\mathbb{G}_a} X]^{SL_2}$ . If  $k[SL_2 *_{\mathbb{G}_a} X]$  were finitely generated then by Nagata's theorem  $k[SL_2 *_{\mathbb{G}_a} X]^{SL_2}$  would be finitely generated as well. Since  $SL_2 \times X$  is everywhere stable,  $k[SL_2 \times X]^{\mathbb{G}_a}$  is necessarily isomorphic to  $k[SL_2 *_{\mathbb{G}_a} X]$ , but this is not finitely generated.

Consider the example due to Daigle-Freudentberg of a  $\mathbb{G}_a$  action on  $\mathbb{C}^5$  with a non-finitely generated ring of invariants (see [DF99]). It is easy to recover their example as the  $\mathbb{G}_a$  action on the closed subvariety  $X$  in  $W = \text{Sym}^3(V) \oplus V \oplus k$ , with coordinates  $w_1, \dots, w_7$ , defined by the equations  $w_7^3 = w_1, w_7^2 = w_5$ . Thus  $X \times SL_2$  can be presented as a codimension 3 closed invariant subvariety in  $\text{Sym}^3(V) \oplus V^{\oplus 3} \oplus I$ , which by the preceding argument cannot have a finitely generated ring of invariants.

In fact,  $X$  being everywhere stable imposes further constraints. Geometrically, a “translate” of  $X$  must contain all of the non-stable points of  $W$ . Let  $c(f)$  denote the constant term of  $f$  (relative to the natural grading on  $k[W]$  preserved by the linear action of  $\mathbb{G}_a$ ).

**Lemma 4.15.** *Let  $X$  be everywhere stable. Let  $f' = f - c(f)$ . Then  $W_{f'} \subset W^s$ .*

*Proof.* In fact we prove the *a priori* stronger statement that  $W_{f'} \subset W^s$ . Assume that some  $w \in W$  is simultaneously not in  $W^s$  and does not satisfy  $f' = 0$ . Let  $Y$  denote an irreducible component of  $W \setminus W^s$  containing  $w$ . Any such  $Y$  is a linear subspace of  $W$ , so in particular contains the origin. But  $f'$  vanishes at the origin. Thus  $f'$  is non-constant on  $Y$  because  $f'(w)$  is non-zero. Since  $Y$  is a closed and hence affine subvariety of  $W$ ,  $f'$  takes every value in  $k$ , in particular  $c(f)$ . Thus  $f(y) = 0$  for some  $y \in Y$ , which violates the definition of everywhere stable.  $\square$

**Theorem 4.16** (Geometric characterization II). *Suppose  $X$  is an everywhere stable hypersurface in a linear representation  $W$  with defining polynomial  $f$ . Then the quotient  $X/\mathbb{G}_a$  is an affine variety if and only if  $W_{f'}/\mathbb{G}_a$  is affine, where  $f' = f - c(f)$  for  $c(f)$  the uniquely defined constant term of  $f$ . Note that  $X \subset W_{f'}$ .*

*Proof.* By assumption,  $X$  is contained in  $W^s$ . But we also know  $X \subset W_{f'}$  and  $W_{f'}$  is contained in  $W^s$  by Lemma 4.15. Therefore, a principal bundle quotient  $W_{f'} \longrightarrow W_{f'}/\mathbb{G}_a$  exists as a scheme.

Now, suppose  $W_{f'}/\mathbb{G}_a$  is affine. Since  $X \subset W_{f'}$  is a  $\mathbb{G}_a$ -invariant closed subscheme, it follows that  $X/\mathbb{G}_a$  is closed in  $W_{f'}/\mathbb{G}_a$  (since the  $X/\mathbb{G}_a$  is a geometric quotient). Therefore,  $X/\mathbb{G}_a$  is affine.

Conversely, assume that  $X/\mathbb{G}_a$  is an affine variety. Since  $X$  is everywhere stable, the defining polynomial  $f$  must have non-zero constant term (the origin is unstable). If  $f' = f - c(f)$ , by Lemma 4.15 we know that  $W_{f'} \subset W^s$ . Therefore, it suffices to show that  $f$  does not intersect the boundary. If  $f$  misses the boundary, then  $f'$  contains the boundary and so  $W_{f'}$  must miss the boundary. Therefore,  $W_{f'}/\mathbb{G}_a$  must be affine by Theorem 4.11.  $\square$

We now give a somewhat “geometric” interpretation for a function that contains the boundary.

**Lemma 4.17.** *Let  $g$  be a  $\mathbb{G}_a$ -invariant function on  $W$ . If  $g$  contains the boundary then  $W_g \subset W^s$ .*

*Proof.* Again we prove the *a priori* stronger statement that  $W_g \subset W^s$ . By Lemma 4.4,  $g$  corresponds to an  $SL_2$ -invariant function  $G$  on  $V \times W$ . Recall that we used the notation  $u, v$  for the coordinate functions on  $V$ , with  $u$  having positive weight and  $v$  negative weight for the action of the torus  $\mathbb{G}_m$  contained in the normalizer of  $\mathbb{G}_a$  in  $SL_2$ . Because  $g$  contains the boundary, each term of  $G$  must contain a factor of  $u$  or  $v$ . Since  $G$  is an  $SL_2$ -invariant it is in particular a  $\mathbb{G}_m$ -invariant, so each term of  $G$  must be weight 0. However,  $g = G|_{u=0, v=1}$ , and hence must consist of terms that are of strictly positive weight for the  $\mathbb{G}_m$  action on  $W$ . By Lemma 4.5 the non-stable set is contained in the locus of points where all coordinate functions of strictly positive weight, and hence all the terms of  $g$ , vanish.  $\square$

For open affines of the form  $W_h$  the cohomological vanishing criterion of Theorem 3.12 has a very simple interpretation.

**Lemma 4.18.** *Let  $h$  be a  $\mathbb{G}_a$ -invariant function on  $W$ . The quotient  $W_h/\mathbb{G}_a$  is affine and  $W_h \rightarrow W_h/\mathbb{G}_a$  is a trivial principal bundle if and only if some power of  $h$  lies in  $Im(D)$ .*

*Proof.* By assumption we have  $h \in Ker(D)$  since  $h$  is a  $\mathbb{G}_a$  invariant. By Theorem 3.12, the quotient map  $W_h \rightarrow W_h/\mathbb{G}_a$  is a trivial principal bundle over the affine variety  $W_h/\mathbb{G}_a$  if and only if  $H^1(\mathfrak{g}_a, k[W_h]) = 0$ . Example 3.13 shows that this happens if and only if there exists a function  $s \in k[W_h]$  such that  $D(s) = 1 \in k[W_h]$ .

Let us first show that if some power of  $h$  lies in  $Im(D)$ , that we can find an  $s$  such that  $D(s) = 1$ . Indeed, suppose  $h^k = D(g)$ . If we set  $s = g/h^k$ ; then, by the quotient rule,  $D(s) = 1$ .

Conversely, assume there exists an  $s \in k[W_h] = k[W][h^{-1}]$  such that  $D(s) = 1$ . In coordinates,

$$s = g_0 + \frac{g_1}{h} + \frac{g_2}{h^2} + \dots + \frac{g_k}{h^k} = \frac{h^k g_0 + h^{k-1} g_1 + \dots + h g_{k-1} + g_k}{h^k}.$$

Then, again by the quotient rule,  $D(s) = D(h^k g_0 + h^{k-1} g_1 + \dots + h g_{k-1} + g_k)/h^k$ . If we have  $D(s) = 1$ , then  $D(h^k g_0 + h^{k-1} g_1 + \dots + h g_{k-1} + g_k) = h^k$ ; in other words,  $h^k \in Im(D)$ .  $\square$

**Lemma 4.19.** *A  $\mathbb{G}_a$ -invariant function  $f'$  contains the boundary if and only if there exists a positive integer  $k$  such that  $(f')^k \in Im(D) \cap Ker(D)$ .*

*Proof.* Since  $f'$  contains the boundary,

$$SL_2 *_{\mathbb{G}_a} W_{f'} = ((V \setminus \{0\}) \times W)_{F'} = (V \times W)_{F'}.$$

In particular  $SL_2 *_{\mathbb{G}_a} W_{f'}$  is affine. The GIT quotient of an affine variety by a reductive group is affine, so  $SL_2 *_{\mathbb{G}_a} W_{f'}/SL_2 = W_{f'}/\mathbb{G}_a$  is therefore affine. By definition of stability and Lemma 4.18, it follows  $(f')^k \in Im(D) \cap Ker(D)$ .

Let  $(f')^k = D(g)$ . As in the first argument in Lemma 4.18, let  $s = g/(f')^k$ , so that  $D(s) = 1$ . Thus  $W_{(f')^k} = W_{f'} \rightarrow W_{f'}/\mathbb{G}_a$  is a trivial principal  $\mathbb{G}_a$ -bundle over the affine variety  $W_{f'}/\mathbb{G}_a$ , and  $W_{f'} \subset W^s$ .

Now, we have the equality  $W_{f'}/\mathbb{G}_a \cong ((V \setminus \{0\}) \times W)_{F'}/SL_2$ . But, because the inclusion  $(V \setminus \{0\})W \times \subset V \times W$  is open with complement of codimension 2 and since  $V \times W$  is normal, all regular functions extend. Since the inclusion is  $SL_2$ -equivariant, this extension sends  $SL_2$ -invariant regular functions to  $SL_2$ -invariant regular functions. Then  $(V \times W)_{F'}/SL_2$  is an affine scheme and its coordinate ring is identified with the coordinate ring of  $((V \setminus \{0\}) \times W)_{F'}/SL_2$ . Therefore, we can identify the quotients  $(V \times W)_{F'}/SL_2$  and  $((V \setminus \{0\}) \times W)_{F'}/SL_2$ .

