

TOWARDS NON-REDUCTIVE GEOMETRIC INVARIANT THEORY

BRENT DORAN AND FRANCES KIRWAN

To Bob MacPherson on his 60th birthday

ABSTRACT. We study linear actions of algebraic groups on smooth projective varieties X . A guiding goal for us is to understand the cohomology of “quotients” under such actions, by generalizing (from reductive to non-reductive group actions) existing methods involving Mumford’s geometric invariant theory (GIT). We concentrate on actions of unipotent groups H , and define sets of stable points X^s and semistable points X^{ss} , often explicitly computable via the methods of reductive GIT, which reduce to the standard definitions going back to Mumford for reductive actions. We compare these with definitions in the literature. Results include (1) a geometric criterion determining whether or not a ring of invariants is finitely generated, (2) the existence of a geometric quotient of X^s , and (3) the existence of a canonical “enveloping quotient” variety of X^{ss} , denoted $X//H$, which (4) has a projective completion given by a reductive GIT quotient and (5) is itself projective and isomorphic to $\text{Proj}(k[X]^H)$ when $k[X]^H$ is finitely generated.

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1. INTRODUCTION

Geometric invariant theory (GIT) is a powerful theoretical and computational tool for the study of reductive algebraic group actions. On the theory side, it provides a good notion of a quotient of an affine or projective variety; many key moduli spaces admit a description as a GIT quotient. More computationally, information about the geometry of projective GIT quotients, in particular their cohomology (indeed, K-theory, algebraic cobordism, motivic cohomology or any oriented cohomology theory [2, 30, 13, 6, 12]) can often be extracted using not much more than the theory of weights. As an added bonus, the “stable” structures identified by GIT tend to have very nice (differential) geometric properties [49]. A drawback is that GIT requires the group which acts to be reductive, whereas many interesting problems in moduli theory, affine geometry, and classical invariant theory are rooted in non-reductive actions. This paper addresses the question of how to develop an effective version of GIT for general affine algebraic group actions, including some of the difficulties, context, and motivating problems, with the eventual goal of computing the cohomology of quotients of nonsingular projective varieties by such actions.

Two key properties of actions of a reductive group G on an affine variety X are crucial to GIT. Firstly, by a theorem of Nagata, the ring of invariants $k[X]^G$ is finitely generated. Secondly, given any two disjoint closed G -invariant subvarieties of X , there exists an invariant function which separates them. These lead to the existence of a “good categorical quotient” $\pi : X \rightarrow X//G = \text{Spec}(k[X]^G)$, along with a distinguished open subset X^s of *stable* orbits such that $\pi(X^s) \subseteq X//G$ is a “geometric quotient”, that is, an orbit space with nice properties. Quotients of more general varieties X (equipped with linearizations of the group actions) are constructed by patching together suitable open affine pieces to get a categorical quotient $X//G$ of the *semistable* subset X^{ss} of X , with an open subset $X^s/G \subseteq X//G$ giving a geometric quotient of X^s .

Less essential, but still very useful, is a third property of reductive actions, that every invariant extends to an invariant of the ambient affine (or projective) space. This effectively reduces GIT to the study of representations of reductive groups on the affine (or projective) space itself. Fourthly and finally, the quotients $X//G$ of affine or projective varieties are again affine or projective, respectively, with the quotient maps $X^{ss} \rightarrow X//G$ surjective.

Each of these properties fails for non-reductive groups. Most famously, examples of non-finitely generated rings of invariants (counter-examples to Hilbert’s Fourteenth Problem) were first discovered by Nagata [42]. One might hope, however, that a good generalization of GIT to non-reductive group actions would, as in the reductive theory, satisfy the following properties:

- It would depend only on the data of a G -linearization of X ; that is, an affine or projective embedding of X , together with a lift of the action of G to the (homogeneous) coordinate ring $k[X]$.
- It would use the invariants in $k[X]$ to separate as many closed orbits as possible.

- It would provide notions of stable and semistable subsets X^s and X^{ss} with a canonical G -invariant morphism $X^{ss} \rightarrow X//G$ (ideally a categorical quotient) restricting to a geometric quotient of X^s .

It would also be convenient to have a good ‘change of groups’ formalism relating $X//H$ and $(G \times_H X)//G$ if H is a closed subgroup of G .

With this in mind, we discuss some generalizations of GIT to non-reductive group actions, including various notions of “stable” and “semistable” points. A direct approach is to patch together “nice enough” affine opens which admit finitely generated rings of invariants. Another method is to pass to reductive GIT by considering the associated reductive G -action on (projective completions of) $G \times_H X$; given a linearized H -action on X in \mathbb{P}^n , we can consider

$$G \times_H X \subseteq G \times_H \mathbb{P}^n \cong (G/H) \times \mathbb{P}^n,$$

for G a reductive group extending the action of H on \mathbb{P}^n , and reductive GIT on $G \times_H \mathbb{P}^n$ or an appropriate projective completion $\overline{G \times_H \mathbb{P}^n}$. In each approach, and conceptually perhaps this is the crucial point, only finitely many invariants are really being used to define the quotients. Somewhat remarkably, when properly formulated these approaches are compatible, and most of the conditions above are satisfied — though the “enveloping quotient” $X^{ss} \rightarrow X//H$ which we obtain fails to be a categorical quotient in general, and its image may not be a subvariety of $X//H$ but only a dense constructible subset. As a byproduct we obtain a geometric criterion for deciding whether or not the ring of invariants $k[X]^H$ is finitely generated. When X is projective and $k[X]^H$ is finitely generated then $X//H = \text{Proj}(k[X]^H)$ is a projective variety; more generally we obtain projective completions $\overline{X//H}$ of $X//H$ which are themselves reductive GIT quotients $\overline{G \times_H X} // G$ and hence, in principle, amenable to standard methods for understanding their geometry and topology.

For convenience we assume X is projective and work over an algebraically closed base field k of characteristic 0, which in examples will be taken to be \mathbb{C} . Over a characteristic 0 field any algebraic group splits as a semi-direct product of a reductive group and the unipotent radical. Effectively, one may first quotient by the unipotent action, then by the induced reductive action, provided that the unipotent quotient is sufficiently canonical to inherit an induced linear action of the reductive group. So what really must be understood is unipotent actions, and we shall concentrate on these.

This paper is intended to be sufficiently self-contained that readers with minimal background knowledge will find it accessible, so it is at times, by necessity or design, somewhat informal. After this introduction §2 discusses some problems in mathematics where non-reductive actions appear: moduli spaces, exotic affine spaces, Hilbert’s fourteenth problem, and so forth. §3 is a brief summary of the idea behind geometric invariant theory — using invariants to parametrize orbits — followed by an explanation of how key facts about reductive actions fail for non-reductive ones, and a summary of the main definitions and results of reductive GIT needed later. §4 discusses various ‘intrinsic’ ways to describe open subsets on which non-reductive actions admit nice quotients, and compares these to existing definitions and approaches in the literature. §5 develops the approach to non-reductive GIT via reductive GIT on an auxiliary space; there are three main results. Theorem 5.3.1 summarizes the relationships between the different notions of stable points,

semistable points, and their “quotients”, while Theorem 5.3.4 provides a stronger conclusion when a particular criterion is satisfied, which can often be arranged in practice (for example, when $k[X]^H$ is finitely generated). With these in mind, Definition 5.3.6 sets our notion of stable and semistable points. Theorem 5.3.16 gives a geometric criterion for deciding when a ring of invariants is finitely generated, relating it to a stability computation in reductive GIT. Finally §6 discusses a family of \mathbb{C}^+ -actions as a straightforward example and computes the (intersection) cohomology of their enveloping quotients $X//H$.

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2. MOTIVATION

Reductive group actions are of great significance in algebraic geometry, but non-reductive actions appear in many important problems. Over \mathbb{C} a group G is reductive if and only if it is the complexification of a maximal compact subgroup K , and many nice properties can be obtained by exploiting this underlying compactness. The simplest example here is of course the complexification \mathbb{C}^* of the circle S^1 , and more generally $GL(n; \mathbb{C})$ viewed as the complexification of the unitary group $U(n)$; in contrast \mathbb{C}^+ has no nontrivial compact subgroups. Given the ubiquity of translation actions, it is not surprising that non-reductive groups appear in so many problems, though sometimes they are well hidden. Two very recent examples include Bridgeland–Douglas stability conditions in derived categories of sheaves, via a duality discussed in [26], and the study of hyperbolic varieties via Griffiths–Green jet bundles, after Demailly, et al. [46]; some more classical examples are given below.

2.1. Moduli spaces. In the preface to the first edition of [41], Mumford states that his goal is “to construct moduli schemes for various types of algebraic objects” and that this problem “appears to be, in essence, a special and highly non-trivial case” of the problem of constructing orbit spaces for algebraic group actions. More precisely, when a family of objects with parameter space S has the local universal property for a given moduli problem, and a group acts on S such that objects parametrized by points in S are equivalent if and only if the points lie in the same orbit, then the construction of a coarse moduli space is equivalent to the construction of a categorical quotient which is an orbit space for the action (cf. [43, Proposition 2.13]). There are many cases of moduli problems involving non-reductive group actions. Here are a few examples.

- Moduli of singularities (or modules over the local ring of a singularity) as studied in [21, 22]; here the rough idea is that translation actions arise from unfoldings of singularities.
- Moduli of suitable maps; for example, degree d maps between projective spaces or with additional constraints (see [14, pp. 80–83]).
- Moduli of hypersurfaces in toric varieties.

This last example generalizes the very classical moduli problem of hypersurfaces in projective space \mathbb{P}^n . There the space of hypersurfaces of degree d is parametrized by $\mathbb{P}(\text{Sym}^d(\mathbb{C}^{n+1}))$ and the equivalence is given by the natural linear action of the reductive group $SL(n+1; \mathbb{C})$. There is a canonical (categorical) projective GIT quotient with an open subset which is a coarse moduli space for stable hypersurfaces.

For a general toric variety X , the analogous parameter space is also a projective space, but the action is a linear action of a non-reductive group. Such spaces arise, for example, in the study of moduli spaces of Calabi-Yau varieties and mirror symmetry [7]. Lacking a theory of non-reductive quotients, the standard trick is to study an associated “simplified” moduli space which arises as a quotient of a torus action; it is a branched cover of the actual moduli space and understanding the geometry of its compactification is complicated and not entirely naturally related to the actual moduli problem.

Example 2.1.1. Let X be the weighted projective space $\mathbb{P}(1, 1, 2)$, with homogeneous coordinates x, y, z , and consider the moduli problem of weighted degree 4 hypersurfaces (which are elliptic curves) in this toric variety X . The relevant group action is that of the automorphism group of X , which lifts to a semidirect product of the unipotent group $(\mathbb{C}^+)^3$ acting via

$$[x : y : z] \mapsto [x : y : z + \lambda x^2 + \mu xy + \nu y^2] \quad \text{for } (\lambda, \mu, \nu) \in (\mathbb{C}^+)^3$$

and the reductive group $GL(2; \mathbb{C}) \times GL(1; \mathbb{C})$ acting on the (x, y) coordinates and the z coordinate. A basis for weighted degree 4 polynomials is

$$\{x^4, x^3y, x^2y^2, xy^3, y^4, x^2z, xyz, y^2z, z^2\}.$$

With respect to this basis, the $(\mathbb{C}^+)^3$ action is linearly represented as:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & \lambda^2 \\ 0 & 1 & 0 & 0 & 0 & \mu & \lambda & 0 & 2\lambda\mu \\ 0 & 0 & 1 & 0 & 0 & \nu & \mu & \lambda & 2\lambda\nu + \mu^2 \\ 0 & 0 & 0 & 1 & 0 & 0 & \nu & \mu & 2\mu\nu \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \nu & \nu^2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2\lambda \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2\mu \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2\nu \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

2.2. Affine geometry. An abundant supply of translation symmetries means that non-reductive actions occur everywhere in affine geometry. In particular the automorphism group $\text{Aut}(\mathbb{A}^n)$ of n -dimensional affine space is a surprisingly rich and mysterious structure, which ties into many of the famous questions in the area. To name one example, the *Jacobian Conjecture* is essentially the statement that automorphisms of \mathbb{A}^n naturally induce locally nilpotent derivations, i.e. unipotent actions [50, §2.2]. Another instance is the existence of *exotic affine spaces* (varieties diffeomorphic but not algebraically isomorphic to \mathbb{A}^n), where constructions of examples directly or implicitly make heavy use of unipotent actions. Perhaps the cleanest to write down is the Russell cubic three-fold:

$$X = \{(w, x, y, z) \in \mathbb{C}^4 \mid x^2w + x + y^2 + z^3 = 0\},$$

which Makar-Limanov [37] proves is exotic effectively by showing it has a smaller set of \mathbb{C}^+ actions than \mathbb{C}^3 does. Perhaps more strikingly, some exotic affine spaces arise as quotients by unipotent actions [52, 1].

Example 2.2.1. Let $\phi_t : \mathbb{C}^5 \rightarrow \mathbb{C}^5$ be the free \mathbb{C}^+ quadratic action given by:

$$\phi_t(w_1, w_2, w_3, w_4, w_5) = (w_1, w_2, w_3 + tw_1, w_4 + tw_2, w_5 + t(1 + w_1w_4 - w_2w_3)).$$

This action yields a geometric quotient of \mathbb{C}^5 which is diffeomorphic to \mathbb{C}^4 and quasi-affine but *not* affine.

Moreover Winkelmann [53] has shown that the study of the coordinate rings of quasi-affine varieties over any field k is precisely the study of the invariant subrings of affine varieties under affine algebraic group actions (indeed k^+ actions).

Theorem 2.2.2. (Winkelmann) *If R is an integrally closed k -algebra then the following are equivalent:*

- *there exists a quasi-affine irreducible, reduced k -variety X such that $R \cong k[X]$;*
- *there exists an irreducible, reduced k -variety X and a subgroup G of $\text{Aut}(X)$ such that $R \cong k[X]^G$;*
- *there exists an affine irreducible, reduced k -variety X and a regular action of $G = k^+$ on X such that $R \cong k[X]^G$.*

2.3. Classical invariant theory. The fourteenth of Hilbert's problems posed at the 1900 ICM was the following question: If an algebraic group acts linearly on a polynomial ring in finitely many variables, is the ring of invariants always finitely generated? The answer is yes for reductive groups (and for some non-reductive groups, in particular for \mathbb{C}^+), but Nagata [42] showed that the answer is no, in general, though counterexamples have not been easy to find. Nagata's original counterexample was an action of $(\mathbb{C}^+)^{13}$. Much later Mukai found an action of $(\mathbb{C}^+)^3$ where the ring of invariants is not finitely generated [38], and made further generalizations in [39, 40]; see [8] for some very recent related results on $(\mathbb{C}^+)^2$ -actions. Popov [44] used Nagata's counterexample to show that if an algebraic group G is not reductive then there is an affine variety X such that $k[X]^G$ is not finitely generated.

3. GIT BASICS

If X is a normal quasi-projective G -variety, then there exists a G -linearization; that is, a G -equivariant embedding in some projective space together with a lift of the G -action to the (homogeneous) coordinate ring. The main goal of geometric invariant theory is to provide a natural algebraic variety (depending only on the choice of linearization) which parametrizes G -orbits in an affine or projective variety X by using invariant functions (or sections) in the (homogeneous) coordinate ring $A = k[X]$. When the ring A^G of *all* invariants is a finitely generated k -algebra, as is the case if G is reductive, we obtain an affine variety $\text{Spec}(A^G)$ and a morphism

$$F : X = \text{Spec}(A) \longrightarrow \text{Spec}(A^G)$$

for affine X , or

$$F : X = \text{Proj}(A) \dashrightarrow \text{Proj}(A^G)$$

for projective X , induced by the inclusion $A^G \rightarrow A$. In the affine case F is a dominant morphism, whereas in the projective case it is just a dominant rational map; F is not defined precisely where all the invariants vanish, known as the *unstable set*. If X is affine and $\{f_i\}_{1 \leq i \leq n}$ is a finite generating set for A^G , then one may realize F as the morphism to affine space $F = (f_1, \dots, f_n) : X \rightarrow \mathbb{A}^n$, whose image is $\text{Spec}(A^G)$; a similar statement holds in the projective case.

When A^G is *not* finitely generated we can still consider $\text{Spec}(A^G)$ and $\text{Proj}(A^G)$ but only as schemes, not varieties.

Definition 3.0.1. Let X be a quasi-projective variety, and L an ample line bundle endowed with a lifting of the G -action. A *finite separating set of invariants* is a collection of invariant sections $\{f_1, \dots, f_n\}$ of positive tensor powers of L such that, if x, y are any two points of X , and f is an arbitrary invariant section of $L^{\otimes k}$ for some $k > 0$, then

$$f(x) = f(y) \Leftrightarrow f_i(x) = f_i(y), \forall i = 1, \dots, n.$$

The Noetherian property guarantees that finite separating sets of invariants always exist. When X is affine any finite collection of invariant functions f_1, \dots, f_n defines a G -invariant map $(f_1, \dots, f_n) : X \rightarrow \mathbb{A}^n$ (likewise in the projective case), and one could hope that when $\{f_1, \dots, f_n\}$ is a finite separating set of invariants the image of (f_1, \dots, f_n) might be independent of the choice of $\{f_1, \dots, f_n\}$. Unfortunately this is not true in general, but a refinement of the idea does work (cf. Proposition 4.2.8 below).

For a general affine G over a characteristic zero field, there is a semi-direct product decomposition $G \cong G_r \ltimes G_u$ into a reductive subgroup G_r and the unipotent radical G_u of G , and the corresponding rings of invariants satisfy $A^G = (A^{G_u})^{G_r}$. So the key to non-reductive invariant theory lies in unipotent actions.

3.1. Comparison of reductive and non-reductive. To see how non-reductive actions differ from reductive ones, it suffices to work with unipotent groups. Unipotent actions have several appealing features. Firstly, every orbit is closed, so unipotent actions are always closed in the sense of Mumford; this is helpful for GIT since invariants cannot distinguish an orbit from another orbit in its closure. Secondly, every orbit is isomorphic to an affine space. Thirdly, for connected unipotent groups, there are no non-trivial finite isotropy groups. However there are also several ways in which unipotent actions behave less well than reductive actions. We list some key properties of reductive invariant theory which make GIT so effective, and follow them with unipotent counter-examples.

Proposition 3.1.1. [41, Corollary 1.2] *Let X be an affine variety with a reductive G -action. Given two disjoint G -invariant closed subvarieties Y_1 and Y_2 in X , there exists an invariant $f \in k[X]^G$ such that $f(Y_1) = 0$ and $f(Y_2) = 1$.*

Proposition 3.1.2. [41, Theorem 1.1(3)] *Let X be a G -invariant affine subvariety of \mathbb{A}^n where G is reductive and acts on \mathbb{A}^n . Then given $f \in k[X]^G$ there exists an extension F of f to an invariant on \mathbb{A}^n . More precisely, if I_X is the defining ideal of X , then $k[X]^G = k[\mathbb{A}^n]^G / I_X \cap k[\mathbb{A}^n]^G$.*

Example 3.1.3. Let \mathbb{C}^+ act on $X = \text{Sym}^2(V)$ via its inclusion as upper triangular matrices in the defining two-dimensional representation V of $SL(2; \mathbb{C})$.

Here the invariants are easy to describe: $A^{\mathbb{C}^+} = \mathbb{C}[x_0, x_1^2 - 2x_0x_2]$. Observe that, even though this is an action on an affine space, the invariants do *not* separate all closed orbits – the pairs of lines $\{x_0 = 0, x_1 = \pm a\}$ for any given $a \neq 0$ are not distinguished, nor are any of the fixed points of the line $\{x_0 = 0, x_1 = 0\}$. Furthermore, observe that *not all* invariants of the hyperplane defined by $x_0 = 0$ lift to invariants of $X \cong \mathbb{C}^3$; for example x_1 is an invariant function of the \mathbb{C}^+ action on the hyperplane, but does not extend to an invariant of the action on X . (The fact that some power, specifically x_1^2 , is the restriction of an invariant on X is more

FIGURE 1. \mathbb{C}^+ -orbits in $X = \text{Sym}^2(V)$

generally true whenever the group action on the subvariety can be extended to a reductive group action on the subvariety, in this case to the action of $SL(2; \mathbb{C})$.)

This example illustrates two important principles: firstly, that non-closed orbits are not the only source of non-separated orbit spaces; and secondly, that set-theoretic freeness does not imply scheme-theoretic freeness, as the action of \mathbb{C}^+ on the complement of the fixed point line $\{x_0 = 0, x_1 = 0\}$ is not even separated — because the parabolic orbits in Figure 3.1.3 degenerate into pairs of disjoint orbits in the (x_0, x_1) -plane — let alone scheme-theoretically free [41, Definition 0.8] (that is, set-theoretically free and proper [12, §6.3 Lemma 8]).

Proposition 3.1.4. [9, Theorem 3.3] (*Nagata*) *Let G be a reductive group acting on an affine variety $X = \text{Spec}(A)$. Then A^G is a finitely generated k -algebra.*

Example 3.1.5 (Nagata counter-example). Consider the n -fold direct sum of the defining representation for $SL(2; \mathbb{C})$, and look at the subgroup corresponding to the direct sum of the n Borel subgroups (each one is a semi-direct product of \mathbb{C}^+ and \mathbb{C}^*):

$$\begin{pmatrix} c_1 & a_1 & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & c_1 & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & c_2 & a_2 & 0 & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & c_2 & 0 & \dots & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & \dots & \dots & 0 & 0 & c_n \end{pmatrix}$$

Demand that $\prod_i c_i = 1$. Then for appropriate n and a closed subgroup G cut out by an appropriate set of linear conditions on the a_i , the ring of invariants $\mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]^G$ is not finitely generated [9, Theorem 4.3].

Proposition 3.1.6. *Let G be a reductive group acting on an affine variety $X = \text{Spec}(A)$. Then the quotient maps $q : \text{Spec}(A) \rightarrow \text{Spec}(A^G)$ and $q : \text{Proj}(A)^{ss} \rightarrow \text{Proj}(A^G)$ are surjective.*

Example 3.1.7. This proposition says that the GIT quotient of an affine (projective) variety by a reductive group action is affine (projective), whereas in contrast Example 2.2.1 provides an affine variety whose quotient by a non-reductive group action is quasi-affine but *not* affine.

3.2. Geometric and categorical quotients. Recall (from [43, Chapter 2, §4], for example) that when a group G acts on a variety X a *categorical quotient* of X

by G is a morphism $\phi : X \rightarrow Y$ from X to a variety Y which is G -invariant (that is, constant on G -orbits) and has the property that any other G -invariant morphism $\tilde{\phi} : X \rightarrow \tilde{Y}$ factors as $\tilde{\phi} = \chi \circ \phi$ for a unique morphism $\chi : Y \rightarrow \tilde{Y}$. An *orbit space* for the action is a categorical quotient $\phi : X \rightarrow Y$ such that $\phi^{-1}(y)$ is a single G -orbit for each $y \in Y$, and a *geometric quotient* is an orbit space $\phi : X \rightarrow Y$ with the following *good* properties: it is an affine morphism such that

(i) if U is open in Y then

$$\phi^* : k[U] \rightarrow k[\phi^{-1}(U)]$$

induces an isomorphism of $k[U]$ onto $k[\phi^{-1}(U)]^G$, and

(ii) if W_1 and W_2 are disjoint closed G -invariant subvarieties of X then their images $\phi(W_1)$ and $\phi(W_2)$ in Y are disjoint closed subvarieties of Y .

Thus an orbit space is a geometric quotient if and only if it is a good categorical quotient.

3.2.1. GIT for reductive group actions on projective varieties. Let X be a projective variety over k and let G be a reductive group acting on X . For Mumford's geometric invariant theory we also require a *linearization* of the action; that is, a line bundle L on X and a lift¹ of the action of G to L . When L is ample and X is normal we can assume without essential loss of generality that for some projective embedding $X \subseteq \mathbb{P}^n$ the action of G on X extends to an action on \mathbb{P}^n given by a representation

$$\rho : G \rightarrow GL(n+1; k),$$

where L is the hyperplane line bundle on \mathbb{P}^n . There is an induced action of G on the homogeneous coordinate ring

$$A = k[X] \stackrel{def}{=} \bigoplus_{k \geq 0} H^0(X, L^{\otimes k})$$

of X , which, when $X \subseteq \mathbb{P}^n$ as above, is the quotient $k[x_0, \dots, x_n]/\mathcal{I}_X$ of the polynomial ring $k[x_0, \dots, x_n]$ by the ideal \mathcal{I}_X generated by the homogeneous polynomials vanishing on X . The subring A^G of A consisting of the elements of A left invariant by G is a graded k -algebra, which by Nagata's theorem is finitely generated because G is reductive [42], so we can define $X//G$ (or $X//_L G$ when the dependence on the linearization L is to be made explicit) to be the variety $\text{Proj}(A^G)$ associated to this ring of invariants A^G . The inclusion of A^G in A defines a rational map ϕ from X to $X//G$, but because there may be points of $X \subseteq \mathbb{P}^n$ where every nonconstant G -invariant homogeneous polynomial vanishes, this map will not in general be a morphism.

We define the set $X^{ss}(L)$ (abbreviated to X^{ss} when there is no risk of confusion) of *semistable* points for the action of G on X with respect to the ample linearization L to consist of those $x \in X$ for which there exists some $k > 0$ and $s \in H^0(X, L^{\otimes k})^G$ not vanishing at x . Then the rational map ϕ restricts to a surjective G -invariant morphism from the open subset X^{ss} of X to the projective variety $X//G$, and $\phi : X^{ss} \rightarrow X//G$ is a categorical quotient for the action of G on X^{ss} . Set-theoretically, $X//G$ is the quotient of X^{ss} by the equivalence relation for which x and y in X^{ss} are equivalent if and only if the closures $\overline{O_G(x)}$ and $\overline{O_G(y)}$ of the G -orbits of x and y meet in X^{ss} .

¹When there is no risk of confusion we will use L to denote the linearization as well as the underlying line bundle.

In order to obtain a geometric quotient, we define a *stable* point for the linear action of G on X to be a point x of X^{ss} with a neighborhood in X^{ss} such that every G -orbit meeting this neighborhood is closed in X^{ss} , and is of maximal dimension equal to the dimension of G (a “properly stable point” in the sense of [41, Definition 1.8]). If U is any G -invariant open subset of the set $X^s = X^s(L)$ of stable points of X , then $\phi(U)$ is an open subset of $X//G$ and the restriction $\phi|_U : U \rightarrow \phi(U)$ of ϕ to U is a universal geometric quotient (it remains a geometric quotient under base change) for the action of G on U . In particular, there is a geometric quotient $X^s/G = \phi(X^s)$ for the action of G on X^s , and $X//G$ can be thought of as a projective completion of the quasi-projective variety X^s/G :

$$\begin{array}{ccccc} X^s & \subseteq & X^{ss} & \subseteq & X \\ & \text{open} & & \text{open} & \\ \downarrow & & \downarrow & & \\ X^s/G & \subseteq & X^{ss}/\sim & = & X//G. \\ & \text{open} & & & \end{array}$$

Remark 3.2.1. Each of X^s , X^{ss} , and $X//G$ remains unchanged if the linearization L is replaced by $L^{\otimes m}$ for any $m > 0$.