It follows that any point  $x \in (\{0\} \times W)_{F'}$  has to be in the closure of an orbit  $\mathcal{O}$  of  $((V \setminus \{0\}) \times W)_{F'}$ , which violates the definition of stability of  $\mathcal{O}$ . This implies  $(\{0\} \times W)_{F'}$  is in fact empty and therefore that  $f'$  contains the boundary.  $\square$

Combining these lemmas, we are led to a simple algebraic characterization of everywhere stability.

**Theorem 4.20** (Algebraic characterization). *Suppose  $X$  is a  $\mathbb{G}_a$ -invariant hypersurface defined by a  $\mathbb{G}_a$ -invariant polynomial  $f$ . Then  $X$  is everywhere stable if and only if there is a canonical decomposition  $f = F_{0,0} + g$  satisfying*

- i) *Either the function  $F_{0,0}$  is constant or  $F_{0,0}$  is a stable  $SL_2$ -invariant with non-zero constant term (we will say that  $F_{0,0}$  is a stable  $SL_2$ -invariant if  $F_{0,0}(x)$  is non-vanishing only if  $x \in W^s$ ), and*
- ii) *the function  $g$  is a  $\mathbb{G}_a$ -invariant that contains the boundary, and some positive power of  $g$  lies in  $Im(D)$ .*

Furthermore, in the above situation, the quotient  $X/\mathbb{G}_a$  is affine if and only if  $F_{0,0}$  is constant.

*Remark 4.21.* In fact, these lemmas show that any  $\mathbb{G}_a$ -invariant function admits a natural decomposition into an  $SL_2$  invariant function  $F_{0,0}$  and a  $g$  such that  $g^k \in im(D) \cap ker(D)$  for some positive integer  $k$ . The everywhere stable condition simply constrains  $F_{0,0}$  to be a “stable” invariant. It would be interesting to know whether this decomposition of  $\mathbb{G}_a$  invariants can be deduced from classical  $SL_2$ -covariant theory.

When enough “ingredients” in this picture are smooth, then the quotients have a particularly clean presentation as open subsets of hypersurfaces in  $\text{Spec } k[W]^{\mathbb{G}_a}$ .

**Lemma 4.22.** *Assume that the  $\mathbb{G}_a$ -invariant hypersurface  $X$  in  $W$ , defined by  $f = 0$ , is smooth. If the variety in  $W$  defined by  $F_{0,0} = 0$  is smooth, then the variety in  $V \times W$  defined by  $F = 0$  is smooth.*

*Proof.* We can assume that  $k$  is algebraically closed as  $X$  is smooth if and only if its base extension to  $\bar{k}$  is non-singular. If  $X$  is a smooth affine scheme, it follows by Corollary 7.5 and the fact that  $SL_2/\mathbb{G}_a$  is smooth that the induced scheme  $SL_2 *_{\mathbb{G}_a} X$  is smooth as well. Let us fix coordinates  $u, v$  on  $V$  as we have above. Now, the singular locus of the hypersurface defined

by the polynomial  $F$  must lie in the locus of points where  $u = 0$  and  $v = 0$ , i.e. in  $\{0\} \times W$ . We may now apply the Jacobian criterion for smoothness.

The singular locus of the hypersurface defined by  $F$  is the simultaneous vanishing locus of the partial derivatives  $\frac{\partial F}{\partial v}$ ,  $\frac{\partial F}{\partial u}$ , and  $\frac{\partial F}{\partial w_i}$  for all  $i$ . Now,  $\frac{\partial F}{\partial w_i} = \frac{\partial F_{0,0}}{\partial w_i} + u(\cdots) + v(\cdots)$ . It follows that any singular points must be singularities of  $\frac{\partial F_{0,0}}{\partial w_i} = 0$ .  $\square$

**Theorem 4.23.** *Let  $X$  be a smooth everywhere stable hypersurface in  $W$  defined by the vanishing of a  $\mathbb{G}_a$ -invariant polynomial  $f$ . Let  $F_{0,0}$  be the  $SL_2$ -invariant polynomial appearing in the decomposition of  $f$  as in Theorem 4.20. Assume the variety  $Y$  in  $W$  defined by  $F_{0,0}$  is smooth. Then the quotient  $X/\mathbb{G}_a$  is an open subvariety of the normal variety  $\text{Spec } k[W]^{\mathbb{G}_a}/(f)$ . Furthermore, the boundary of  $X/\mathbb{G}_a$  in  $\text{Spec } k[W]^{\mathbb{G}_a}/(f)$  is of codimension 2 and is identified with the quotient  $Y/SL_2$ , which has at most finite quotient singularities.*

*Proof.* By Lemma 4.22 the variety defined by the vanishing of  $F$ , which we denote by  $\overline{SL *_{\mathbb{G}_a} X}$ , is smooth. Hence any  $SL_2$ -invariant function on  $SL_2 *_{\mathbb{G}_a} X$  extends to an  $SL_2$ -invariant function on  $\overline{SL_2 *_{\mathbb{G}_a} X}$ . Any  $SL_2$ -invariant function on an affine subvariety extends to an  $SL_2$ -invariant function on the ambient affine space; here this is given by  $V \times W$ .

Consequently, we have isomorphisms  $k[X]^{\mathbb{G}_a} \cong k[V \times W]^{SL_2}/(F) \cong k[W]^{\mathbb{G}_a}/(f)$ . Thus the quotient  $X/\mathbb{G}_a = SL_2 *_{\mathbb{G}_a} X/SL_2$  is an open subscheme of  $\overline{SL_2 *_{\mathbb{G}_a} X}/SL_2 = \text{Spec}(k[V \times W]/(F))^{SL_2} \cong \text{Spec } k[V \times W]^{SL_2}/(F) \cong \text{Spec } k[W]^{\mathbb{G}_a}/(f)$ . This latter variety is normal because  $\overline{SL *_{\mathbb{G}_a} X}$  is smooth, hence normal, so the categorical quotient  $\overline{SL_2 *_{\mathbb{G}_a} X}/SL_2$  is normal.

By Theorem 4.20,  $F_{0,0}$  is a stable  $SL_2$ -invariant. Thus the boundary  $F_{0,0} = 0$ , and hence  $F = 0$  is in fact everywhere stable. Thus, we have an identification  $\overline{SL_2 *_{\mathbb{G}_a} X}/SL_2 = \overline{SL_2 *_{\mathbb{G}_a} X}/SL_2$ , so in particular the boundary of the quotient is isomorphic to the quotient  $F_{0,0}/SL_2$ .  $\square$

*Remark 4.24.* For this remark, let us assume that  $X$  is defined over  $\mathbb{C}$  and let  $X(\mathbb{C})$  denote the associated analytic space. It follows from Theorem 4.23 that when the quotient  $X/\mathbb{G}_a$  is quasi-affine, the analytic space  $X/\mathbb{G}_a(\mathbb{C})$  is not Stein because it is the complement of a codimension 2 analytic subspace in a normal Stein space.

It is thus easy to produce examples of affine or strictly quasi-affine quotients; indeed, any possible example can be produced through the  $SL_2$ -invariant (and covariant) theory of  $W$ .

## A quasi-affine quotient

Let us illustrate the algebraic characterization by giving the simplest example of a strictly quasi-affine quotient of an everywhere stable action.

*Example 4.25.* Consider  $SL_2$  as a hypersurface  $X$  in  $W = V \oplus V$  determined by the vanishing of the polynomial  $f = 1 - w_0w_3 + w_1w_2$ . Note that  $f$  is a stable- $SL_2$ -invariant with non-zero constant term because the unstable locus is the set of all points  $w_0 = w_2 = 0$ . Therefore, by Theorem 4.20, the hypersurface defined by  $f$  is everywhere stable and has a quasi-affine quotient. Here, the invariants are generated by  $w_0$  and  $w_2$  and the image  $X/\mathbb{G}_a$  is isomorphic to  $\mathbb{A}^2 \setminus \{0\}$ .

Similarly, if  $\phi$  is a degree  $d$  function of one variable with no repeated roots and no constant term, then let  $f = 1 - \phi(w_0w_3 - w_1w_2)$ . The boundary in  $F = 0$  consists of  $d$  disjoint isomorphic

copies of  $SL_2$ . The quotient is identified with  $\mathbb{A}^2 \setminus \{p_1, \dots, p_d\}$ , where  $p_i$  are points representing the boundary components. Since the action of the automorphism group  $Aut_k(\mathbb{A}^2)$  on  $\mathbb{A}^2$  is  $d$ -transitive for any positive integer  $d$ , it follows that, up to isomorphism, the complement in  $\mathbb{A}^2$  of any set of  $d$ -points may always be realized as a  $\mathbb{G}_a$ -quotient.

## 5 Examples: $\mathbb{A}^1$ -contractible strictly quasi-affine varieties

In this section, we use Theorem 4.20 to give many examples of  $\mathbb{A}^1$ -contractible smooth varieties in every dimension  $\geq 4$ . In order to do this, we will construct everywhere stable actions of  $\mathbb{G}_{\mathbf{a}}$  on affine spaces with strictly quasi-affine quotients.

### $SL_2$ -representations

Let  $V$  denote the standard 2-dimensional representation of  $SL_2$ . Since every linear representation of  $\mathbb{G}_{\mathbf{a}}$  extends to an  $SL_2$ -representation, we will index representations of  $\mathbb{G}_{\mathbf{a}}$  by the corresponding representations of  $SL_2$ . Fix coordinates  $w_0, \dots, w_k$  on  $\mathrm{Sym}^k V$  corresponding to a basis of weights with  $w_0$  being  $\mathbb{G}_{\mathbf{a}}$ -fixed. Let  $\partial_i$  denote the derivation  $\frac{\partial}{\partial w_i}$ . In these coordinates, the locally nilpotent derivation defining the  $\mathbb{G}_{\mathbf{a}}$ -action on  $\mathrm{Sym}^k V$  is given by

$$D_{\mathrm{Sym}^k V} = \sum_{i=0}^{k-1} (k-i)w_i \partial_{i+1}.$$

A general linear representation  $W$  of  $\mathbb{G}_{\mathbf{a}}$ , extends to an  $SL_2$  representation that decomposes as  $W = \bigoplus \mathrm{Sym}^{k_j} V$ . The locally nilpotent derivation  $D$  determining the  $\mathbb{G}_{\mathbf{a}}$ -action on  $W$  then takes the form

$$D = \sum_{i \in I} c_i w_i \partial_{i+1}$$

where the set  $I$  and the coefficients  $c_i$  are determined by the  $k_i$ .