Remark 3.2.2. When X is a connected nonsingular projective variety and G is a connected reductive group acting on X with generic stabilizer having dimension 0, then a G -invariant open subset U of X is proper and there exists a quasi-projective geometric quotient $\phi : U \rightarrow Y$ if and only if $U \subseteq X^s(L)$ for some linearization L of the action of G (see [41, Converse 1.13]).

Remark 3.2.3. Let T be a maximal torus of a reductive group G acting linearly on X . Then the subsets X^{ss} and X^s of X are characterised by the following properties (essentially the Hilbert-Mumford criteria for stability and semistability) which make them easy to identify:

(i) A point $x \in X$ is semistable (respectively stable) for the linear action of G on X if and only if gx is semistable (respectively stable) for the action of T on X for every $g \in G$.

(ii) If the maximal torus T of G acts diagonally on $X \subseteq \mathbb{P}^n$ with weights $\alpha_0, \dots, \alpha_n$, then a point $x = [x_0, \dots, x_n] \in X$ is semistable (respectively stable) for the action of T if and only if the convex hull

$$\text{Conv}\{\alpha_i : x_i \neq 0\}$$

in \mathfrak{t}^* contains 0 (respectively contains 0 in its interior) where \mathfrak{t}^* is the Lie algebra of T .

Example 3.2.4. Consider the linear action of \mathbb{C}^* on $X = \mathbb{P}^n$, with respect to the hyperplane line bundle on \mathbb{P}^n , where the linearization L_0 is given by the representation

$$t \mapsto \text{diag}(t^2, t^2, \dots, t^2, 1)$$

of \mathbb{C}^* in $GL(n+1; \mathbb{C})$. The same action of \mathbb{C}^* on \mathbb{P}^n has other linearizations with respect to the hyperplane line bundle; let L_+ denote the linearization given by the representation

$$t \mapsto \text{diag}(t^3, t^3, \dots, t^3, t)$$

of \mathbb{C}^* in $GL(n+1; \mathbb{C})$ and let L_- denote the linearization given by the representation

$$t \mapsto \text{diag}(t, t, \dots, t, t^{-1})$$

of \mathbb{C}^* in $GL(n+1; \mathbb{C})$. Then

$$X^{ss}(L_+) = X^s(L_+) = \emptyset,$$

while

$$X^{ss}(L_0) = \mathbb{C}^n, \quad X^s(L_0) = \emptyset,$$

and

$$X^{ss}(L_-) = X^s(L_-) = \mathbb{C}^n \setminus \{0\}.$$

3.2.2. Reductive GIT and quasi-projective varieties. If a G -action on a variety X has a categorical quotient $\phi : X \rightarrow Y$ then its restriction to a G -invariant open subset of X is not necessarily a categorical quotient for the action of G on U , as the following simple example shows.

Example 3.2.5. Let the multiplicative group \mathbb{C}^* of \mathbb{C} act on \mathbb{C}^n as multiplication by scalars. Since the origin lies in the closure of every orbit, it follows that every G -invariant morphism $\phi : \mathbb{C}^n \rightarrow Y$ is constant and hence that the constant map from \mathbb{C}^n to a point is a categorical quotient for the action. However the restriction of this constant map to $\mathbb{C}^n \setminus \{0\}$ is not a categorical quotient, since the natural map $\mathbb{C}^n \setminus \{0\} \rightarrow \mathbb{P}^{n-1}$ is a nonconstant G -invariant morphism, and indeed is a categorical quotient for the action of \mathbb{C}^* on $\mathbb{C}^n \setminus \{0\}$. Thus in Example 3.2.4 above we have $\mathbb{P}^n //_{L_-} \mathbb{C}^* = \mathbb{P}^{n-1}$ while $\mathbb{P}^n //_{L_+} \mathbb{C}^*$ is empty and $\mathbb{P}^n //_{L_0} \mathbb{C}^*$ is a point.

Similarly, although following Mumford [41] we can define stable and semistable points for any linear action of a reductive group G on a quasi-projective variety X , it is not necessarily the case that $U^{ss} = U \cap X^{ss}$ or that $U^s = U \cap X^s$ whenever U is a G -invariant open subset of X .

Definition 3.2.6. Let X be a quasi-projective variety with an action of a reductive group G which lifts to a line bundle L on X . Then $y \in X$ is *semistable* for this linear action if there exists some $m \geq 0$ and $f \in H^0(X, L^{\otimes m})^G$ not vanishing at y such that the open subset

$$X_f := \{x \in X \mid f(x) \neq 0\}$$

is affine, and y is *stable* if in addition the action of G on X_f is closed with all stabilizers of dimension 0.

Remark 3.2.7. Note that the line bundle L is not required to be ample here. When X is projective and L is ample and $f \in H^0(X, L^{\otimes m})^G \setminus \{0\}$ for some $m \geq 0$, then X_f is affine if and only if f is nonconstant or equivalently $m > 0$. Thus Definition 3.2.6 agrees with §3.2.1 for projective X and ample L . Moreover when $X \subseteq \mathbb{A}^n \subseteq \mathbb{P}^n$ is *affine* and the linear G -action is given by a representation of G in $GL(n; k)$ embedded in $GL(n+1; k)$ in the usual way by taking a direct sum with a one-dimensional trivial representation of G , then we have

$$X^{ss} = X$$

so that $X//G = \text{Spec}(k[X]^G)$ is a categorical quotient of X , and $x \in X$ is stable if and only if there is some $f \in k[X]^G$ such that the action of G on X_f is closed with all stabilizers having dimension 0.

Remark 3.2.8. One can see from these definitions that three things can go wrong in relating stable and semistable points for open immersions $U \subseteq X$: (1) invariants do not necessarily extend, and even if an invariant f extends then (2) affineness of X_f is neither a necessary nor a sufficient for U_f to be affine, and (3) the action on U_f may be closed with all stabilizers of dimension 0 without the same being true of X_f .

Theorem 3.2.9 (Mumford). [41, Theorem 1.10] *Let X be an algebraic scheme over k with an action of a reductive group G which lifts to a line bundle L on X . Then X^{ss} has a quasi-projective categorical quotient $\phi : X^{ss} \rightarrow X//G$ which restricts to a geometric quotient $\phi : X^s \rightarrow \phi(X^s)$ of X^s , where $\phi(X^s) = X^s/G$ is an open subset of $X//G$.*

Remark 3.2.10. The maximality property of X^s described in Remark 3.2.2 extends to nonsingular quasi-projective varieties X .

4. GENERALIZING GIT: INTRINSIC VIEWPOINTS

Throughout, unless otherwise stated, X will be a projective variety endowed with an ample line bundle L , on which an affine algebraic group H acts linearly. As discussed previously, H may be assumed to be connected unipotent.

4.1. Global approaches. This subsection will be fairly informal, serving three purposes – to set the stage as naively as possible, to present some definitions, and to give a brief survey of some of the literature.

The inclusion $k[X]^H \hookrightarrow k[X]$ induces a rational “quotient” map of schemes $q : X \dashrightarrow \text{Proj}(k[X]^H)$. The image of q is a constructible subset, i.e. a finite union of locally closed subschemes. The goal is to understand q well enough to construct canonically associated “quotient” varieties. There are a couple of interrelated approaches to analyzing q and possible quotients:

- take large enough finitely generated approximations to $k[X]^H$, and
- understand q restricted to open sets on which the action is nice.

For simplicity of notation we shall assume that the generic stabilizer has dimension 0; the discussion in general is not significantly different.

4.1.1. Naive stability. As in reductive GIT for projective varieties with ample line bundles, we might think of the semistable points as the points where some H -invariant does not vanish; that is, the points in the domain of definition of q .

Definition 4.1.1. A point $x \in X$ is *naively semistable* if $q(x)$ is well-defined. The set of naively semistable points X^{nss} is the domain of definition of q .

Consider an increasing filtration of $A = k[X]^H$ by finitely generated subrings A_i with $\varinjlim (A_i) = A$. The inclusions $A_i \rightarrow A_{i+1} \rightarrow \cdots \rightarrow A \rightarrow k[X]$ induce rational maps

$$q_i : X \dashrightarrow \text{Proj}(A) \dashrightarrow \cdots \dashrightarrow \text{Proj}(A_{i+1}) \dashrightarrow \text{Proj}(A_i).$$

Because X is Noetherian, the unstable set $X \setminus X^{nss}$ is cut out by finitely many invariants, so for large enough i , the domain of definition of q_i is X^{nss} . Furthermore, when i is large enough, A_i contains a finite separating set of invariants; the natural rational map $q_{i+1,i} : \text{Proj}(k[X]_{i+1}^H) \dashrightarrow \text{Proj}(k[X]_i^H)$ then maps $q_{i+1}(X^{nss})$ bijectively onto $q_i(X^{nss})$.

It therefore makes sense to look for open subsets U of X^{nss} whose images $q_i(U)$ are varieties independent (up to isomorphism) of i , once i is sufficiently large. Upper semi-continuity of q_i for large i determines a finite filtration of X^{nss} by opens U_j indexed according to the maximal dimension $j \geq \dim(H)$ of components of the fiber at a point; provided that A_i contains a separating set of invariants this filtration is independent of i . Restricting one's attention to the smallest of these open subvarieties, $U_{\dim(H)}$, is a good way to find analogs of “stable” points, since q can only be an orbit map (and hence possibly a geometric quotient) on subsets of $U_{\dim(H)}$.

Definition 4.1.2. Let X be a projective variety with a linear H action. A point $x \in X^{nss}$ is said to be *almost stable* if its stabilizer group is trivial and $\dim(q^{-1}(q(x))) = \dim(H)$. The set of almost stable points we denote by X^{as} .

Remark 4.1.3. If $k[X]^H$ is finitely generated and $\text{Proj}(k[X]^H)$ is normal, then by standard results (see [5, Proposition 18.4]) the morphism q restricted to X^{as} is an open mapping, so its image is a variety. Restricting q further to $U = q^{-1}(V)$ where V is the union of open subsets of $q(X^{as})$ over which all geometric fibres are single orbits — what might be called the “*naively stable*” set — it follows [5, Proposition 6.6] that we obtain a geometric quotient (in fact this holds in general, see Remark 4.2.5).

Remark 4.1.4. Fauntleroy [15] considered an analog of this concept of almost stability: for Y quasi-affine and normal on which a connected unipotent group H acts with finite stabilizers, and $p : Y \rightarrow \text{Spec}(k[Y]^H)$ the canonical morphism, if $\dim(p^{-1}(p(y))) = \dim(H)$ then he calls y “semistable” for the action of H . He shows that p restricted to the open subvariety of Y consisting of these semistable points is a categorical quotient (in the category of varieties). Fauntleroy's results applied to open subvarieties of an affine cone over a normal projective variety X with a linear H -action show that $q(X^{as})$ is a variety.

Another option is to find “quotient” varieties Q , obtained by throwing out selected closed H -invariant subvarieties of $\text{Proj}(A_i)$ for sufficiently large i , which are as close as possible to the image of q_i — ideally while retaining some other good feature, such as $k[X]^H = k[Q]$. To that end, removing those (finitely many) codimension 1 divisors in $\text{Proj}(A_i)$ which contain dense open subsets disjoint from $\text{im}(q_i)$ is a reasonable approach, although some points of $\text{im}(q_i)$ are typically removed in the process and some points not in $\text{im}(q_i)$ may remain: this is, very roughly, what we do later with strong reductive envelopes (see Definition 5.2.7 and, with more stringent conditions, Theorem 5.3.4). In any such construction, whether the quotient variety is suitably canonical becomes a serious issue.

Remark 4.1.5. In the case when X is affine, Winkelmann [53] showed there exists a quasi-affine “quotient” variety Q which admits a *rational* map from X and such that $k[Q] = k[X]^H$. The idea here, too, is removal of codimension 1 divisors in $\text{Spec}(A_i)$ for large enough i .

4.1.2. *Quotients by free actions.* One can also make use of auxiliary properties of a group action which ensure the existence of a well-behaved quotient; specifically, recall that in reductive GIT the action on the stable locus is *proper*. In the case of a connected unipotent group over a characteristic 0 field, the proper actions (being also set-theoretically free) are exactly [12, §6.3 Lemma 8] the *scheme-theoretically*

free actions [41, Definition 0.8]. By Artin [3] and Kollár [36], proper actions admit geometric quotients in the category of algebraic spaces. So by considering only those open subvarieties on which the action is proper, coupled with some condition that ensures the quotient is a variety, one obtains a notion of stable sets, though finding a canonical choice of stable set is an issue here. Fauntleroy [14] defines a notion of *properly stable actions* of connected unipotent groups on quasi-affine normal varieties over algebraically closed fields, which yield geometric quotients in the category of varieties. Conversely, sufficiently well behaved geometric quotients come from properly stable actions.

Definition 4.1.6. A linear action of a connected unipotent group H on a normal quasi-affine variety U is *properly stable* if the H -action on U is proper and every $x \in U$ has a G -invariant open neighbourhood W such that the nonempty fibres of the natural map of schemes $q : W \rightarrow \text{Spec}(k[W]^H)$ are H -orbits.

Proposition 4.1.7 ([14]). *Let U be a normal quasi-affine variety over the algebraically closed field k , on which the connected unipotent group H acts. Assume that the action of H on U is properly stable. Then the quotient map $q : U \rightarrow q(U) \subseteq \text{Spec}(k[U]^H)$ is affine and a principal H -bundle (i.e. a locally trivial geometric quotient by [41, Proposition 0.9]). Conversely, if Y is a variety and $q : U \rightarrow Y$ is a principal H -bundle, then the action of H on U is properly stable.*

Remark 4.1.8. Note that H is a special group, in the sense of Serre and Grothendieck, so all principal H -bundles are in fact Zariski, not just étale, locally trivial [45, §2.6].