### Everywhere stable embeddings of affine space

We are interested in realizations of  $\mathbb{A}^n$  as an everywhere stable hypersurface in a linear representation  $W$ . In order to do this, choose coordinates  $z_1, \dots, z_n$  on  $\mathbb{A}^n$ , and  $w_0, \dots, w_n$  on  $W$ . As usual, an embedding  $\mathbb{A}^n \hookrightarrow W$  is determined by specifying polynomials  $w_i = w_i(z_1, \dots, z_n)$ . The embedding being  $\mathbb{G}_{\mathbf{a}}$ -equivariant is equivalent to the locally nilpotent derivation  $D$  defining the  $\mathbb{G}_{\mathbf{a}}$ -action on  $W$  restricting to a locally nilpotent derivation on  $\mathbb{A}^n$ ; by abuse of notation, we will also call this restricted derivation  $D$ .

Let  $f$  be the polynomial relation between  $w_i$ ; this defines a hypersurface  $X$  in  $W$ . By Lemma 4.6, the polynomial  $f$  is necessarily  $\mathbb{G}_{\mathbf{a}}$ -invariant and hence  $D(f)$  must vanish identically. This imposes the condition

$$\sum_{i \in I} c_i w_i \partial_{i+1} f = 0,$$

and  $I$  is an index set as above.

We now construct a class of  $\mathbb{G}_{\mathbf{a}}$ -equivariant embeddings of  $\mathbb{A}^n$  into  $W$  as follows. Consider the morphism  $\mathbb{A}^n \rightarrow W$  defined by setting  $w_i = z_i$  for all  $i \neq 0$ , and let  $w_0 = h(z_1, \dots, z_n)$  where  $h$  is a  $\mathbb{G}_{\mathbf{a}}$ -invariant function; observe that this morphism is actually an embedding. The corresponding hypersurface  $X$  in  $W$  is then the vanishing locus of the polynomial  $f = h(w_1, \dots, w_n) - w_0$ . By Theorem 4.20, in order that  $X$  be an everywhere stable hypersurface with quasi-affine quotient,

the invariant  $h$  must decompose as  $F_{0,0} + g$  where i)  $F_{0,0}$  is a stable  $SL_2$ -invariant with non-vanishing constant term and ii) some strictly positive power of  $g$  is in the image of  $D$ .

**Theorem 5.1.** *For every  $m \geq 4$ , there exists a denumerably infinite collection of pairwise non-isomorphic exotic  $\mathbb{A}^1$ -contractible varieties, each admitting an embedding into a smooth affine variety with pure codimension 2 smooth boundary.*

*Proof.* Consider the representation  $W = \text{Sym}^q \oplus \text{Sym}^{2p+1}(V) \oplus \oplus W'$ , where  $p, q \geq 1$  and  $W'$  is any linear  $\mathbb{G}_a$  representation. Choose coordinates  $w_0, \dots, w_{2p+q+\dim W'+2}$  which are a basis of  $\mathbb{G}_m$ -eigenvectors for an  $SL_2$ -action on  $W$  extending the given  $\mathbb{G}_a$ -action; by convention, the first  $q+1$ -coordinates will be coordinates on  $\text{Sym}^q(V)$ , the next  $2p+2$ -coordinates will be coordinates on  $\text{Sym}^{2p+1}(V)$ , and the remaining  $w_i$  will be coordinates on  $W'$ .

Because  $\text{Sym}^{2p+1}(V)$  has an  $SL_2$ -stable point, it has an  $SL_2$  GIT quotient of dimension  $2p-1 \geq 1$ , so in particular there exists a non-constant homogeneous  $SL_2$  invariant  $\Delta$  on  $W$  (for example, the discriminant). Furthermore, because  $\text{Sym}^{2p+1}(V)$  is even dimensional, all of its points are either stable or unstable (in that all homogeneous invariants vanish on the unstable set); consequently all homogeneous  $SL_2$  invariants are in fact stable  $SL_2$  invariants. Let  $w_0$  denote the invariant coordinate in direct summand  $\text{Sym}^q(V)$  of  $W$ ; note that  $w_0 = D(w_1)$  (and  $w_1$  is an element of  $\text{Sym}^q(V)$  because  $q \geq 1$ ). In particular, this explains why we must choose  $q \geq 1$ .

Suppose  $\varphi$  is a polynomial in one variable of strictly positive degree and with no multiple roots. Let  $h_\varphi = 1 + \varphi(\Delta)$ . Note that  $h_\varphi$  is a stable  $SL_2$ -invariant with non-vanishing constant term. Therefore, if  $f = h_\varphi - w_0$ , the discussion just prior to the theorem says that  $f$  defines a hypersurface isomorphic to  $\mathbb{A}^{m+1}$  in  $W$ , where  $m = 2p + q + \dim(W') + 1$ . By Theorem 4.20 the quotient  $X_\varphi = \mathbb{A}^{m+1}/\mathbb{G}_a$  is strictly quasi-affine.

We now claim that if  $\varphi$  and  $\varphi'$  are two one-variable polynomials whose degrees differ, then the resulting quotients are non-isomorphic. Note that  $X_\varphi$  is an open subvariety of the affine variety  $\text{Spec } k[z_1, \dots, z_n]^{\mathbb{G}_a}$  (which is finitely generated by Corollary 4.13) with complement of codimension at least 2. Let us denote the boundary by  $B_\varphi$ .

Suppose  $\varphi$  and  $\varphi'$  are a pair of strictly positive degree polynomials in one variable with no repeated roots. Let  $d = \deg(\varphi)$ . If  $F_\varphi$  denotes the  $SL_2$ -invariant hypersurface equation attached to  $h_\varphi$ , then the boundary of the hypersurface  $F_\varphi$  in  $V \times W$  is given by  $F_{\varphi,0,0} = 0$ . Here  $F_{\varphi,0,0} = \varphi(\Delta)$ . Hence the boundary  $B_\varphi$  is the  $SL_2$ -quotient of the vanishing locus of  $F_{\varphi,0,0}$ . This equips  $B_\varphi$  with the structure of vector bundle over the vanishing locus of  $h_\varphi$ . Since  $\varphi$  has  $d$  distinct roots, the components are of the form  $\Delta = c_i$  where  $i$  runs from 1 to  $d$ ; these components are obviously disjoint. Taking the  $SL_2$ -quotient, we see that  $B_\varphi$  therefore has  $d$  distinct components as well.

Finally, note that because  $W$  has no points of finite isotropy in  $SL_2$ , the variety  $(1 + \Delta = 0)/SL_2$  is smooth, and for generic  $\varphi$  the variety  $h_\varphi/S_L_2$  is smooth. Likewise,  $V \times W$  has no points of finite isotropy, so for generic  $\varphi$  the variety  $(F_\varphi = 0)/SL_2$  is smooth. The smoothness assertions in the theorem follow.  $\square$

*Remark 5.2.* Note that the second summand in  $\text{Sym}^q V \oplus \text{Sym}^{2p+1} V \oplus W'$  can be replaced by any  $\mathbb{G}_a$  representation with at least one stable  $SL_2$ -invariant – for example,  $V \oplus V$  with its unique

quadratic  $SL_2$ -invariant. If one takes  $q = 1$  and  $W' = \{0\}$ , then the hypersurface defined by the vanishing of  $w_0 = 1 + (w_2w_5 - w_3w_4)$  is a  $\mathbb{G}_a$ -equivariant linear embedding of Winkelmann's example (see [Win90] §2) and the quotient of course then agrees with his. Specifically, by Corollary 4.13 the  $\mathbb{G}_a$ -invariants for  $X$  here are just restrictions from  $k[W]^{\mathbb{G}_a}$ , namely:

$$w_0, w_2, w_4, w_0w_3 - w_1w_2, w_0w_5 - w_1w_4, w_2w_5 - w_3w_4.$$

Imposing the hypersurface equation for  $X$  there are only 5 generating invariants, and one relation. The relation gives the hypersurface equation  $x_1x_4 - x_2x_3 - x_5(x_5 + 1) = 0$  in  $\mathbb{A}^5$ , where  $x_1, \dots, x_5$  are identified, in order, with the last five invariants above.

**Theorem 5.3.** *For every  $m \geq 6$  and every  $n > 0$ :*

- *there exists a connected  $n$ -dimensional scheme  $S$  and a smooth morphism  $f : X \rightarrow S$  of relative dimension  $m$ , whose fibers over  $k$ -points are  $\mathbb{A}^1$ -contractible and quasi-affine, not affine, and pairwise non-isomorphic.*
- *The morphism  $f : X \rightarrow S$  admits a partial compactification to a flat family  $\bar{f} : \bar{X} \rightarrow S$  whose fibers over  $k$ -points are smooth affine varieties. Furthermore, for any  $k$ -point  $t \in S$ , the map  $X_t \rightarrow \bar{X}_t$  is an open immersion with a smooth complement of codimension  $\geq 2$*

*Proof.* Let  $W$  be a linear  $\mathbb{G}_a$ -representation. Suppose we choose a family  $f_t$  of polynomials in  $W^s$  parameterized by some base scheme  $T$ , such that the corresponding hypersurfaces  $X_t$  are all everywhere stable. Let  $F_t$  is the corresponding induced family of polynomials in  $G *_U W$  and let  $F_{t,0,0}$  be the polynomial defining the boundary. Let  $\overline{SL_2 *_{\mathbb{G}_a} X_t}$  be the vanishing locus of  $F_t$  in  $V \times W$ . Assuming  $X_t$  are all smooth, we know the quotients  $X_t/\mathbb{G}_a$  are all smooth. Furthermore, by Lemma 4.22, if the vanishing loci of  $F_{t,0,0}$  define smooth subvarieties of  $W$ , it follows that  $F_t$  are all smooth varieties as well. Denote these vanishing loci by  $B_t$ .