Remark 4.1.9. Unfortunately, a maximal properly stable open subset of an affine variety X is not in general unique, as Example 4.1.10 below shows, though when $k[X]$ is a UFD then being properly stable turns out to be a local property, so the union of all properly stable open subsets is the unique maximal properly stable open subset [14, Thm 2.4]. By [24, II Prop 6.2, II Cor 6.16] when X is an affine variety the condition that $k[X]$ be a UFD is the same as $\text{Pic}(X)$ being trivial. When working with a projective variety X the analogous condition is $\text{Pic}(X) = \mathbb{Z}$.

Example 4.1.10. Let $H = \mathbb{C}^+$ act on \mathbb{P}^3 via

$$t[x_0 : x_1 : x_2 : x_3] = [x_0 : x_1 : x_2 + tx_1 : x_3 + t(2x_2 + x_0) + t^2x_1]$$

and let X be the smooth irreducible surface

$$x_1x_3 - x_2^2 - x_0x_2 = 0.$$

Then \mathbb{C}^+ acts set-theoretically freely but not properly on the affine variety $W = X \cap \mathbb{A}^3$ where $\mathbb{A}^3 \subseteq \mathbb{P}^3$ is defined by $x_0 \neq 0$, and the (non-separated) quotient is the affine line \mathbb{A}^1 with a doubled point; W is the union of two invariant open subsets on each of which the action is properly stable with quotient \mathbb{A}^1 (see [16]; cf. [10] for a similar example). Here points of an open subset of X on which the H -action is properly stable are not necessarily naively stable in the sense of Definition 4.2.1.

4.2. Gluing local quotients $X_f//H$. In reductive GIT, the set X^s of G -stable points of X is a union of affine invariant hypersurface complements X_f on which the action is closed and orbits have maximal dimension, and the geometric quotient of X^s can be constructed by patching together the corresponding affine varieties $\text{Spec}(k[X_f]^G)$. Unipotent actions on affine varieties are always closed, but when a unipotent group H acts linearly on a projective variety X and f is an invariant

section there is no guarantee that $\text{Spec}(k[X_f]^H)$ is a finitely generated k -algebra, that the natural quotient map q from X_f to $\text{Spec}(k[X_f]^H)$ is surjective, or even that the image of q is a variety.

However one knows that the *field* of invariant rational functions $k(X)^H$ is finitely generated, so there is an invariant open, X_f , for which $k[X_f]^H$ is finitely generated. When g is another invariant then $X_f \cap X_g = X_{fg}$ has a finitely generated ring of invariants $k[X_f]^H[g^{-1}]$. We will see that taking the union of all such open affines X_f for which the natural map $q : X_f \rightarrow \text{Spec}(k[X_f]^H)$ has sufficiently good properties, and patching the associated maps q , yields canonical open sets with nice quotients. Patching works here, in contrast with Example 4.1.10, since any orbit in X_f is distinguished from any orbit in the complement of X_f by the invariant f itself.

Since we would like a stable set to have a geometric quotient, it is natural to impose the condition that $q : X_f \rightarrow \text{Spec}(k[X_f]^H)$ should be a geometric quotient.

Definition 4.2.1. Let X be a projective variety and let H be a connected unipotent group acting linearly on X with respect to an ample line bundle L . The set of *naively stable* points of X (with respect to the linearization L) is

$$X^{ns} = \bigcup_{f \in I^{ns}} X_f$$

where

$$I^{ns} = \{f \in H^0(X, L^{\otimes m})^H \text{ for some } m > 0 \mid k[X_f]^H \text{ is finitely generated, and } q : X_f \rightarrow \text{Spec}(k[X_f]^H) \text{ is a geometric quotient}\}.$$

Proposition 4.2.2. $q(X^{ns})$ is a quasi-projective variety and $q : X^{ns} \rightarrow q(X^{ns})$ is a geometric quotient.

Proof. If $f \in I^{ns}$ then $q(X_f) = \text{Spec}(k[X_f]^H)$ is an affine variety. By the Noetherian property we can choose $f_0, \dots, f_r \in I^{ns}$ such that $X^{ns} = \bigcup_{j=0}^r X_{f_j}$, and without loss of generality we can assume that there is some $m > 0$ such that $f_j \in H^0(X, L^{\otimes m})^H$ for $0 \leq j \leq r$. For each j by the definition of I^{ns} we can choose finitely many generators $\{f_{ij} : 1 \leq i \leq q_j\}$ for $k[X_{f_j}]^H$, which we can express as $f_{ij} = g_{ij}/(f_j)^\ell$ with $g_{ij} \in H^0(X, L^{\otimes \ell m})^H$ for some large $\ell > 0$.

Now let $M = r + \sum_{j=0}^r q_j$ and define $s : X^{ns} \rightarrow \mathbb{P}^M$ to have homogeneous coordinates given by the sections f_j^ℓ and g_{ij} (for $0 \leq j \leq r$ and $1 \leq i \leq q_j$) of $L^{\otimes \ell m}$. If y_0, \dots, y_r are the first $r+1$ homogeneous coordinates on \mathbb{P}^M which pull back to $f_0^\ell, \dots, f_r^\ell$ on X^{ns} , then s maps X^{ns} into $\bigcup_{j=0}^r (\mathbb{P}^M)_{y_j}$ and for $0 \leq j_0 \leq r$ we have

$$s^{-1}((\mathbb{P}^M)_{y_{j_0}}) = X_{f_{j_0}}$$

where

$$s : X_{f_{j_0}} \rightarrow (\mathbb{P}^M)_{y_{j_0}} \cong \mathbb{A}^M$$

is the composition of $q : X_{f_{j_0}} \rightarrow q(X_{f_{j_0}}) = \text{Spec}(k[X_{f_{j_0}}]^H)$ with the embedding of $\text{Spec}(k[X_{f_{j_0}}]^H)$ as a closed subvariety of \mathbb{A}^M defined by the subset

$$\left\{ \frac{f_0^\ell}{f_{j_0}^\ell}, \dots, \frac{f_r^\ell}{f_{j_0}^\ell} \right\} \cup \left\{ \frac{g_{ij}}{f_{j_0}^\ell} : 0 \leq j \leq r, 1 \leq i \leq q_j \right\}$$

of $k[X_{f_{j_0}}]^H$ which includes the generators

$$\{f_{ij} = \frac{g_{ij}}{f_{j_0}^{\ell}} : 0 \leq i \leq q_{j_0}\}$$

of $k[X_{f_{j_0}}]^H$. It follows that s is the composition of $q : X^{ns} \rightarrow q(X^{ns}) = \bigcup_{j=0}^r q(X_{f_j})$ with an embedding of $q(X^{ns})$ as a locally closed subvariety of \mathbb{P}^M , and moreover that $q : X^{ns} \rightarrow q(X^{ns})$ is a geometric quotient since $q : X_{f_j} \rightarrow q(X_{f_j})$ is a geometric quotient for $0 \leq j \leq r$ by the definition of I^{ns} . \square

We can also consider a more stringent condition, that $q : X_f \rightarrow \text{Spec}(k[X_f]^H)$ be a locally trivial geometric quotient (that is, an H -principal bundle with base $\text{Spec}(k[X_f]^H)$).

Definition 4.2.3. The set of *locally trivial stable points* is

$$X^{lts} = \bigcup_{f \in I^{lts}} X_f$$

where

$$I^{lts} = \{f \in H^0(X, L^{\otimes m})^H \text{ for some } m > 0 \mid k[X_f]^H \text{ is finitely generated, and } q : X_f \rightarrow \text{Spec}(k[X_f]^H) \text{ is a locally trivial geometric quotient}\}.$$

Then X^{lts} is an open subset of X^{ns} , and by patching together the natural maps $q : X_f \rightarrow \text{Spec}(k[X_f]^H)$ as in Proposition 4.2.2 we obtain a locally trivial geometric quotient $q : X^{lts} \rightarrow q(X^{lts})$ where $q(X^{lts}) \cong X^{lts}/H$ is an open subvariety of $q(X^{ns}) \cong X^{ns}/H$.

Remark 4.2.4. Note that for a *reductive* group acting with trivial isotropy on an invariant open subvariety X_f local triviality is *not* a more stringent condition; that is, $X^{lts} = X^{ns}$. Indeed the quotient map is affine, and so by Mumford [41, Proposition 0.8] it is proper, and so here is scheme-theoretically free [12, §6.3 Lemma 8]; then by [41, Proposition 0.9] it is a principal bundle.

Remark 4.2.5. Another option is to ignore the finitely generated condition on $k[X_f]^H$ altogether, and work with $\text{Spec}(k[X]^H)$ and $\text{Spec}(k[X_f]^H)$ as schemes and the natural map $q : X_f \rightarrow \text{Spec}(k[X_f]^H) \subseteq \text{Spec}(k[X]^H)$ as a map of schemes. This is the approach taken by Greuel and Pfister [19], coinciding with that of Dixmier and Raynaud [10]. For stability they require less than Definition 4.2.1; they demand that the image $q(X_f)$ should be an open subscheme of $\text{Spec}(k[X]^H)$ with $q : X_f \rightarrow q(X_f)$ an open map whose geometric fibres are just the H -orbits in X_f . (Strictly speaking their definition is for affine varieties, or quasi-affine with a choice of affine embedding). However they show, using [10, Proposition 2.2.2], that the open subscheme $q(X_f)$ of $\text{Spec}(k[X]^H)$ is in fact then a variety, and furthermore that the restriction of the natural map of schemes $q : X \rightarrow \text{Spec}(k[X]^H)$ to their stable set onto its image in $\text{Spec}(k[X]^H)$ is a geometric quotient in the category of varieties. It follows that their definition of stability is essentially equivalent to X^{ns} .

In a similar manner, we can mimic the GIT construction of X^{ss} and its categorical quotient by patching together the affine varieties $\text{Spec}(k[X_f]^H)$ for all those open subvarieties X_f with finitely generated rings of H -invariants.

Definition 4.2.6. We define the *finitely generated semistable set* by

$$X^{ss,fg} = \bigcup_{f \in I^{ss,fg}} X_f$$

where

$$I^{ss,fg} = \{f \in H^0(X, L^{\otimes m})^H \text{ for some } m > 0 \mid k[X_f]^H \text{ is finitely generated}\}.$$

It is not difficult to check using the proof of Proposition 4.2.2 that the affine varieties $\text{Spec}(k[X_f]^H)$ for $f \in I^{ss,fg}$ patch to give a quasi-projective variety which is an open subscheme of the scheme $\text{Proj}(k[X]^H)$; see Proposition 4.2.8 below. We introduce some terminology for such quotients:

Definition 4.2.7. Let $q : X^{ss,fg} \rightarrow \text{Proj}(k[X]^H)$ be the natural morphism of schemes. Then the *enveloped quotient* of $X^{ss,fg}$ is $q : X^{ss,fg} \rightarrow q(X^{ss,fg})$, where $q(X^{ss,fg})$ is a dense constructible subset of the *enveloping quotient*

$$X//H = \bigcup_{f \in I^{ss,fg}} \text{Spec}(k[X_f]^H)$$

of $X^{ss,fg}$.

Proposition 4.2.8. *The enveloping quotient $X//H$ is a quasi-projective variety with an ample line bundle $L_H \rightarrow X//H$ which pulls back to a positive tensor power of L under the natural map $q : X^{ss,fg} \rightarrow X//H$.*

Proof. This follows from a trivial modification of the proof of Proposition 4.2.2, replacing I^{ns} with $I^{ss,fg}$, replacing $q(X^{ns})$ with $X//H$ and replacing $q(X_f)$ with $\text{Spec}(k[X_f]^H)$ for any invariant f throughout. \square

Proposition 4.2.9. *If $k[X]^H$ is finitely generated then $X//H$ is the projective variety $\text{Proj}(k[X]^H)$.*

Proof. When $k[X]^H$ is finitely generated then so is $k[X_f]^H$ for any $f \in I = \bigcup_{m>0} H^0(X, L^{\otimes m})^H$, so $I^{ss,fg} = I$ and hence

$$X//H = \bigcup_{f \in I} \text{Spec}(k[X_f]^H) = \text{Proj}(k[X]^H).$$

\square

Lemma 4.2.10. *If X is normal, then $X^{as} \subset X^{ss,fg}$.*

Proof. By definition $X^{as} \subseteq X^{nss} = \bigcup_{f \in I} X_f$ where $I = \bigcup_{m>0} H^0(X, L^{\otimes m})^H$. If $f \in I$ then X_f is affine, so we know by [15] (where normality is assumed) that $q(X^{as} \cap X_f)$ is an open subvariety of $\text{Spec}(k[X_f]^H)$. Thus it is the complement of a closed subscheme cut out by functions $f_i \in k[X_f]^H$ which we can assume are of the form $f_i = g_i/f^\ell$ for some large $\ell \geq 0$ where $g_i \in k[X]^H$, and hence is a union of affine schemes $\text{Spec}(k[X_f]^H)_{f_i} = \text{Spec}(k[X_{g_i f}]^H)$; as subschemes of the variety $q(X^{as} \cap X_f)$ these are themselves affine varieties, and hence each $k[X_{g_i f}]^H$ is finitely generated. But then X^{as} is the union of the corresponding opens $X_{g_i f} = q^{-1}(\text{Spec}(k[X_{g_i f}]^H))$ of X , and so $X^{as} \subseteq X^{ss,fg}$ as required. \square

5. GENERALIZING GIT: INDUCED REDUCTIVE ACTIONS – FROM H TO G

5.1. **Stability for $G \times_H X$.** Choose a reductive group G with the connected unipotent group H as a closed subgroup. A linearization of an H -action on a projective (or quasi-projective) X with respect to an embedding of X in \mathbb{P}^n gives us a representation of H in $SL(n+1; k)$; if we assume, as without loss of generality we may, that this representation is faithful, then we can always choose G to be $SL(n+1; k)$. The H -action on X induces a G -action on $G \times_H X$ where H acts on G by left multiplication and G acts on itself by right multiplication. When X is affine we have

$$k[G \times_H X]^G \cong k[X]^H.$$

When X is projective (or more generally quasi-projective), and $L \rightarrow X$ is a very ample H -linearization inducing an embedding of X in \mathbb{P}^n , and G is a subgroup of $SL(n+1; k)$, then we get a very ample G -linearization on $G \times_H X$ via the sequence:

$$G \times_H X \hookrightarrow \mathbb{P}^n \times_H G \cong \mathbb{P}^n \times G/H \rightarrow \mathbb{P}^n,$$

taking the trivial bundle on G/H . By abuse of notation let us also call this G -linearization L . Then

$$k[G \times_H X]^G = \bigoplus_{m \geq 0} H^0(G \times_H X, L^{\otimes m})^G \cong \bigoplus_{m \geq 0} H^0(X, L^{\otimes m})^H = k[X]^H.$$

Remark 5.1.1. If the action of H on X extends to G we have $G \times_H X \cong G/H \times X$, in which case this identification of rings of invariants is sometimes known as the Borel transfer principle (see [9, Lemma 4.1]).