If the  $F_t$  are smooth, then the categorical quotients  $\overline{SL_2 *_{\mathbb{G}_a} X_t}/SL_2$  are normal varieties. Furthermore, if  $B_t$  is smooth, then  $B_t//SL_2$  is a normal variety as well. Now, any isomorphism  $\psi : X_t/\mathbb{G}_a \xrightarrow{\sim} X_{t'}/\mathbb{G}_a$  extends to an isomorphism  $\bar{\psi} : \overline{SL_2 *_{\mathbb{G}_a} X_t} \xrightarrow{\sim} \overline{SL_2 *_{\mathbb{G}_a} X_{t'}}$  by normality. The isomorphism  $\bar{\psi}$  then would restrict to an isomorphism  $\hat{\psi} : B_t \xrightarrow{\sim} B_{t'}$ . Therefore, if  $B_t//SL_2$  and  $B_{t'}//SL_2$  are non-isomorphic, the quotients  $X_{t'}/\mathbb{G}_a$  and  $X_t/\mathbb{G}_a$  are not isomorphic.

If  $w_0, \dots, w_n$  are coordinates on a linear representation  $W$ , as in Theorem 5.1, we will use invariants of the form  $f_t = h(w_1, \dots, w_n) - w_0$ , where  $h$  is a stable  $SL_2$ -invariant. Indeed, suppose we can find  $\{\Delta_i\}$ ,  $i = 1, \dots, j$  a collection of stable  $SL_2$ -invariants in  $W$  depending only on  $w_1, \dots, w_n$  that are algebraically independent. Then if we fix a family of polynomials  $\varphi_t$  in  $j$  variables, the function  $h_t = 1 + \varphi_t(\Delta_1, \dots, \Delta_j) - w_0$  gives a family of embeddings of affine space into  $W$  parameterized by  $t$ . Furthermore, as  $\Delta_i$  are invariants and algebraically independent, we can treat the  $\Delta_i$  as coordinates on the quotient  $B_t//SL_2$ . In other words, we can view  $B_t//SL_2$  as a closed subvariety of  $W//SL_2$  defined by the vanishing of  $\varphi_t(\Delta_1, \dots, \Delta_j) + 1$ .

Assuming these families admit  $n$ -dimensional moduli, which we defer (along with a discussion of smoothness) to Lemma 5.5 (and Lemma 5.6), let us complete the proof of the theorem. We can assume that  $t$  takes values in a non-singular curve  $T$ . Then, the assignment  $X_t \rightarrow t$  defines an algebraic family of varieties  $\pi : X \rightarrow T$  over a curve. Furthermore, the closures

$\overline{SL_2 *_{\mathbb{G}_a} X_t} // SL_2$  define an algebraic family of normal schemes (see [Har77] Ch. III Defn. 9.10) in which  $X_t$  are open of codimension  $\geq 2$ , in other words we get a morphism  $\bar{\pi} : \overline{SL_2 *_{\mathbb{G}_a} X} // SL_2 \rightarrow T$  factoring  $\pi$ . Such families are necessarily flat by [Har77] Ch. III Thm. 9.11. The family  $X_t$  is in fact smooth as the geometric fibers  $X_t$  are all smooth. Therefore, we get a family  $X' \rightarrow C$  which is smooth. Taking products of such families, we can increase the dimension of the parameter space of the family as we wish.  $\square$

*Remark 5.4.* This discussion recovers Winkelmann's example of a family of quasi-affine quotients of  $\mathbb{A}^7$  (see [Win90] §4). There is, however, a gap in Winkelmann's argument for existence of families of quasi-affine quotients. If  $X$  and  $X'$  are two quasi-affine schemes which are realized as closed subschemes of affine schemes  $\overline{X}$  and  $\overline{X'}$ , then an isomorphism between  $X$  and  $X'$  extends uniquely to the normalizations of  $\overline{X}$  and  $\overline{X'}$ . Winkelmann then claims that this extended isomorphism induces an isomorphism of the normalizations of the boundaries  $\overline{X'} \setminus X'$  and  $\overline{X} \setminus X$ . However, this is not necessarily true. Indeed, it is possible that  $\overline{X}$  is normal (so that its normalization is trivial) and  $\overline{X} \setminus X$  is not normal. Furthermore, even if  $\overline{X} \setminus X$  were normal, unless  $\overline{X}$  is normal there may well be moduli of  $\overline{X} \setminus X$  all of which have the same inverse image in the normalization of  $\overline{X}$ . For us, all of these problems are circumvented by application of Lemma 4.22.

The remaining step in the proof of Theorem 5.3 is to establish the intuitively "clear" statement that there exist boundaries  $B_t/SL_2$ , expressed as  $\varphi_t(\Delta_1, \dots, \Delta_j) - 1 = 0$  in  $\text{Spec } k[W]^{SL_2}$ , which admit many deformations. Rather than argue in complete generality, we now consider a simple special case sufficient for the theorem; the more general formulation would follow exactly the same lines.

**Lemma 5.5.** *Let  $W = V^{\oplus k}$ , for  $V$  the standard two-dimensional representation of  $SL_2$ , and for  $k \geq 4$ . Then for any  $p$  there exists a  $p$ -dimensional family  $X \rightarrow T$  of non-isomorphic smooth hypersurfaces in  $\text{Spec } k[W]^{SL_2}$  defined by  $\varphi_t(\Delta_1, \dots, \Delta_j) + 1 = 0$ .*

*Proof.* Let  $x_{i,\varepsilon}$ , for  $i \in \{1, \dots, n\}$  and  $\varepsilon$  either 0 or 1, be coordinates on  $W$ ; identify them with the usual coordinates via  $w_{2(i-1)+\varepsilon} = x_{i,\varepsilon}$ . The ring  $k[W]^{SL_2}$  is generated by  $\binom{k}{2}$  quadratic stable invariants:  $x_{i,0}x_{j,1} - x_{i,1}x_{j,0}$ . (That all  $SL_2$  invariants in this representation are stable follows from the Hilbert-Mumford criterion.) Consider the three invariants, call them  $\Delta_1, \Delta_2, \Delta_3$ , associated with the last 3 direct summands in  $W = V^{\oplus k}$ . They are the coordinate functions of  $V^{\oplus 3} // SL_2 \cong \mathbb{A}^3$ .

Consider a hypersurface  $B_t$  in  $W$  defined by  $\varphi_t(\Delta_1, \Delta_2, \Delta_3) + 1$ ; then  $B_t \cong V^{\oplus k-3} \times Y_t$ , where  $Y_t$  denotes the variety defined by  $\varphi_t(\Delta_1, \Delta_2, \Delta_3) + 1 = 0$  in  $V^{\oplus 3}$ . Observe that  $SL_2$  acts freely on any  $SL_2$ -invariant everywhere stable hypersurface. Note that the projection morphism  $B_t \rightarrow Y_t$  equips  $B_t$  with the structure of a trivial  $SL_2$ -equivariant vector bundle over  $Y_t$ . Since the  $SL_2$ -action is free on  $Y_t$ , the projection morphism  $B_t \rightarrow Y_t$  descends to give the morphism  $B_t/SL_2 \rightarrow Y_t/SL_2$  the structure of a vector bundle with fiber  $V^{k-3}$ , i.e., of rank  $2(k-3)$ . In particular  $B_t/SL_2$  is  $\mathbb{A}^1$ -weakly equivalent to  $Y_t/SL_2$  (e.g by Lemma 3.3).

The condition that the hypersurface defined by  $\varphi_t(\Delta_1, \Delta_2, \Delta_3) + 1 = 0$  be everywhere stable in  $W$  is satisfied as long as the constant term is non-zero. A generic hypersurface in  $\mathbb{A}^3$  is smooth,

and hence any  $SL_2$ -bundle  $Y_t$  over it is smooth; it follows that a generic choice of  $\varphi_t$  determines a smooth everywhere stable  $B_t$  in  $W$ . Furthermore, we can choose  $\varphi_t$  so that the corresponding affine varieties occur in arbitrary dimensional families. Indeed, we know that hypersurfaces in  $\mathbb{P}^3$  of degree  $d \geq 5$  are generically smooth and admit arbitrary dimensional moduli. Fixing a hyperplane section  $H \subset \mathbb{P}^3$ , one can obtain hypersurfaces in  $\mathbb{A}^3$  with the same property.

Let  $Y_t/SL_2$  and  $Y_s/SL_2$  be two such non-isomorphic hyperbolic surfaces in  $\mathbb{A}^3$ . If  $B_t/SL_2$  is isomorphic to  $B_s/SL_2$  then  $Y_t/SL_2$  and  $Y_s/SL_2$  are  $\mathbb{A}^1$ -weakly equivalent; by assumption however, they are  $\mathbb{A}^1$ -rigid and so would have to be isomorphic. Thus  $B_t/SL_2$  is not isomorphic to  $B_s/SL_2$ . Consequently, any  $p$ -dimensional moduli of algebraically hyperbolic surfaces in  $\mathbb{A}^3$  induces  $p$ -dimensional moduli of  $B_t/SL_2$ .  $\square$

**Lemma 5.6.** *There exist families as described in Theorem 5.3 such that the quasi-affine quotients each admit a smooth affine closure in  $\text{Spec } k[W]^{\text{Ga}}$  with smooth pure codimension 2 boundary.*

*Proof.* In the proof of Lemma 5.5 the variety  $B_t$  could be chosen to be smooth, i.e.,  $F_{0,0} = 0$  is smooth. Now Lemma 4.22 implies  $F = 0$  in  $V \times W$  is smooth. But for  $W = V^{\oplus k}$ , there are no points with finite isotropy for the  $SL_2$  action on  $V \times W$ . Therefore  $SL_2$  acts freely on  $F = 0$  in  $V \times W$ , so the quotient is smooth.  $\square$

### Quotients of Dimension $\leq 2$

For this rest of this section, let us suppose  $k$  is the field of complex numbers.