A couple of fundamental results on algebraic homogeneous spaces will be useful (see [45, Theorem 4.17] and [23, Corollary 2.8], respectively).

Proposition 5.1.2 (Matsushima). *Given a reductive G , then G/H is affine if and only if H is reductive.*

Proposition 5.1.3. *If H is unipotent, then H is an observable subgroup of G ; that is, G/H is a quasi-affine variety.*

Corollary 5.1.4. *If H is a positive dimensional unipotent group and G is reductive, then G/H is quasi-affine but not affine.*

Let $i : X \rightarrow G \times_H X$ be the closed immersion given by $x \mapsto [e, x]$.

Definition 5.1.5. Let the set of *Mumford stable points* for a linear H -action on a quasi-projective variety X be $X^{ms} = i^{-1}((G \times_H X)^s)$. Similarly, the set of *Mumford semistable points* for the linear H -action on X is $X^{mss} = i^{-1}((G \times_H X)^{ss})$. Here $(G \times_H X)^s$ and $(G \times_H X)^{ss}$ are defined as in Definition 3.2.6 for the induced linear action of G on $G \times_H X$.

These sets admit geometric and categorical quotients, respectively, by standard reductive GIT. But because H is unipotent something quite interesting happens.

Lemma 5.1.6. *Every Mumford semistable point is Mumford stable, that is, $X^{ms} = X^{mss}$.*

Proof. Let O be an orbit in $X^{mss} \setminus X^{ms}$, so that $(G \times_H O)$ is an orbit in an affine hypersurface $(G \times_H X)_F$ which is either closed or non-maximal dimension or not closed in $(G \times_H X)_F$. In each case there is a unique closed orbit $(G \times_H O')$ in

the closure of $(G \times_H O)$ in $(G \times_H X)_F$, which is of non-maximal dimension. In an affine variety a closed G -orbit is of course affine so by Proposition 5.1.3 the corresponding stabilizer is a reductive subgroup of G . Hence $G \times_H O'$ must have a positive dimensional reductive stabilizer, which is a contradiction since the stabilizer is a subgroup of H and so is unipotent. Thus there are no Mumford semistable points which are not Mumford stable; that is, $X^{ms} = X^{mss}$. \square

Remark 5.1.7. When coupled with the fact that $(G \times_H X)_F$ is always quasi-affine (see Corollary 5.2.9), it follows that whatever notion one comes up with for “strictly semistable” orbits that are not Mumford stable, such orbits must lie in *quasi-affine but not affine* $(G \times_H X)_F$. This is a useful geometric observation, to be used again in the main theorem below.

A priori the definitions of X^{ms} and X^{mss} depend on the choice of the reductive group G , but in fact they do not.

Proposition 5.1.8. *The open subsets X^{ms} and X^{mss} of X are intrinsically defined, in that they depend only on the linear H -action on X and are independent of the choice of G .*

Proof. By Lemma 5.1.6 it is equivalent to show X^{mss} is independent of G . Pick G_1 and G_2 both containing H and contained in some G . Let f be an H -invariant, and let F_1, F_2 and F be the associated G_1, G_2 and G invariants on $G_1 \times_H X, G_2 \times_H X$, and $G \times_H X$, respectively. By definition of X^{mss} , we want to show that $(G_1 \times_H X)_{F_1}$ is affine if and only if $(G_2 \times_H X)_{F_2}$ is affine. But then it suffices to show $(G_1 \times_H X)_{F_1}$ is affine if and only if $(G \times_H X)_F$ is affine.

If $(G_1 \times_H X)_{F_1}$ is affine, then because $((G_1 \times_H X)_{F_1}) \times_{G_1} G = ((G_1 \times_H X) \times_{G_1} G)_F = (G \times_H X)_F$, and both G_1 and G are affine, the latter is affine [45, p. 196]. Conversely, if $(G \times_H X)_F$ is affine, then so is any closed subvariety. But the inclusion of $(G_1 \times_H X)_{F_1}$ in $(G \times_H X)_F$ is a closed immersion, so it must be an affine variety. \square

The intrinsic nature of X^{ms} and X^{mss} can be seen more geometrically by the following characterization.

Proposition 5.1.9. *The Mumford stable points are precisely the locally trivially stable points; that is, $X^{ms} = X^{lts}$.*

Proof. Consider $x \in X^{lts}$. Then $x \in X_f$ for f an H -invariant section of a positive tensor power of L , where $\phi : X_f \rightarrow X_f/H$ is a principal H -bundle and $X_f/H = \text{Spec}(k[X]^H)_f$ is an affine variety. By étale descent, ϕ is a principal H -bundle if and only if $\pi : (G \times_H X)_F \rightarrow X_f/H$ is a principal G -bundle, for F the corresponding G -invariant. But by [41, Proposition 0.7] this means π is an affine morphism, so $(G \times_H X)_F$ is affine. It follows that $x \in X^{ms}$.

In the reverse direction we see that $X^{ms} \subseteq X^{lts}$ as follows. G acts schematically freely on $(G \times_H X)^s$ because the action is proper and set-theoretically free [12, §6.3 Lemma 8]. Furthermore by reductive GIT this action has a geometric quotient $(G \times_H X)^s \rightarrow (G \times_H X)^s/G \cong X^{ms}/H$. Hence by Mumford [41, Proposition 0.9] $(G \times_H X)^s \rightarrow X^{ms}/H$ is a G -principal bundle, so by descent $X^{ms} \rightarrow X^{ms}/H$ is an H -principal bundle, and furthermore it is locally described by $X_f \rightarrow X_f/H$ for some collection f of H -invariants. \square

Remark 5.1.10. It follows that any H -orbit in $X^{ns} \setminus X^{lts}$ has the property that the corresponding G -orbit lies in a quasi-affine but not affine $(G \times_H X)_F$ (cf. Remark 5.1.7 above).

Remark 5.1.11. When it is convenient to do so we can choose G to be semisimple throughout this section; indeed we can make a canonical choice of G as $SL(n+1; k)$, but it is often easier to work with a smaller semisimple or reductive group G .

5.2. Completions and reductive envelopes. The techniques of reductive GIT are most effective when applied to projective varieties, so it makes sense to choose a G -equivariant projective completion of $G \times_H X$ together with an extension of the G -linearization. However, as discussed in Section 3.2.2, GIT does not behave well with respect to G -equivariant open inclusions. Recalling from Lemma 5.1.6 that the G -semistable points of $G \times_H X$ are all G -stable, we may summarize the issues as follows:

- (1) Some invariants may not extend to the boundary, so G -stable points of $G \times_H X$ could become unstable in $\overline{G \times_H X}$.
- (2) Even if a given invariant F does extend from $G \times_H X$ to $\overline{G \times_H X}$, the open subvariety $(G \times_H X)_F$ may be affine when $(\overline{G \times_H X})_F$ is not, so again G -stable points in $G \times_H X$ could become unstable in $\overline{G \times_H X}$.
- (3) Alternatively, if an invariant F extends from $G \times_H X$ to $\overline{G \times_H X}$, then $(\overline{G \times_H X})_F$ might be affine when $(G \times_H X)_F$ is not, so unstable points of $G \times_H X$ could become semistable in $\overline{G \times_H X}$.
- (4) If the G -action on $(G \times_H X)_F$ is closed the G -action on $(\overline{G \times_H X})_F$ need not be, so G -stable points in $G \times_H X$ may become strictly semistable in $\overline{G \times_H X}$.

Remark 5.2.1. Recall that if $L \rightarrow Y$ is an ample line bundle and F is a section of L , then Y_F is affine. So the second issue above is not a problem for ample extensions of the G -linearization. For ample extensions, it also follows that the third issue above can be refined since unstable points of $G \times_H X$ cannot become stable in $\overline{G \times_H X}$: it follows from Proposition 3.1.1 that they can only be unstable or strictly semistable in $\overline{G \times_H X}$.

The first task is to find a completion $\overline{G \times_H X}$ together with an extension of the line bundle L , such that “sufficiently many” invariants F_i extend over the boundary with $\overline{G \times_H X}_{F_i}$ affine.

Definition 5.2.2. Let X be a quasi-projective variety with an H -action with linearization L , and let G be any reductive group containing H . A finite separating set S (in the sense of Definition 3.0.1) of invariant sections of positive tensor powers of L is a *finite fully separating set of invariants* for the linear H -action on X if

- (i) for every $x \in X^{ms}$ there exists $f \in S$ with associated G -invariant F over $G \times_H X$ such that $x \in (G \times_H X)_F$ and $(G \times_H X)_F$ is affine; and
- (ii) for every $x \in X^{ss, fg}$ there exists $f \in S$ such that $x \in X_f$ and S is a generating set for $k[X_f]^H$.

Remark 5.2.3. The proof of Proposition 5.1.8 shows that this definition is independent of the choice of G .

We now make a sequence of definitions to address the issues enumerated above.

Definition 5.2.4. Let X be a quasi-projective variety with a linear H -action with respect to an ample line bundle L on X . Consider a choice of reductive group $G \geq H$ and a G -equivariant projective completion $\overline{G \times_H X}$ of $G \times_H X$ together with a G -linearization L' which restricts to the H -linearization L on X . If every H -invariant f in some finite fully separating set of invariants S for the H -action on X (in the sense of Definition 5.2.2) extends to a G -invariant section of a tensor power of L' over $\overline{G \times_H X}$, we will call $\overline{G \times_H X}$ together with the G -linearization L' a *reductive envelope* of the linear H -action on X (with respect to the finite fully separating set of invariants S).

Definition 5.2.5. With notation as above, if there exists such an S for which every $f \in S$ extends to a G -invariant section F over $\overline{G \times_H X}$ such that $(\overline{G \times_H X})_F$ is affine, then we say that $(\overline{G \times_H X}, L')$ is a *fine reductive envelope*. If L' is ample then we say this is an *ample reductive envelope*.

Remark 5.2.6. Note that an ample reductive envelope is automatically fine.

Issue (4) above can be overcome when the following stringent condition holds, from which other good properties will also follow (see Theorem 5.3.4).

Definition 5.2.7. With the notation of Definition 5.2.4 and given a reductive envelope $(\overline{G \times_H X}, L')$ with respect to a finite fully separating set of invariants S , let D_1, \dots, D_r denote the codimension 1 components of the boundary of $G \times_H X$ in $\overline{G \times_H X}$. If every $f \in S$ extends to a G -invariant F over $\overline{G \times_H X}$ which vanishes on each component D_j , then $(\overline{G \times_H X}, L')$ will be called a *strong reductive envelope* of the linear H -action on X .

Proposition 5.2.8. *An ample reductive envelope for an H -action on X with an ample linearization L always exists.*

Proof. By the Noetherian property a finite fully separating set of invariants S exists. Moreover there is some $m > 0$ such that $L^{\otimes m}$ is very ample and there is a finite fully separating set S such that every $f \in S$ is an invariant section of $L^{\otimes m}$. Take the union of S and a basis of sections of $L^{\otimes m}$ over $G \times_H X$. This set of sections defines a projective embedding of $G \times_H X$ into some large \mathbb{P}^n . By construction the closure of $G \times_H X$ in \mathbb{P}^n together with the restriction of $\mathcal{O}_{\mathbb{P}^n}(1)$ to $\overline{G \times_H X}$ is an ample reductive envelope. \square

Corollary 5.2.9. *Let f be an H -invariant of (X, L) , and F be the corresponding G invariant over $X \times_H G$. Then $(X \times_H G)_F$ is a quasi-affine variety.*

Proof. Any F may be included in the set of invariants from the proof of Proposition 5.2.8. Because the result is an ample reductive envelope, $(\overline{G \times_H X})_F$ is affine. \square

Remark 5.2.10. We will see in Remark 5.3.10 that even for smooth varieties X smooth ample reductive envelopes do not always exist. The heart of the matter is contained in the observation that if a normal affine variety X has a fine strong reductive envelope $(\overline{G \times_H X}, L')$ with respect to a finite fully separating set of invariants S which includes a nonzero constant function on X , then $k[X]^H$ is finitely generated. To see this, first note that without loss of generality, one may take a normalization of the reductive envelope. Let $f \in S$ be a nonzero constant function on X and F denote its associated G -invariant on $\overline{G \times_H X}$. Then the affine variety $(\overline{G \times_H X})_F$ contains $(G \times_H X)_F$ as a codimension 2 complement, so by normality

G -invariant sections over the latter canonically extend to sections over the former. Because G is reductive, we know the space of G -invariant sections over the affine variety $(\overline{G \times_H X})_F$ is finitely generated. But $(G \times_H X)_F = G \times_H X$ since F restricts to a nonzero constant function over X , and G -invariant sections over $G \times_H X$ correspond exactly to H -invariant sections over X .