**Claim 5.7.** *There is a unique up to isomorphism  $\mathbb{A}^1$ -contractible smooth scheme of dimension 1; namely  $\mathbb{A}^1$ . Hence, any everywhere stable action of an  $n$ -dimensional unipotent group  $U$  on  $\mathbb{A}^{n+1}$  is isomorphic to  $\mathbb{A}^1$ .*

*Proof.* The analytic space  $X(\mathbb{C})$  is necessarily a contractible complex algebraic curve and in particular a contractible topological surface. Every such surface is necessarily homeomorphic to the open unit disc. It follows that  $X$  is isomorphic to  $\mathbb{A}^1$ .

Suppose we have an everywhere stable action of  $U$  on  $\mathbb{A}^{n+1}$ . By Corollary 3.16 (iii), we know that  $\mathbb{A}^n/U$  is an affine variety. Hence by Corollary 3.16 (ii), we have an isomorphism  $\mathbb{A}^n \cong X \times U$ . Now,  $X$  is a smooth affine,  $\mathbb{A}^1$ -contractible variety of dimension 1.  $\square$

**Claim 5.8.** *Suppose  $U$  is an  $n$ -dimensional unipotent group. The quotient of any everywhere stable action of  $U$  on  $\mathbb{A}^{n+2}$  is isomorphic to affine space.*

*Proof.* Suppose we have an everywhere stable action of  $U$  on  $\mathbb{A}^{n+2}$ . Let  $X$  be the quotient  $\mathbb{A}^{n+2}/U$ . Note that  $X$  is a smooth quasi-affine  $\mathbb{A}^1$ -contractible surface; in particular it is a smooth quasi-projective surface. The topological space  $X(\mathbb{C})$  is therefore acyclic and hence by a result of Fujita (see [Zai99] Lemma 2.1)  $X$  is necessarily a smooth affine surface. Furthermore, we know that the morphism  $\mathbb{A}^{n+2} \rightarrow X$  is in fact a trivial principal bundle isomorphic to  $X_{\mathbb{C}} \times U$  by Corollary 3.16 (ii). Now, since  $X \times \mathbb{A}^n \cong \mathbb{A}^{n+2}$ , it follows that  $X$  is fact isomorphic to  $\mathbb{A}^2$  by Fujita's proof of Zariski cancellation (see [Fuj79]) in dimension 2.  $\square$

*Remark 5.9.* If  $X$  is a smooth algebraic variety, let  $\kappa(X)$  denote the logarithmic Kodaira dimension of  $X$ . Suppose  $f : Y \rightarrow X$  is a  $U$ -torsor. By the Iitaka Easy Addition theorem (see e.g. [Zai99] Theorem 2.5 (c)),  $\kappa(Y) \leq \kappa(U) + \dim(X)$ , and hence  $\kappa(Y)$  must be  $-\infty$ . In other words, if a unipotent group acts everywhere stably on a variety  $Y$ , then  $\kappa(Y) = -\infty$ . However, the quotient of an everywhere stable action of a unipotent group on a smooth variety can have arbitrary logarithmic Kodaira dimension. Indeed, let  $X$  be a smooth affine variety with logarithmic Kodaira dimension  $k$ . Then the usual translation action of  $\mathbb{G}_a$  on  $\mathbb{A}^1 \times X$  (acting trivially on  $X$ ) is everywhere stable with quotient isomorphic to  $X$ .

### Quotients of dimension 3

Again, suppose  $k$  is the field  $\mathbb{C}$ . In order to produce an example of a strictly quasi-affine quotient in dimension 3 by our method, we would have to fix an embedding of  $\mathbb{A}^4$  as a hypersurface in  $\mathbb{A}^5$ . The restriction of the locally nilpotent derivation  $D$  defining the  $\mathbb{G}_a$ -action on  $\mathbb{A}^5$  to  $\mathbb{A}^4$  is triangular. It follows from results of Deveney, Finston and van Rossum (see [DFvR04] Theorem 2.1) that *any* everywhere stable action of  $\mathbb{G}_a$  on  $\mathbb{A}^4$  defined by a triangular locally nilpotent derivation has quotient isomorphic to  $\mathbb{A}^3$ . This shows in particular, that the examples produced by the method of proof of Theorem 5.1 are of minimal dimension.

**Question 5.10.** *Does there exist a 3-dimensional quasi-affine quotient of  $\mathbb{A}^4$  by  $\mathbb{G}_a$ ?*

## 6 Consequences, Conjectures, and Comments

In this section, we emphasize some formal consequences of  $\mathbb{A}^1$ -contractibility and discuss some conjectures regarding the structure of  $\mathbb{A}^1$ -contractible smooth schemes.

### Cohomology Computations

The motivic homotopy category was constructed to study cohomology theories on the category of algebraic varieties. For any  $\mathbb{A}^1$ -contractible scheme  $X$ , and any space  $\mathcal{Y}$ , the sets of  $\mathbb{A}^1$ -homotopy classes of maps  $[X, \mathcal{Y}]_{\mathbb{A}^1}$  and  $[\mathcal{Y}, X]_{\mathbb{A}^1}$  are isomorphic to  $[\mathrm{Spec} k, \mathcal{Y}]_{\mathbb{A}^1}$  and  $[\mathcal{Y}, \mathrm{Spec} k]_{\mathbb{A}^1}$  respectively. Here are some trivial consequences of these facts.

**Corollary 6.1.** *Suppose  $X$  is an  $\mathbb{A}^1$ -contractible smooth scheme. The structure map  $X \rightarrow \mathrm{Spec} k$  induces isomorphisms  $H^{*,*}(\mathrm{Spec} k, \mathbb{Z}) \xrightarrow{\sim} H^{*,*}(X, \mathbb{Z})$ ,  $K^*(\mathrm{Spec} k) \xrightarrow{\sim} K^*(X)$ , and  $MGL^{*,*}(\mathrm{Spec} k) \xrightarrow{\sim} MGL^{*,*}(\mathrm{Spec} k)$ .*

*Proof.* All of these facts follow from the observation that the corresponding cohomology groups can be defined, unstably, as maps into an appropriate space: motivic Eilenberg-MacLane spaces  $K(\mathbb{Z}(q), p)$  for motivic cohomology (see [Voe98]),  $BGL_{\infty}$  for algebraic  $K$ -theory (see [MV99] §4 Proposition 3.9), and  $MGL_{\infty}$  for algebraic cobordism (see [Voe98]).  $\square$

*Example 6.2.* Suppose  $X$  is an  $\mathbb{A}^1$ -contractible smooth scheme. Then  $\mathrm{Pic}(X)$  is trivial,  $\mathcal{O}^*(X)$  is isomorphic to  $k^*$ , and  $K^2(X) \cong K_M^2(k)$  (where  $K_M^i(k)$  is the  $i$ -th Milnor  $K$ -theory group of the field  $k$ ).

### Quillen-Suslin style theorems

The classical theorem of Quillen-Suslin states that all algebraic vector bundles on affine space are in fact trivial bundles. In topology, isomorphism classes of rank  $n$  complex vector bundles on a space  $X$  are in bijection with homotopy classes of maps to the classifying space  $BU(n)$ . Analogously, one expects that the set of isomorphism classes of algebraic vector bundles of rank  $n$  on a smooth scheme  $X$  will be in bijection with the set of  $\mathbb{A}^1$ -homotopy classes of maps  $[X, BGL_n]_{\mathbb{A}^1}$ . In particular, when  $X$  is  $\mathbb{A}^1$ -contractible, this set should be in bijection with  $[\mathrm{Spec} k, BGL_n]_{\mathbb{A}^1}$  and this leads to the following conjecture.

**Conjecture 6.3.** *Suppose  $X$  is an  $\mathbb{A}^1$ -contractible, smooth algebraic variety. Then all algebraic vector bundles on  $X$  are in fact trivial.*

*Remark 6.4.* Suppose  $X$  is a smooth algebraic variety over  $\mathbb{C}$  whose underlying topological space is contractible. The generalized Serre problem (see [Zai99] §8) asks whether all algebraic vector bundles on  $X$  are in fact trivial. If the generalized Serre problem admits a positive solution, then  $[X, BGL_n]_{\mathbb{A}^1}$  should be trivial for all  $n > 0$ .

*Remark 6.5.* Recent results of Morel indicate that the above conjecture is true at least when  $X$  is affine and with certain restrictions on  $n$  (see [Morb]). According to [Morb] Theorem 1 and the discussion thereafter, suppose  $X$  is a smooth, affine, algebraic variety. Then, for all  $n \geq 1$ ,

$n \neq 2$ , there is a bijection between the set of isomorphism classes of rank  $n$  algebraic vector bundles on  $X$  and the set of  $\mathbb{A}^1$ -homotopy classes of maps  $[X, BGL_n]_{\mathbb{A}^1}$ . It follows that if  $X$  is a smooth, affine,  $\mathbb{A}^1$ -contractible variety, then for all  $n \neq 2$ , every rank  $n$  algebraic vector bundle is algebraically isomorphic to a trivial bundle.

### Remarks about motives

Suppose  $X$  is any smooth algebraic variety. Analogous to ordinary topology, one can study for  $X$  motivic homology and motivic homology with compact supports (i.e. Borel-Moore motivic homology). Voevodsky denotes the corresponding objects in the derived category of mixed motives by  $M(X)$  and  $M^c(X)$ ; these objects can be thought of as analogous to the usual singular chain complex and the singular chains with locally finite supports viewed as objects in the derived category of  $\mathbb{Z}$ -modules. We refer the reader to [Voe00] for details about derived categories of motives. We denote by  $DM_{Nis}^{eff,-}(k, \mathbb{Z})$  and  $\mathbf{DM}(\mathrm{Spec} k, \mathbb{Z})$  Voevodsky's derived category of effective motivic complexes and derived category of geometric motives respectively (the Tate motive  $\mathbb{Z}(1)$  is inverted in the latter category).