Definition 5.2.11. Let X be a nonsingular projective variety with a linear H -action and a reductive envelope $\overline{G \times_H X}$ in the sense of Definition 5.2.4. Label the inclusions $i : X \hookrightarrow G \times_H X$ and $j : G \times_H X \hookrightarrow \overline{G \times_H X}$. Let the *completely stable points* of X with respect to the reductive envelope be the set

$$X^{\overline{s}} = (j \circ i)^{-1}(\overline{G \times_H X}^{\overline{s}}).$$

Let the set of *completely semistable points* be

$$X^{\overline{ss}} = (j \circ i)^{-1}(\overline{G \times_H X}^{\overline{ss}}).$$

Remark 5.2.12. It will follow from Theorems 5.3.1 and 5.3.4 that if $\overline{G \times_H X}$ is normal and together with G -linearization L' forms a fine strong reductive envelope, then $X^{\overline{s}}$ and $X^{\overline{ss}}$ are independent of the choice of $\overline{G \times_H X}$ and L' .

5.3. Main Theorem and Corollaries. Let

$$i : X \hookrightarrow G \times_H X \text{ and } j : G \times_H X \hookrightarrow \overline{G \times_H X}$$

denote the natural injections for a given reductive envelope $\overline{G \times_H X}$. Throughout

$$\pi : \overline{G \times_H X} \rightarrow \overline{G \times_H X} // G$$

will denote the GIT quotient map for the reductive envelope, restricted to open subsets (by abuse of notation) as appropriate.

Theorem 5.3.1. *Let X be a projective variety with a linear H -action, for H a connected unipotent group, and let $(\overline{G \times_H X}, L')$ be any fine reductive envelope as in Definition 5.2.5. Then*

$$\begin{array}{ccccccc} X^{\overline{s}} & \subseteq & X^{lts} = X^{ms} = X^{mss} & \subseteq & X^{ns} & \subseteq & X^{as} \subseteq X^{ss,fg} \subseteq X^{\overline{ss}} = X^{nss} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X^{\overline{s}}/H & \subseteq & X^{lts}/H & \subseteq & X^{ns}/H & \subseteq & X//H \subseteq \text{Proj}(k[X]^H) \end{array}$$

The stable sets $X^{\overline{s}}$, $X^{lts} = X^{ms} = X^{mss}$ and X^{ns} admit quasi-projective geometric quotients, given by restrictions of the quotient map $q = \pi \circ j \circ i$. The quotient map $q = \pi \circ j \circ i$ restricted to the open subvariety $X^{ss,fg}$ is an enveloped quotient with $q : X^{ss,fg} \rightarrow X//H$ an enveloping quotient in the sense of Definition 4.2.7. The enveloping quotient $X//H$ is an open subvariety of $\overline{G \times_H X} // G$ and there is an ample line bundle L_H on $X//H$ which pulls back to a tensor power $L^{\otimes k}$ of the line bundle L for some $k > 0$, such that for any $m > 0$ the sections of $L_H^{\otimes m}$ on $X//H$ are precisely the H -invariant sections of the tensor power $L^{\otimes km}$ of L on $X^{ss,fg}$.

Proof. By definition $X^{ss,fg} \subseteq X^{nss}$ and $X^{ns} \subseteq X^{as}$. Lemma 4.2.10 shows that $X^{as} \subseteq X^{ss,fg}$ and Lemma 4.2.1 shows $X^{lts} \subseteq X^{ns}$. Lemma 5.1.6 gives $X^{ms} = X^{mss}$ and by Proposition 5.1.9, $X^{lts} = X^{ms}$.

Let $x \in X^{\overline{s}}$ and embed X in $\overline{G \times_H X}$ in the natural way. Then for some G -invariant F restricting to an H -invariant f on X there is an affine subvariety $(\overline{G \times_H X})_F$ containing x on which the G action is proper, and which admits a geometric quotient π to $(\overline{G \times_H X})_F // G$. The boundary in $(\overline{G \times_H X})_F$ is a G -invariant

closed set, so the restriction of π to the complement of the boundary (that is, to $(G \times_H X)_F = G \times_H X_f$) is also a geometric quotient; furthermore the restricted action is proper because properness is a local property on the base, and hence, since it is set-theoretically free, it is in fact scheme-theoretically free [12, §6.3 Lemma 8]. It follows from [41, Proposition 0.9] that π restricted to $G \times_H X_f$ is a principal G -bundle, and hence by étale descent the morphism $q : X_f \rightarrow (G \times_H X_f)/G = X_f/H$ is a principal H -bundle. In particular, $x \in X^{ls}$.

It is clear that $X^{\overline{ss}} \subseteq X^{nss}$ because any G -invariant section F over $\overline{G \times_H X}$ restricts to a G -invariant section over $G \times_H X$ and hence to an H -invariant section f over X . Now we argue the reverse containment. By Definition 5.2.5, there exists a finite fully separating set of H -invariants $\{f_i : 1 \leq i \leq r\}$ defining G -invariant sections F_i which extend over the reductive envelope $\overline{G \times_H X}$ such that $(\overline{G \times_H X})_{F_i}$ is affine. Thus $\cup_i X_{f_i} \subseteq X^{\overline{ss}}$. By Definition 5.2.2, given $x \in X$, all the invariant sections $f \in k[X]^H$ vanish at x if and only if $f_i(x) = 0$ for all i . Thus $X^{nss} = \bigcup_{i=1}^r X_{f_i} \subseteq X^{\overline{ss}}$.

X^{ns} has a geometric quotient by Proposition 4.2.2, and hence the same is true of any G -invariant open subset of X^{ns} , while $q : X^{ss,fg} \rightarrow X//H$ is an enveloping quotient by Proposition 4.2.8. It follows from the proof of Proposition 4.2.8 together with condition (ii) in the definition 5.2.2 of a finite fully separating set of invariants that for some $m, M > 0$ there exist H -invariants $f_0, \dots, f_M \in H^0(X, L^{\otimes m})^H$ such that

$$x \mapsto [f_0(x) : \dots : f_M(x)]$$

induces an embedding of $X//H$ as a quasi-projective subvariety of \mathbb{P}^M , and moreover f_0, \dots, f_M extend to G -invariant sections F_0, \dots, F_M of $(L')^{\otimes m}$ over $\overline{G \times_H X}$. Since $k[\overline{G \times_H X}]^G$ is finitely generated, by adding restrictions of G -invariants over $\overline{G \times_H X}$ to f_0, \dots, f_M we can assume that F_0, \dots, F_M define an embedding

$$y \mapsto [F_0(y) : \dots : F_M(y)]$$

of $(\overline{G \times_H X})//G = \text{Proj}(k[\overline{G \times_H X}]^G)$ as a projective subvariety of the same projective space \mathbb{P}^M . The latter embedding is the composition of the first embedding with the natural map $X//H \rightarrow (\overline{G \times_H X})//G$, so the natural map embeds $X//H$ as a subvariety of $(\overline{G \times_H X})//G$.

Finally, the existence of an ample line bundle L_H on $X//H$ with the desired properties is proved in Proposition 4.2.8. The identification between invariant sections of L over $X^{ss,fg}$ and sections of L_H over $X//H$ follows trivially by restriction and patching. \square

Remark 5.3.2. Recall that a fine reductive envelope always exists for any linear H -action on a projective variety X by Proposition 5.2.8 and Remark 5.2.6. A choice of fine reductive envelope $\overline{G \times_H X}$ provides a projective completion

$$\overline{X//H} = \overline{G \times_H X} // G$$

of the enveloping quotient $X//H$, which in general depends on the choice of reductive envelope. Note however that when $k[X]^H$ is finitely generated then $X//H = \text{Proj}(k[X]^H)$ is itself projective by Proposition 4.2.9 and hence

$$X//H = \overline{G \times_H X} // G$$

for any fine reductive envelope $\overline{G \times_H X}$.

Remark 5.3.3. The Hilbert-Mumford criterion from reductive GIT (see Remark 3.2.5) can be used (at least when L' is ample, but see also [25]) to determine $X^{\bar{s}}$ and $X^{\overline{ss}}$ and to analyze the orbit structure of $X^{\overline{ss}} \setminus X^{\bar{s}}$ and the quotient map $q : X^{\overline{ss}} \rightarrow X//H$ (cf. [34]), which leads in examples to identification of the intermediate stable and semistable sets $X^{ms} = X^{mss} = X^{lts} \subseteq X^{ns} \subseteq X^{as} \subseteq X^{ss,fg}$. In especially good situations we have $X^{\overline{ss}} = X^{\bar{s}}$ (as in the example in the final section §6 below with n odd) and then

$$X^{\bar{s}} = X^{ms} = X^{mss} = X^{lts} = X^{ns} = X^{as} = X^{ss,fg} = X^{\overline{ss}}.$$

Theorem 5.3.4. *If $\overline{G \times_H X}$ is normal and together with a line bundle L' provides a fine strong reductive envelope for the linear H -action on X , then $X^{\bar{s}} = X^{ms}$ and $X^{ss,fg} = X^{nss}$.*

Proof. We show that $X^{ms} \subseteq X^{\bar{s}}$. Consider $x \in X^{ms} = X^{mss}$ and embed X in $G \times_H X$ in the natural way. Then there exists an $F \in H^0(G \times_H X, L^{\otimes m})^G$ for some $m > 0$ such that $F(x) \neq 0$ and $(G \times_H X)_F$ is affine and the action of G on $(G \times_H X)_F$ is closed with all stabilizers of dimension 0. The section F extends to a G -invariant section over the reductive envelope of the H -action on X and F vanishes on every codimension 1 component D_j of the boundary. We claim that $(\overline{G \times_H X})_F = (G \times_H X)_F$: they are both affine, so they are determined by their coordinate rings, and they differ at most in codimension 2, so by normality their coordinate rings agree; therefore since one is contained in the other, they must in fact be equal. Thus $x \in X^{\bar{s}}$.

We prove now that $X^{nss} \subseteq X^{ss,fg}$. Consider a finite fully separating set $S = \{f_i : 1 \leq i \leq r\}$ associated to the reductive envelope, and label the corresponding G -invariant sections of powers of L' over $\overline{G \times_H X}$ by F_i . Then $x \in (\overline{G \times_H X})_{F_i}$ for one of these G -invariants F_i . Since G is reductive and $(\overline{G \times_H X})_{F_i}$ is an affine variety, it has a finitely generated ring of invariants. Furthermore, by the definition of a strong reductive envelope, each F_i vanishes on the codimension 1 components D_j of the boundary of $G \times_H X$ in $\overline{G \times_H X}$. It follows that the complement of the quasi-affine $(G \times_H X)_{F_i}$ in the affine variety $\overline{G \times_H X}_{F_i}$ has codimension 2. Thus by normality we have $k[X_f]^H = k[(G \times_H X)_F] = k[(\overline{G \times_H X})_F]^G$ which is finitely generated, so $x \in X^{ss,fg}$. The result now follows from Theorem 5.3.1. \square

We will see in §5.3.1 that fairly mild conditions on the singularities of a completion $\overline{G \times_H X}$ are sufficient to prove the existence of a fine strong reductive envelope.

Remark 5.3.5. Let X be an affine variety; then we can define $q : X^{ss,fg} \rightarrow X//H \subseteq \text{Spec}(k[X]^H)$ by analogy with the definition of $q : X^{ss,fg} \rightarrow X//H \subseteq \text{Proj}(k[X]^H)$ when X is projective, and $X//H$ is a quasi-affine variety with coordinate ring $k[X//H] \cong k[X^{ss,fg}]^H$. When we have an ample strong reductive envelope which is normal then by Theorem 5.3.4 $X^{ss,fg} = X^{nss}$ where $X^{nss} = X$ since X is affine (cf. Remark 3.2.12), so $X//H$ is an explicit construction of Winkelmann's quasi-affine "quotient" variety [53], and the quotient morphism $q : X \rightarrow X//H$ plays the rôle of the morphism (rather than just a rational map) Winkelmann hoped for.

Definition 5.3.6. Let X be a projective variety equipped with a linear H -action. We say that a point $x \in X$ is *stable* if $x \in X^{ms}$ and that it is *semistable* if $x \in X^{ss,fg}$.

5.3.1. *Constructing strong reductive envelopes.* Although Proposition 5.2.8 guarantees the existence of an ample (and hence fine) reductive envelope, its method of

construction requires us to identify a finite fully separating set of invariants, and it is not very clear how to do this in practice. The difficulty in building a fine reductive envelope lies in the trade-offs between the choice of completion, finding a line bundle so that enough sections extend, and guaranteeing that the $F = 0$ complements are affine. In this subsection we discuss how, for a fixed sufficiently nice (for example, smooth) completion, one can find a line bundle so that the sections extend; when this line bundle is ample, we get a strong ample reductive envelope. The technique, with slight adjustment, will be applied in §5.3.2 and §5.3.3.

Definition 5.3.7. Let Y be a normal quasi-projective variety. We say that a completion \overline{Y} is *gentle* if it is normal and some integral multiple of each boundary Weil divisor is Cartier.