### Consequences of a very general Hodge-type conjecture

In this section we work with varieties over  $\mathbb{C}$ . As we noted above, any smooth,  $\mathbb{A}^1$ -contractible  $\mathbb{C}$ -algebraic variety has trivial motivic cohomology. Via realization functors, this statement has consequences for the ordinary topology of such varieties.

The usual Hodge conjecture predicts that the embedding of Grothendieck's category of homological motives (with  $\mathbb{Q}$ -coefficients) into the category of  $\mathbb{Q}$ -Hodge structures is a fully-faithful embedding. One can construct a Hodge realization functor  $R_{\mathcal{H}}$  on Voevodsky's derived category of motives (see [Hub] §3). Huber deduces the following result from "standard conjectures" (see [Hub] Proposition 3.4.1)

**Proposition 6.6.** *If  $H^i(R_{\mathcal{H}}(M(X))) = 0$  for all  $i$ , then  $M(X)$  is equivalent to a point in the category  $DM_{Nis}^{eff,-}(k, \mathbb{Q})$ .*

In particular, this conjecture implies that if a smooth variety is rationally acyclic, then it must also have rational motivic homology isomorphic to that of a point. This condition, together with the generalized Serre problem of the previous section, imposes topological restrictions on  $\mathbb{A}^1$ -contractible smooth varieties. In particular, if both the Hodge conjecture and the generalized Serre problem admit positive solutions, then any smooth contractible  $\mathbb{C}$ -algebraic variety has rational motivic homology isomorphic to that of a point, and all algebraic vector bundles are trivial. We are therefore led to consider the following question.

**Question 6.7.** *Do there exist contractible smooth  $\mathbb{C}$ -algebraic varieties that are not  $\mathbb{A}^1$ -contractible?*

*Remark 6.8.* Let  $X$  be a smooth, contractible  $\mathbb{C}$ -algebraic variety. It is, a priori, possible that  $[X, BGL_n]_{\mathbb{A}^1}$  is trivial for all  $n$  and that  $[X, K(\mathbb{Q}(q), p)]$  is trivial for all integers  $p, q$  while  $X$  does not have the motivic homotopy type of a point.

### Motivic topology at infinity: a dream

As we noted in the introduction, the examples of open contractible  $n$ -manifolds  $n \geq 4$  (PL or smooth) were non-simply connected at infinity. However, it is natural to study the whole “homotopy type at infinity (see [HR96] Chapter 9).” A first step in this direction is to study the homology at infinity (see [HR96] Chapter 3 for a definition of this notion). It follows essentially from considerations involving Poincaré duality that the homology at infinity of a smooth open contractible  $n$ -manifold is that of an  $n - 1$ -sphere. The main goal of this section is to develop some notions of motivic topology at infinity. The first thing to do is to study *motivic homology at  $\infty$* ; this notion has been introduced by Wildeshaus (see [Wil]).

Notation as in [Voe00], the motive  $M(X)$  and the compactly supported motive  $M^c(X)$  of a scheme  $X$  are the objects in the derived category of motives  $DM_{Nis}^{eff,-}(k, \mathbb{Z})$  corresponding to the complexes  $C_*\mathbb{Z}_{tr}(X)$  and  $C_*\mathbb{Z}_{tr}^c(X)$ . The functor  $C_*$  is exact and we can then define the *boundary motive* or the *motive at infinity*, denoted  $M^\infty(X)$  as the object in the derived category of motives corresponding to  $C_*(Coker(\iota_X))[-1]$ .

The motive at infinity is analogous to the *singular chain complex at infinity* (see [HR96] Definition 3.8 (i)). Similar to its topological analogue, the motive at infinity measures the extent to which  $X$  fails to be compact: if  $X$  is proper, the motive at infinity is trivial. The following result shows that any  $\mathbb{A}^1$ -contractible variety has motive at infinity that of a “motivic sphere” of appropriate dimension. Let  $\mathbf{DM}(\mathrm{Spec} k, \mathbb{Z})$  denote Voevodsky’s derived category of mixed motives with the Tate motive inverted (see [Voe00] p. 192).

**Lemma 6.9.** *Suppose  $X$  is an  $m$ -dimensional  $\mathbb{A}^1$ -contractible smooth algebraic variety. Then  $M^\infty(X) \cong \mathbb{Z} \oplus \mathbb{Z}(m)[2m - 1]$  (non-canonically) as objects in  $\mathbf{DM}(\mathrm{Spec} k, \mathbb{Z})$ .*

*Proof.* By definition of the motive at infinity, there is a distinguished triangle in  $\mathbf{DM}(\mathrm{Spec} k, \mathbb{Z})$  of the form

$$M^c(X)[-1] \longrightarrow M^\infty(X) \longrightarrow M(X) \longrightarrow M^c(X).$$

As  $X$  is  $\mathbb{A}^1$ -contractible, it follows that  $M(X) \cong \mathbb{Z}$ . Using motivic Poincaré duality (see [Voe00] Theorem 4.3.7), we see that

$$M^c(X) \cong M(X)^* \otimes \mathbb{Z}(m)[2m] \cong \mathbb{Z}(m)[2m].$$

Finally, we know that  $Hom_{\mathbf{DM}(\mathrm{Spec} k, \mathbb{Z})}(\mathbb{Z}, \mathbb{Z}(m)[2m]) \cong H^{2m, m}(\mathrm{Spec} k, \mathbb{Z})$  which vanishes for all  $m > 1$ . Therefore, we can find a splitting of the distinguished triangle just mentioned and we obtain an isomorphism  $M^\infty(X) \cong \mathbb{Z} \oplus \mathbb{Z}(m)[2m - 1]$ .  $\square$

## 7 Appendix

In this appendix, we review some aspects of faithfully flat descent, discuss Borel transfer, and prove the cohomological criterion for affine quotients (Theorem 3.12). These results are used in the main body of the text but we have presented them here so as not to interrupt the narrative flow.

### Faithfully flat descent

Let us begin by recalling one form of faithfully flat descent; this material is well known, but we collect it here for the convenience of the reader. A general reference for the material in this section is [SGA71] Expose VIII. Let  $G$  be a group scheme. For this section only, we will say  $Y$  is a *left* (resp. *right*)  $G$ -scheme if  $Y$  is a scheme equipped with a left (resp. right)  $G$ -action. If we do not specify the chirality of an action in a theorem, we mean the result holds for both left and right actions. We will use both left and right torsors. Finally, if  $Y$  is a  $G$ -scheme, we let  $\mathrm{Qcoh}^G(Y)$  denote the category of  $G$ -equivariant sheaves on  $Y$ .

**Theorem 7.1.** *Suppose  $G$  is a linear algebraic group and that  $f : \mathcal{P} \rightarrow X$  is a  $G$ -torsor over a scheme  $X$ . Then the the functor*

$$f^* : \mathrm{Qcoh}(X) \rightarrow \mathrm{Qcoh}^G(\mathcal{P})$$

*is an equivalence of categories. The functor  $\mathcal{F} \mapsto (f_*(\mathcal{F}))^G$  defines an explicit quasi-inverse to  $f^*$ .*

*Sketch of Proof.* Since  $f : \mathcal{P} \rightarrow X$  is a principal  $G$ -bundle, we have a canonical identification  $G \times \mathcal{P} \xrightarrow{\sim} \mathcal{P} \times_X \mathcal{P}$ . In particular since  $f$  is faithfully flat, we see that a descent datum for  $\mathcal{F}$  is equivalent to specifying a  $G$ -equivariant structure on a quasi-coherent sheaf.  $\square$

If  $Y$  is a  $G$ -scheme, we let  $\mathcal{A}^G(Y)$  denote the category of  $G$ -schemes affine over  $Y$  such that the structure morphism is  $G$ -equivariant. Suppose now that  $G$  is a linear algebraic group and  $H$  is a closed algebraic subgroup. Then we know the homogeneous space quotient  $G/H$  exists and that the morphism  $\pi : G \rightarrow G/H$  is a right  $H$ -torsor which is  $G$ -equivariant for the natural left  $G$ -actions on  $G$  and  $H$ . Furthermore, the structure morphism  $s : G \rightarrow \mathrm{Spec} k$  is a left  $G$ -torsor.

**Corollary 7.2.** *The functor  $\mathcal{F} \mapsto f_*(s^*\mathcal{F})^H$  defines an equivalence of categories*

$$\mathrm{Qcoh}^H(\mathrm{Spec} k) \xrightarrow{\sim} \mathrm{Qcoh}^G(G/H).$$

*This functor also determines an equivalence of categories*

$$\mathcal{A}^H(\mathrm{Spec} k) \xrightarrow{\sim} \mathcal{A}^G(G/H).$$

*An inverse to the last functor can be obtained by taking the scheme-theoretic fiber of a morphism  $f : X \rightarrow G/H$  over the identity coset of  $G/H$ .*

*Proof.* The first statement follows immediately from Theorem 7.1. The second statement follows because there is an equivalence of categories between schemes affine over  $Y$  and quasi-coherent sheaves of  $\mathcal{O}_Y$ -algebras. For more details see [SGA71] Expose VIII Thm 2.1.  $\square$

For an affine  $H$ -scheme  $X$ , we write  $G *_H X$  for the image of  $X$  under the functor of Corollary 7.2; the scheme  $G *_H X$  is referred to as a contracted product scheme. By construction it can be identified with the quotient of the product  $G \times X$  by the action of  $H$  defined by  $h \cdot (g, x) = (gh, h \cdot x)$ . Given a morphism  $f : X \rightarrow X'$  of affine  $H$ -schemes, we denote by  $G *_H f$  the induced morphism  $G *_H X \rightarrow G *_H X'$ .