Remark 5.3.8. Over a characteristic 0 field, as we have assumed from the start, a gentle completion can always be arranged by resolution of singularities.

Given a gentle G -equivariant completion $\overline{G \times_H X}$ of $G \times_H X$, some positive tensor power of any line bundle on $G \times_H X$ will extend non-canonically across the boundary. One can check the appropriate cocycle condition [41, proof of Converse 1.13, page 41] to verify that in fact the G -linearization extends as well. If the boundary is codimension at least 2, then by normality all invariant sections extend uniquely to invariant sections over the whole of $\overline{G \times_H X}$. If the boundary has codimension 1 components, the basic idea is to weight them heavily enough that any given invariant section extends uniquely over the boundary, vanishing on the codimension 1 components.

To make this precise, let $\{D_j : 1 \leq j \leq r\}$ denote the collection of codimension 1 components of the boundary of $G \times_H X$ in $\overline{G \times_H X}$, let L be the G -linearization on $G \times_H X$ discussed in §5.3.2, and let L' be some chosen extension to a G -linearization over a projective completion $\overline{G \times_H X}$. Denote by L'_N the induced G -linearization on $L'[N \sum_{j=1}^r D_j]$ when N is such that ND_j is Cartier for $1 \leq j \leq r$. Then we have the following proposition.

Proposition 5.3.9. *Let $\overline{G \times_H X}$ be a gentle G -equivariant completion of $G \times_H X$, with a G -linearization L' of the G -action which extends the given linearization L . Given a finite fully separating set S of invariants on X , then $(\overline{G \times_H X}, L'_N)$ is a strong reductive envelope with respect to S for suitable sufficiently large N . If moreover $(\overline{G \times_H X}, L')$ is a fine reductive envelope with respect to S then $(\overline{G \times_H X}, L'_N)$ is a fine strong reductive envelope with respect to S , and hence Theorems 5.3.1 and 5.3.4 apply.*

Proof. (Cf. [41, proof of Converse 1.13, page 41]). For any given $f \in S$, there exists an N_f such that f extends to a section of L'_{N_f} over the codimension 1 components D_j of the boundary of $G \times_H X$ in $\overline{G \times_H X}$, and thus by normality extends to an invariant section F over the whole of $\overline{G \times_H X}$. We can identify $H^0(\overline{G \times_H X}, L'_n)$ with a subspace of $H^0(\overline{G \times_H X}, L'_{n+1})$ for any n , so that if f extends to a section F of L'_n then it vanishes on each D_j as a section of L'_{n+1} . Thus taking $N > N_f$ forces F to vanish on each of the codimension 1 components D_j of the boundary. Consequently, since S is finite, simply take $N > \max_{f \in S}(N_f)$; then $(\overline{G \times_H X}, L'_N)$ is a strong reductive envelope for the linear H -action on X .

For the final part, observe that the complement of a Cartier divisor in an affine variety is affine, because any line bundle on an affine variety is ample [24, Example

II.7.4.2]. Thus the complement of the D_j in each affine $(\overline{G \times_H X})_F$ is affine, and the result follows. \square

Remark 5.3.10. Unfortunately the following three properties do not in general hold simultaneously, although we can arrange any two of them: the completion $\overline{G \times_H X}$ being gentle, the line bundle L' being ample, and the completion together with the line bundle forming a reductive envelope (that is, enough invariants extending). More concretely, if X is affine and there exists a gentle ample reductive envelope, then $k[X]^H$ must be finitely generated, since by Remark 5.2.10, it suffices to show there exists a fine strong reductive envelope with respect to a finite fully separating set S of invariants which includes a nonzero constant function over X . Because the completion is gentle, we can construct L'_N as in Proposition 5.3.9 above for any such S , and for large enough N it defines a fine strong reductive envelope.

In particular, the Nagata counter-examples admit no gentle ample reductive envelopes.

Observe that essentially the same argument shows that if G/H has a gentle G -equivariant affine completion $\overline{G/H}^{aff}$ then G/H can be represented as a codimension 2 complement in an affine G -variety, and hence $k[G/H] = k[G]^H$ is finitely generated (cf. [23, §4]).

Remark 5.3.11. Suppose that L' is chosen to be ample in Proposition 5.3.9. If the line bundles L'_N are also ample for large N (for example, if the divisors given by positive integral multiples of the D_j are ample), then we can compare the G -linearizations $(\overline{G \times_H X}, L'_N)$ for different $N \geq 0$ using the theory of variation of GIT quotients [48, 11]. Since GIT is unaffected when we replace a line bundle by any positive tensor power of itself we can consider L'_N for all positive rational values of N . We know [48, 11] that the interval $(0, \infty)$ can be divided into (rational) subintervals I_j such that when N lies in the interior of a subinterval I_j the GIT quotient (and the stable and semistable sets) defined with respect to the linearization L'_N depends only on the subinterval in which N lies. Moreover if N lies on the boundary between two subintervals then there is a nonempty set of points which are semistable with respect to the linearization L'_N (for this particular N) but not semistable for the whole family, having as stabilizer a reductive subgroup R of dimension at least one in G . Any maximal torus of R acts with zero weights on the fibres of L'_N at all semistable fixed points of R . Up to conjugacy only finitely many subgroups can occur as stabilizers, and their fixed point sets have only finitely many connected components. From this and the requirement of zero weights on the fibres of L'_N , it follows that only finitely many positive N can occur as the boundary between two subintervals I_j , and hence that there are only finitely many such subintervals.

This proves:

Lemma 5.3.12. *For $N \geq 0$ let $(\overline{G \times_H X}, L'_N)$ be an ample strong reductive envelope for a linear H -action on a projective variety X . Then the GIT quotient $\overline{G \times_H X} // G$ and the stable and semistable sets in $\overline{G \times_H X}$ defined with respect to the G -linearizations L'_N are independent of N when N is sufficiently large.*

5.3.2. *Important case: H -action extends to G -action.* If a linear H -action on a projective variety X extends to a linear reductive G -action on X , then since $G \times_H X \cong (G/H) \times X$ (cf. Remark 5.1.1), the problem of constructing an ample reductive

envelope reduces to understanding completions of G/H and extensions of the trivial line bundle over these completions. Since G/H is quasi-affine, the natural choice is to take a G -equivariant affine closure $\overline{G/H}^{\text{aff}}$, and then projectively complete with the hypersurface at infinity. Many practical applications of non-reductive GIT, for example to moduli spaces of hypersurfaces in toric varieties, fall into this setting. Two scenarios work out especially well:

- if a G -equivariant affine closure $\overline{G/H}^{\text{aff}}$ contains G/H as a codimension 2 complement, and
- if a G -equivariant affine closure $\overline{G/H}^{\text{aff}}$ is nonsingular.

These are both special cases of a third:

- if a G -equivariant affine closure $\overline{G/H}^{\text{aff}}$ is gentle (that is, given a Weil divisor in the boundary, some integer multiple is Cartier),

though in fact this apparent extra generality is spurious, as we observed in Remark 5.3.10: the existence of a gentle G -equivariant affine closure implies that G/H can be embedded as a codimension 2 complement in an affine G -variety, which in turn implies that $k[G/H] = k[G]^H$ is finitely generated [23, §4], and by the Borel transfer principle that

$$k[X]^H \cong k[(G/H) \times X]^G$$

is finitely generated whenever the linear action of H on X extends to a linear G -action.

Note that the affine closure $\overline{G/H}^{\text{aff}}$ of G/H can always be chosen to be normal, so we assume that throughout. We will see that any of the three conditions above implies the existence of a strong ample reductive envelope when the H -action extends to a linear G -action, and hence, by Theorem 5.3.4, that $X^{\overline{s}} = X^{ms}$ and $X^{ss,fg} = X^{\overline{ss}} = X^{nss}$ for such an envelope, while $X//H = \text{Proj}(k[X]^H)$ by Proposition 4.2.9.

Let $\overline{G/H}$ denote the normalized projective completion of $\overline{G/H}^{\text{aff}}$. Let L' denote the G -linearization obtained by taking the hyperplane line bundle on $\overline{G/H}$ (with its natural G -linearization) tensored with the given G -linearized ample line bundle L on X . Let D_∞ represent the hypersurface at infinity in $\overline{G/H}$. Note that D_∞ is a Cartier divisor on $\overline{G/H}$ (corresponding to the hyperplane line bundle), so $\overline{G/H}$ is a gentle completion of G/H if and only if $\overline{G/H}^{\text{aff}}$ is gentle.

Lemma 5.3.13. *If the given linear H -action on X extends to a linear G -action, and if G/H has a gentle G -equivariant affine closure $\overline{G/H}^{\text{aff}}$, then $(\overline{G/H} \times X, L'_N)$ is a strong ample reductive envelope for suitable large N , where L'_N is defined as in Proposition 5.3.9.*

Proof. For L'_N as in Proposition 5.3.9 to be defined requires that ND_j should be a Cartier divisor for each codimension 1 component D_j of the boundary of $G \times_H X$ in $\overline{G} \times_H \overline{X}$. Since any line bundle on an affine variety is ample, and since the hypersurface at infinity is an ample divisor, L'_N is ample for every such N . The result now follows immediately from Proposition 5.3.9. \square

Proposition 5.3.14. *If the linear H -action on a projective variety X extends to a linear G -action, and if $\overline{G/H}^{\text{aff}}$ is gentle, then $X^{ms} = X^{\overline{s}}$ and $X^{nss} = X^{\overline{ss}} =$*

$X^{ss,fg}$ has a canonical enveloping quotient $X//H = \text{Proj}(k[X]^H)$ which is realized as a reductive GIT quotient.

Proof. This follows from Lemma 5.3.13, Proposition 4.2.9 and Theorems 5.3.1 and 5.3.4. \square

Even when the linear H -action fails to extend to a linear G -action on X , this is true for the ambient projective space \mathbb{P}^n , so we obtain the following corollary.

Corollary 5.3.15. *If $\overline{G/H}^{\text{aff}}$ is gentle, then*

$$X^{\bar{s}} = X \cap (\mathbb{P}^n)^{ms}$$

is independent of the choice of $\overline{G/H}^{\text{aff}}$.

Proof. Since $(G/H) \times X$ is closed in $(G/H) \times \mathbb{P}^n$, by reductive GIT [41, Theorem 1.19] we have $X^{\bar{s}} = X \cap (\mathbb{P}^n)^{\bar{s}}$, and by Proposition 5.3.14 applied to \mathbb{P}^n we have $(\mathbb{P}^n)^{\bar{s}} = (\mathbb{P}^n)^{ms}$. \square

5.3.3. Consequences for finite generation. Let us now assume that X is nonsingular, so that by resolution of singularities we can find a nonsingular (and hence gentle) projective completion $\overline{G \times_H X}$. The construction of L'_N in Proposition 5.3.9 gives us a necessary and sufficient condition for the ring of invariants $k[X]^H$ to be finitely generated; when L'_N is ample, this condition is that for large enough N the codimension 1 components of the boundary of $G \times_H X$ in $\overline{G \times_H X}$ are all *unstable*.

Theorem 5.3.16. *Let X be a nonsingular projective variety acted on by a connected unipotent group H , and let L be a linearization of the H -action with respect to a projective embedding $X \subseteq \mathbb{P}^n$ of X . Let $G \geq H$ be a reductive subgroup of $SL(n+1; k)$. Let L be the induced G -linearization over $G \times_H X$ and let L' be an extension of L over a gentle completion $(\overline{G \times_H X})$.*

Then the ring of invariants $k[X]^H$ is finitely generated if and only if there exists N such that, for all $N' > N$ for which $L'_{N'}$ is defined, any G -invariant section of a positive tensor power of $L'_{N'}$ vanishes on every codimension 1 component D_j in the boundary of $G \times_H X$ in $\overline{G \times_H X}$, where $L'_N = L'[N \sum_j D_j]$.

Proof. By restriction $k[\overline{G \times_H X}]^G \subseteq k[G \times_H X]^G = k[X]^H$. The forward direction is then a consequence of the construction of Proposition 5.3.9 as follows. For large enough N , any given invariant section over $G \times_H X$ extends and vanishes on each D_j . So for large enough N the finitely many generators of the ring of invariants will all vanish on every D_j , hence every element of $k[\overline{G \times_H X}]^G$ vanishes on every D_j .

The reverse direction follows by proving that for any such N the ring of invariants $k[X]^H \cong k[G \times_H X]^G$ is isomorphic to the ring of invariants $k[\overline{G \times_H X}]^G$, defined with respect to the linearization L'_N , which is finitely generated since $\overline{G \times_H X}$ is a projective variety acted on linearly by the reductive group G . This isomorphism arises since any invariant section s over $G \times_H X$ of L'_N extends as in the proof of Proposition 5.3.9 above to an invariant section of $L'_{N'}$ over $\overline{G \times_H X}$ for some $N' > N$. By hypothesis this section vanishes on each D_j and hence defines a section of $L'_{N'-1}$ extending s . Repeating this argument enough times we find that s extends to a section of L'_N . The same argument applies to any invariant section s over $G \times_H X$ of a positive tensor power $(L'_N)^{\otimes m}$ of L'_N , so we have $k[G \times_H X]^G \cong k[\overline{G \times_H X}]^G$ as required. \square

Corollary 5.3.17. *In the setting of Theorem 5.3.16, if the L'_N are ample for all $N' > N$, then $k[X]^H$ is finitely generated if and only if every D_i is unstable for all such N' .*

Proof. For an ample bundle, the complement of the zero set of a section is affine. In particular, given an ample G -linearization, the set on which all invariant sections vanish is precisely the unstable set. \square

Remark 5.3.18. For H -actions extending to G -actions on a nonsingular projective variety X , this necessary and sufficient condition can be made effective by an explicit construction of a suitable projective completion for G/H as in § 5.3.2, together with a careful analysis of Hilbert-Mumford in this setting (cf. Remark 3.2.5), at least when the bundles L'_N are ample. When the bundles L'_N are not ample the analysis of stability is less straightforward (cf. [25]). Unfortunately ampleness does not always occur, as can be seen when $H = \mathbb{C}^+$ and $G = SL(2; \mathbb{C})$ and $\overline{G/H}$ is the blow-up of \mathbb{P}^2 at 0.