*Remark 7.3.* In fact by Corollaire [SGA71] Expose VIII Cor. 7.9, if  $X$  is a *quasi-affine*  $H$ -scheme, then  $G *_H X$  exists as a scheme.

*Remark 7.4.* Through the paper we have assumed that the term “scheme” is synonymous with separated scheme, locally of finite type; for this remark we lift this convention. Theorem 7.1 and Corollary 7.2 above make no reference to any finiteness assumptions and hold as well for affine schemes that are not locally of finite type.

Let  $P(f)$  be a property of a morphism  $f : X \rightarrow Y$  of schemes that is stable by base change and local for the étale topology. The following properties of morphisms of schemes are of this form:

- surjective, radiciel, universally bijective, universally open, universally submersive, separated, quasi-compact, locally of finite type, finite type, open immersion, closed immersion, affine, quasi-affine, integral, geometrically irreducible fibers, geometrically reduced, geometrically connected fibers, flat, or smooth.

For a more complete list of such properties, we refer the reader to [SGA71] Expose VIII §3-8.

**Corollary 7.5.** *A morphism of quasi-affine  $H$ -schemes  $f : X \rightarrow X'$  has a property  $P(f)$  as in the previous paragraph if and only if the induced morphism  $G *_H f$  has the same property.*

*Proof.* This follows from Corollary 7.2 and Remark 7.3. We refer the reader to [SGA71] Exposes VIII §3, §4.  $\square$

*Remark 7.6.* The construction of Remark 7.3 and Corollary 7.5 then gives an equivalence of categories between the category of  $G$ -schemes quasi-affine over  $G/H$  (with quasi-affine morphisms) and the category of quasi-affine  $H$ -schemes (with quasi-affine morphisms).

### Naturality of contracted products

We now explore the naturality properties of the contracted product construction of the previous section. In the main body of the text, we will only consider the situation where  $H$  is a connected unipotent group or  $H$  is a reductive group.

**Theorem 7.7.** *Let  $G$  be a reductive linear algebraic group. Then the quotient  $G/H$  is affine if and only if  $H$  is a reductive subgroup of  $G$ . If  $U$  is a unipotent subgroup of  $G$ , then  $G/U$  is strictly quasi-affine.*

*Sketch of Proof.* The first part of this result is the statement of Matsushima's theorem (see [Hab78] Theorem 3.3). Matsushima's theorem also guarantees that if  $U$  is unipotent, then  $G/U$  is not affine. One can check that for any  $k$ -defined parabolic  $P$ , with unipotent radical  $R_u(P)$ , the quotient  $G/R_u(P)$  is strictly quasi-affine. Since  $U$  may be embedded in  $R_u(P)$  for some  $P$ , and the quotient  $R_u(P)$  is isomorphic to affine space, we get an étale locally trivial fiber bundle  $G/U \rightarrow G/R_u(P)$  with fibers isomorphic to affine space. Since the base of this fibration is quasi-affine, it follows that  $G/U$  is necessarily quasi-affine by 7.5. It follows that  $G/U$  is strictly quasi-affine.  $\square$

Suppose we have a sequence of inclusions  $U \hookrightarrow G \hookrightarrow G'$  where  $G'$  and  $G$  are reductive and  $U$  is quasi-affine. If  $X$  is a  $U$ -scheme, then it follows immediately from Corollary 7.5 and 7.7 that both  $G *_U X$  and  $G' *_U X$  are necessarily quasi-affine schemes. Indeed, the morphism  $X \rightarrow \text{Spec } k$  is affine, and the composite of an affine morphism and quasi-affine morphism is quasi-affine. Furthermore we have a canonical identification  $G' *_G (G *_U X) \xrightarrow{\sim} G' *_U X$ . We now study global sections of the structure sheaf on a contracted product scheme.

### Borel Transfer

Henceforth let us write  $k[Y]$  for the  $k$ -algebra of global sections  $\Gamma(Y, \mathcal{O}_Y)$  for any quasi-affine scheme  $Y$ . Suppose  $X$  is a quasi-affine  $U$ -scheme. The closed immersion  $X \hookrightarrow G \times X$  defined by  $x \mapsto (e, x)$  descends to a closed immersion  $\iota : X \hookrightarrow G *_U X$ . Pull-back by  $\iota$  defines a morphism  $k[G *_U X] \rightarrow k[X]$  sending  $G$ -invariant functions to  $U$ -invariant functions. The Borel transfer principle says that induced map

$$\iota^* : k[G *_U X]^G \xrightarrow{\sim} k[X]^U$$

is an isomorphism (this follows from e.g. [SGA71] Exposé VIII Cor. 1.7). This isomorphism is functorial in the following sense. Suppose  $G'$  is another reductive group such that  $G \subset G'$ . We then have induced morphisms  $\iota' : X \hookrightarrow G' *_U X$  and  $\iota_1 : G *_U X \hookrightarrow G' *_G (G *_U X)$ . Finally, we have a canonical identification  $\iota_1 \circ \iota = \iota'$ . Through the rest of this document, we will use the induced isomorphisms without comment.

### Proof of Theorem 3.12

Let us now prove Theorem 3.12. Essentially, this theorem is a long exercise in faithfully flat descent and we felt this justified its deferral to this appendix. Suppose  $\pi : X \rightarrow X/U$  is a principal bundle (in particular it is faithfully flat and affine). We can consider the Čech simplicial scheme  $\check{C}(\pi)$  attached to this morphism:

$$(7.1) \quad \cdots X \times_{X/U} X \rightrightarrows X \rightarrow X/U$$

In other words, the  $n^{\text{th}}$ -term of  $\check{C}(\pi)$  is the  $n + 1$ -fold fiber product of  $X$  over  $X/U$ ; the partial projections and the relative diagonals give the face and degeneracy maps. This is a simplicial scheme augmented toward  $X/U$ . Since  $X \rightarrow X/U$  is a  $U$ -torsor, the Čech simplicial scheme is

isomorphic as a simplicial scheme to the bar simplicial scheme whose  $n^{\text{th}}$  term is  $X \times U^{\times n}$ . We denote this scheme by  $X \times U^\bullet$ . As  $X$  and  $\pi$  are affine, it follows that  $\check{C}(\pi)$  is a simplicial affine scheme.

Given a quasi-coherent sheaf  $\mathcal{F}$  on  $X/U$ , we can consider the simplicial sheaf on  $\mathcal{F}$  on  $\check{C}(\pi)$  induced by pull-back. As  $\check{C}(\pi)$  is an affine simplicial scheme, the spectral sequence of cohomological descent attached to the morphism  $\pi$  (see [SGA72] V.bis 2.5) degenerates and gives us a complex

$$(7.2) \quad \check{C}(\pi, \mathcal{F}) = \Gamma(X/U, \mathcal{F}) \longrightarrow \Gamma(X, \pi^* \mathcal{F}) \longrightarrow \Gamma(X \times U, p_1^* \mathcal{F}) \longrightarrow \cdots$$

whose  $i$ -th cohomology computes the group  $H^i(X, \mathcal{F})$ . Furthermore, the isomorphism of simplicial schemes described in the previous paragraph identifies the groups  $H^i(X/U, \mathcal{F})$  with the group cohomology  $H^i(U, \Gamma(X, \pi^* \mathcal{F}))$ . With these notations, our proof is fairly streamlined.

*Proof of Theorem 3.12.* (i  $\implies$  ii). By the identifications of the previous paragraph, we see that  $0 = H^1(X/U, \mathcal{O}_{X/U}) = H^1(U, \Gamma(X, \mathcal{O}_X))$ . Now, we use the van Est spectral sequence, let us recall the setup (see [Tel98] §6.1). Let  $\text{Rep}(U)$  denote the category of locally finite algebraic  $U$ -modules, and similarly for  $\text{Rep}(\mathfrak{u})$ . There is a restriction functor  $\text{Res}_{\mathfrak{u}}^U : \text{Rep}(U) \longrightarrow \text{Rep}(\mathfrak{u})$ , and this functor possesses a right adjoint functor  $\text{Ind}_{\mathfrak{u}}^U$  of induction. let  $(\cdot)^U$  denote the functor of  $U$ -invariants and let  $(\cdot)^{\mathfrak{u}}$  denote the functor of  $\mathfrak{u}$ -invariants. The composite functor  $(\text{Ind}_{\mathfrak{u}}^U(\cdot))^U$  is isomorphic to the functor  $(\cdot)^{\mathfrak{u}}$ . If  $V$  is a  $\mathfrak{u}$ -module, we obtain a Grothendieck spectral sequence:

$$(7.3) \quad E_2^{p,q} = H^p(U, \mathbf{R}^q \text{Ind}(V)) \implies H^*(\mathfrak{u}, V).$$

The  $U$ -action can be untwisted, and we get an isomorphism

$$(7.4) \quad E_2^{p,q} \cong H^p(U, H_{dR}^q(U) \otimes V) \implies H^*(\mathfrak{u}, V).$$

where  $H_{dR}^q(U)$  is the algebraic de Rham cohomology of  $U$ . Since  $U$  is connected, it acts trivially on  $H_{dR}^q(U)$  and each  $E_2^{p,q}$  factors as a tensor product:  $H^p(U, H_{dR}^q(U) \otimes V) \cong H_{dR}^q(U) \otimes H^p(U, V)$ . The algebraic de Rham cohomology of  $U$  is trivial by homotopy invariance:  $U$  is isomorphic to affine space and hence the spectral sequence degenerates and defines an isomorphism of  $\mathfrak{u}$ -cohomology and  $U$ -cohomology. In particular, we see that  $H^1(U, \Gamma(X, \mathcal{O}_X))$  vanishes if and only if  $H^1(\mathfrak{u}, \Gamma(X, \mathcal{O}_X))$ -vanishes.

(ii)  $\implies$  iii). We can't assume that the quotient  $X/U$  exists as a scheme, but as  $U$  is a smooth  $k$ -group-scheme and  $X$  is a  $k$ -variety we can consider the Artin quotient stack  $[X/U]$  (see [LMB00] Chapter 3). The stack  $[X/U]$  comes equipped with the universal atlas  $X \longrightarrow [X/U]$ . Furthermore, the coherent cohomology of any sheaf on  $[X/U]$  can be computed as the cohomology of the associated Čech complex as described in the paragraph just preceding this proof (See e.g. [LMB00] §13.5). Using the argument of the previous paragraph identifying the  $\mathfrak{u}$ -cohomology of  $\Gamma(X, \mathcal{O}_X)$  and the  $U$ -cohomology of  $\Gamma(X, \mathcal{O}_X)$ , we see that  $H^1([X/U], \mathcal{O}_{[X/U]})$  vanishes.

Since  $H^1([X/U], \mathcal{O}_{X/U})$  vanishes, all  $U$ -torsors on  $[X/U]$  must be trivial, and hence the morphism  $\pi : X \longrightarrow [X/U]$  admits a section, i.e. a morphism  $s : [X/U] \longrightarrow X$ . We claim that such a morphism is necessarily representable. Indeed, as  $s$  is a section,  $s$  has the property

that the two composite morphisms are representable morphisms:  $s\pi = Id_{[X/U]}$  and  $\pi s$  is a morphism of schemes. The ‘morphism  $X \times_{[X/U]} X \rightarrow X \times X$  is representable as well as it is canonically isomorphic to the action map  $X \times U \rightarrow X \times X$ . It follows by [LMB00] Lemme 3.12 c(i), the section  $s$  is a representable morphism. Now, any stack admitting a representable morphism to a scheme must actually be an algebraic space by [LMB00] Lemme 3.12(a) but in this situation, it must actually be a scheme; hence  $[X/U]$  is actually a scheme which we denote by  $X/U$ . Furthermore, the quotient  $\pi : X \rightarrow X/U$  is then a  $U$ -torsor admitting a section, and must therefore be trivial.

(iii  $\implies$  i). Suppose  $\pi : X \rightarrow X/U$  is a trivial principal  $G$ -bundle. Then we have a product decomposition  $X \cong X/U \times U$  and  $\pi$  is identified with the projection onto the first factor. In this case, it follows from the Künneth formula for quasi-coherent sheaves that each term in the complex  $\check{C}(\pi)$  is isomorphic to  $\Gamma(X/U, \mathcal{F}) \otimes_k \Gamma(U^{n+1}, \mathcal{O}_{U^{n+1}})$ . Furthermore, the differential commutes with this identification, so the complex  $\check{C}(\pi)$  is actually isomorphic to  $\Gamma(X/U, \mathcal{F}) \otimes \Gamma(U^\bullet, \mathcal{O}_{U^\bullet})$ . Now, the cohomology of the complex  $\Gamma(U^\bullet, \mathcal{O}_{U^\bullet})$  is trivial for all  $i > 0$ , whence by Serre’s criterion for affine-ness we know  $X/U$  is affine.  $\square$

*Remark 7.8.* As a corollary of our proof, we see that the group  $H^1(\mathfrak{u}, k[X])$  is an *invariant* of the action because it is identified with the group  $H^1([X/U], U)$ .

## References

- [AKO02] Y. André, B. Kahn, and P. O’Sullivan. Nilpotence, radicaux et structures monoïdales. *Rend. Sem. Mat. Univ. Padova*, 108:107–291, 2002.
- [Bor91] A. Borel. *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991.
- [Bou98] N. Bourbaki. *Commutative algebra. Chapters 1–7*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1998. Translated from the French, Reprint of the 1989 English translation.
- [CK65] M. L. Curtis and K-W. Kwun. Infinite sums of manifolds. *Topology*, 3:31–42, 1965.
- [DF95] J. Deveney and D. Finston. A proper  $\mathbb{G}_a$  action on  $\mathbb{C}^5$  which is not locally trivial. *Proc. Amer. Math. Soc.*, 123(3):651–655, 1995.
- [DF99] D. Daigle and G. Freudenburg. A counterexample to Hilbert’s fourteenth problem in dimension 5. *J. Algebra*, 221(2):528–535, 1999.
- [DFvR04] J. K. Deveney, D. R. Finston, and P. van Rossum. Triangular  $\mathbb{G}_a$  actions on  $\mathbb{C}^4$ . *Proc. Amer. Math. Soc.*, 132(10):2841–2848 (electronic), 2004.
- [DI04] D. Dugger and D. C. Isaksen. Topological hypercovers and  $\mathbb{A}^1$ -realizations. *Math. Z.*, 246(4):667–689, 2004.
- [DK] B. Doran and F. Kirwan. Towards non-reductive geometric invariant theory. Preprint.
- [Dug01] D. Dugger. Universal homotopy theories. *Adv. Math.*, 164(1):144–176, 2001.
- [Fuj79] T. Fujita. On Zariski problem. *Proc. Japan Acad. Ser. A Math. Sci.*, 55(3):106–110, 1979.
- [Gla67] L. C. Glaser. Uncountably many contractible open 4-manifolds. *Topology*, 6:37–42, 1967.
- [GP93] G-M. Greuel and G. Pfister. Geometric quotients of unipotent group actions. *Proc. London Math. Soc. (3)*, 67(1):75–105, 1993.
- [Hab78] W. J. Haboush. Homogeneous vector bundles and reductive subgroups of reductive algebraic groups. *Amer. J. Math.*, 100(6):1123–1137, 1978.
- [Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.
- [HR96] B. Hughes and A. Ranicki. *Ends of complexes*, volume 123 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.
- [Hub] A. Huber. Slice filtration on motives and the Hodge conjecture. *Preprint*, available at <http://www.math.uiuc.edu/K-theory/0725/>.
- [KMT74] T. Kambayashi, M. Miyanishi, and M. Takeuchi. *Unipotent algebraic groups*. Springer-Verlag, Berlin, 1974. Lecture Notes in Mathematics, Vol. 414.
- [LMB00] G. Laumon and L. Moret-Bailly. *Champs algébriques*, volume 39 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2000.
- [Lun73] D. Luna. Slices étales. In *Sur les groupes algébriques*, pages 81–105. Bull. Soc. Math. France, Paris, Mémoire 33. Soc. Math. France, Paris, 1973.
- [Maz61] B. Mazur. A note on some contractible 4-manifolds. *Ann. of Math. (2)*, 73:221–228, 1961.
- [McM62] D. R. McMillan, Jr. Some contractible open 3-manifolds. *Trans. Amer. Math. Soc.*, 102:373–382, 1962.
- [MFK94] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]*. Springer-Verlag, Berlin, third edition, 1994.
- [Mil65] J. Milnor. *Lectures on the h-cobordism theorem*. Notes by L. Siebenmann and J. Sondow. Princeton University Press, Princeton, N.J., 1965.

- [Mora] F. Morel.  $\mathbb{A}^1$ -algebraic topology. In *Proceedings of the International Congress of Mathematicians, Vol. ? (Madrid, 2006)*. Preprint, available at <http://www.mathematik.uni-muenchen.de/~morel/listepublications.html>.
- [Morb] F. Morel.  $\mathbb{A}^1$ -homotopy classification of vector bundles over smooth affine schemes. Preprint, available at <http://www.mathematik.uni-muenchen.de/~morel/listepublications.html>.
- [MV99] F. Morel and V. Voevodsky.  $\mathbb{A}^1$ -homotopy theory of schemes. *Inst. Hautes Études Sci. Publ. Math.*, (90):45–143 (2001), 1999.
- [MZ62] D. R. McMillan, Jr. and E. C. Zeeman. On contractible open manifolds. *Proc. Cambridge Philos. Soc.*, 58:221–224, 1962.
- [Poe60] V. Poenaru. Les decompositions de l’hypercube en produit topologique. *Bull. Soc. Math. France*, 88:113–129, 1960.
- [Ram71] C. P. Ramanujam. A topological characterisation of the affine plane as an algebraic variety. *Ann. of Math. (2)*, 94:69–88, 1971.
- [SGA71] *Revêtements étales et groupe fondamental*. Springer-Verlag, Berlin, 1971. Séminaire de Géométrie Algébrique du Bois Marie 1960–1961 (SGA 1), Dirigé par A. Grothendieck. Augmenté de deux exposés de M. Raynaud, Lecture Notes in Mathematics, Vol. 224.
- [SGA72] *Théorie des topos et cohomologie étale des schémas. Tome 2*. Springer-Verlag, Berlin, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat, Lecture Notes in Mathematics, Vol. 270.
- [Sta62] J. Stallings. The piecewise-linear structure of Euclidean space. *Proc. Cambridge Philos. Soc.*, 58:481–488, 1962.
- [Tel98] C. Teleman. Borel-Weil-Bott theory on the moduli stack of  $G$ -bundles over a curve. *Invent. Math.*, 134(1):1–57, 1998.
- [Voe98] V. Voevodsky.  $\mathbb{A}^1$ -homotopy theory. In *Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998)*, number Extra Vol. I, pages 579–604 (electronic), 1998.
- [Voe00] V. Voevodsky. Triangulated categories of motives over a field. In *Cycles, transfers, and motivic homology theories*, volume 143 of *Ann. of Math. Stud.*, pages 188–238. Princeton Univ. Press, Princeton, NJ, 2000.
- [Whi35] J.H.C. Whitehead. A certain open manifold whose group is unity. *Quart. J. Math. Oxford*, (6):268–279, 1935.
- [Wil] J. Wildeshaus. The boundary motive: definition and basic properties. Preprint, available at <http://www.math.uiuc.edu/K-theory/0704/>.
- [Win90] J. Winkelmann. On free holomorphic  $\mathbb{C}$ -actions on  $\mathbb{C}^n$  and homogeneous Stein manifolds. *Math. Ann.*, 286(1-3):593–612, 1990.
- [Zai99] M. Zaïdenberg. Exotic algebraic structures on affine spaces. *Algebra i Analiz*, 11(5):3–73, 1999.