6. EXAMPLE: n UNORDERED POINTS ON \mathbb{P}^1

Let $H = \mathbb{C}^+$, identified with the group of upper triangular matrices of the form

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

in $GL(2; \mathbb{C})$, act linearly on $X = \mathbb{P}^n = \mathbb{P}(\text{Sym}^n(\mathbb{C}^2))$ via the standard representation of $GL(2; \mathbb{C})$ on $\text{Sym}^n(\mathbb{C}^2)$.

Let $G = SL(2; \mathbb{C})$; then we can identify G/H with $\mathbb{C}^2 \setminus \{0\}$ via the usual transitive action of $SL(2; \mathbb{C})$ on $\mathbb{C}^2 \setminus \{0\}$ which extends to a linear action on its projective completion $\mathbb{P}^2 = \overline{G/H}$ with the point $[1 : 1 : 0]$ representing the identity coset H . Since the linear action of H on X extends to G we have $G \times_H X \cong G/H \times X$, and we are in the setting of §5.3.2. Since \mathbb{P}^2 is smooth, this is a gentle completion. Let L denote the hyperplane line bundle on $X = \mathbb{P}^n$ and let L_2 denote the hyperplane line bundle on $\overline{G/H} = \mathbb{P}^2$. For any positive integers p and q there is then an induced linearization of the action of G on $\overline{G/H} \times X$ with respect to the line bundle $L^{\otimes p} \otimes L_2^{\otimes q}$. Note this is a line bundle of the type denoted by L'_q in §5.3.2. In particular, by Lemma 5.3.13 this provides a strong ample reductive envelope for large enough q . By Theorem 5.3.4 this means $X^{\overline{s}} = X^{ms}$ and $X^{nss} = X^{ss,fg} \stackrel{\text{def}}{=} X^{ss} = X^{\overline{ss}}$.

We know that $k[X]^H$ is finitely generated because G/H is a codimension 2 complement in its affine closure \mathbb{A}^2 [23, §4]. We can also see this by applying the finitely generated criterion, Theorem 5.3.16: here there is only one boundary divisor, namely the product of the line at infinity with X ; by the Hilbert-Mumford numerical criterion (see Remark 3.2.3) it is easy to see that this is unstable, and in particular that all the invariants vanish there, for sufficiently large q .

By the Hilbert-Mumford numerical criterion, a point of $\mathbb{P}(\text{Sym}^n(\mathbb{C}^2)) \times \{[1 : 1 : 0]\} \subseteq \mathbb{P}(\text{Sym}^n(\mathbb{C}^2)) \times \mathbb{P}^2$, represented by an unordered sequence p_1, \dots, p_n of points in \mathbb{P}^1 , is stable for this linear action of G provided that

- strictly fewer than $\frac{n}{2} + \frac{q}{p}$ of the points p_1, \dots, p_n coincide anywhere on \mathbb{P}^1 , and
- strictly fewer than $\frac{n}{2}$ of the points p_1, \dots, p_n coincide at $[1 : 0]$;

it is G -semistable unless

- strictly more than $\frac{n}{2} + \frac{q}{p}$ of the points p_1, \dots, p_n coincide anywhere on \mathbb{P}^1 ,
or
- strictly more than $\frac{n}{2}$ of the points p_1, \dots, p_n coincide at $[1 : 0]$.

When q is large compared with p then the first condition is vacuous in each case, and so a point of $\mathbb{P}(\text{Sym}^n(\mathbb{C}^2))$ represented by an unordered sequence p_1, \dots, p_n of points in \mathbb{P}^1 is in $X^{\overline{s}} = X^s$ provided that strictly fewer than $n/2$ of the points coincide at $[1 : 0]$, and is in $X^{\overline{ss}} = X^{ss}$ unless strictly more than $n/2$ coincide there.

Thus when n is odd semistability and stability coincide and we have a geometric quotient $X^s/H = X^{ss}/H$ which is an open subset of $X//H = \text{Proj}(k[X]^H) \cong (\overline{G}/\overline{H} \times X)//G$; its complement can be identified with the reductive quotient $X//G$. Observe that a point of $X = \mathbb{P}(\text{Sym}^n(\mathbb{C}^2))$ represented by an unordered sequence p_1, \dots, p_n of points in \mathbb{P}^1 is G -stable provided that strictly fewer than $n/2$ of the points coincide anywhere on \mathbb{P}^1 , and is G -semistable unless strictly more than $n/2$ coincide anywhere on \mathbb{P}^1 .

When n is even we have a geometric quotient X^s/H which is an open subvariety of $X//H \cong (\overline{G}/\overline{H} \times X)//G$, again with complement $X//G$, but now the image of X^{ss} in $X//H$ is not a subvariety: it is the union of the open subvariety X^s/H and the closed subvariety – which is in fact a point – $(X//G) \setminus (X^{s,G}/G)$ where $X^{s,G}$ is the stable set for the action of G on X .

When n is odd $X^s = X^{ns} = X^{as} = X^{ss}$. Indeed, whenever $n > 2$ then $X^s = X^{ns} = X^{as}$: by continuity, given the \mathbb{C}^+ -action on \mathbb{P}^1 no \mathbb{C}^+ -invariant can distinguish among orbits consisting of points which correspond to configurations with $n/2$ points at $[1 : 0]$, and if $n > 2$ this set is more than 1-dimensional (the dimension of \mathbb{C}^+).

Note that the algorithm based on flattening stratifications described in [19] produces a set of stable points much smaller than X^s ; for these \mathbb{C}^+ -actions the algorithm removes the hyperplane $x_0 = 0$, where x_0 is the unique \mathbb{C}^+ -invariant coordinate function.

6.1. $n = 3$ and 4. Let x_0, \dots, x_n be the usual coordinates on $\text{Sym}^n(\mathbb{C}^2)$, so that if $[x_0, \dots, x_n] \in \mathbb{P}(\text{Sym}^n(\mathbb{C}^2))$ corresponds to an unordered sequence p_1, \dots, p_n of points in \mathbb{P}^1 then x_0, \dots, x_n are the coefficients of a homogeneous polynomial of degree n in two variables whose roots are p_1, \dots, p_n .

When n is small, using Gröbner basis techniques (in particular, an adaptation of an algorithm in [50, Appendix C]) we can explicitly find generators for the ring of H -invariants in the polynomial ring

$$\bigoplus_{j \geq 0} H^0(X, L^{\otimes j}) = \mathbb{C}[x_0 : \dots : x_n].$$

For $n = 3$ we have four generators

$$x_0, \quad 2x_0x_2 - x_1^2, \quad 3x_0^2x_3 - 3x_0x_1x_2 + x_1^3$$

and

$$9x_0^2x_3^2 - 3x_1^2x_2^2 + 6x_1^3x_3 + 8x_0x_2^3 - 18x_0x_1x_2x_3$$

for the invariants, so all the invariants vanish if and only if

$$x_0 = x_1 = 0,$$

or equivalently if and only if at least two of the three points p_1, p_2, p_3 coincide at $[1 : 0]$, as expected from the analysis above. The enveloping quotient $X//H =$

$\text{Proj}(k[X]^H)$ is a degree 6 hypersurface in the weighted projective space $\mathbb{P}(1, 2, 3, 4)$ with equation

$$3y_0^2y_3 = 8y_1^3 + 9y_2^2$$

and $X^s/H = X^{ss}/H$ is the complement of the point $[0 : 0 : 0 : 1]$ in this hypersurface.

When $n = 4$ we have seven generators

$$\begin{aligned} & x_0, \quad 2x_0x_2 - x_1^2, \quad 2x_0x_4 - 2x_1x_3 + x_2^2, \\ & 3x_0^2x_3 - 3x_0x_1x_2 + x_1^3, \quad 8x_0^3x_4 - 8x_0^2x_1x_3 + 4x_0x_1^2x_2 - x_1^4, \\ & -2x_2^3 + 6x_1x_2x_3 - 9x_0x_3^2 - 6x_1^2x_4 + 4x_0x_1x_4 \end{aligned}$$

and

$$9x_0^2x_3^2 - 3x_1^2x_2^2 + 6x_1^3x_3 + 8x_0x_2^3 - 18x_0x_1x_2x_3$$

so all the invariants vanish if and only if

$$x_0 = x_1 = x_2 = 0,$$

or equivalently if and only if at least three of the two points p_1, p_2, p_3, p_4 coincide at $[1 : 0]$, as expected. In this case the boundary $(X//H) \setminus (X^s/H)$ of X^s/H in $X//H$ is \mathbb{P}^1 and the image of X^{ss} in $X//H$ is the union of the open subset X^s/H and the point ∞ in its complement \mathbb{P}^1 .

6.2. Cohomology of quotients. We can study the rational intersection cohomology [17, 18] of any reductive GIT quotient $Y//G$, where Y is a projective variety, by stratifying Y and using equivariant (intersection) cohomology [27, 28, 29, 30, 31, 32, 33]. These methods apply in particular to any ample reductive envelope $Y = \overline{G \times_H X}$ of a linear H -action on a projective variety X over $k = \mathbb{C}$, allowing us to investigate the intersection cohomology of the projective completion $Y//G = \overline{X//H}$ of the enveloping quotient $X//H$; when, as in the examples we are considering, $k[X]^H$ is a finitely generated k -algebra so that $X//H = \text{Proj}(k[X]^H)$ is projective, this is the intersection cohomology of the enveloping quotient $X//H$ itself.

The simplest situation is when $Y = \overline{G \times_H X}$ is nonsingular; this holds in our examples since $Y = \mathbb{P}^2 \times \mathbb{P}^n$. Then we have a G -equivariantly perfect stratification $\{S_\beta : \beta \in \mathcal{B}\}$ of Y by G -invariant locally closed nonsingular subvarieties of Y with $S_0 = Y^{ss}$ as an open stratum, so that

$$(1) \quad P_t^G(Y^{ss}) = P_t^G(Y) - \sum_{\beta \neq 0} t^{2d_\beta} P_t^G(S_\beta)$$

where $P_t^G(Z) = \sum_{i \geq 0} t^i \dim H_G^i(Z; \mathbb{Q})$ is the G -equivariant Poincaré series of a G -space Z and d_β is the (complex) codimension of S_β in Y [30]. When $G = SL(2; \mathbb{C})$ and $Y = \mathbb{P}^2 \times \mathbb{P}^n$ then the stratification $\{S_\beta : \beta \in \mathcal{B}\}$ is given by

$$S_0 = Y^{ss}$$

and for $n/2 < j \leq n$

$$S_{j,0} = \{(w, x) \in \mathbb{P}^2 \times \mathbb{P}^n : x \text{ represents } n \text{ points of } \mathbb{P}^1$$

exactly j of which coincide at $[a : b]$ where $y = [1 : ta : tb]$ for some $t \in \mathbb{C}$

with codimension j in Y , and for $0 \leq j \leq n$

$$S_{j,1} = \{(w, x) \in \mathbb{P}^2 \times \mathbb{P}^n : x \text{ represents } n \text{ points of } \mathbb{P}^1$$

exactly j of which coincide at $[a : b]$ where $y = [0 : a : b]$ with codimension $j + 1$ in Y . Equation (1) gives us

$$P_t^G(Y^{ss}) = (1 + t^2 + t^4)(1 + t^2 + \dots + t^{2n})(1 - t^4)^{-1} - \sum_{\frac{n}{2} < j \leq n} t^{2j}(1 - t^2)^{-1} - \sum_{0 \leq j \leq n} t^{2j+2}(1 - t^2)^{-1}.$$

When n is odd we have $Y^{ss} = Y^s$ and so $Y//G = Y^{ss}/G = Y^s/G$ and the Poincaré polynomial $P_t(X//H) = \sum_{i \geq 0} t^i \dim H^i(X//H; \mathbb{Q})$ of the enveloping quotient $X//H = Y//G$ is given by

$$\begin{aligned} P_t(X//H) &= P_t^G(Y^{ss}) = [(1 + t^2 + t^4)(1 + t^4 + t^8 + \dots + t^{2n-2}) - (t^{n+1} + t^{n+3} + \dots + t^{2n}) \\ &\quad - (t^2 + t^4 + \dots + t^{2n+2})](1 - t^2)^{-1} \\ &= (1 + t^4 + t^8 + \dots + t^{2n-2} - t^{n+1} - t^{n+3} - \dots - t^{2n})(1 - t^2)^{-1} \\ &= 1 + t^2 + 2t^4 + 2t^6 + \dots + [1 + \frac{\min(j, n-1-j)}{2}]t^{2j} + \dots + 2t^{2n-8} + 2t^{2n-6} + t^{2n-4} + t^{2n-2} \end{aligned}$$

where $[\]$ denotes the integer part. Thus $[1 + (\min(j, n-1-j))/2]$ is the j th Betti number of $X//H$ when n is odd; this is also the j th intersection Betti number since $X//H$ is an orbifold and so its rational intersection cohomology is the same as its ordinary rational cohomology. The Poincaré polynomial of the geometric quotient $X^s/H = X^{ss}/H = (X//H) \setminus (X//G)$ is given by

$$P_t(X^s/H) = P_t(X//H) - t^4 P_t(X//G) = 1 + t^2 + t^4 + \dots + t^{n-1}.$$

We can in fact see this directly, since X^s retracts onto $\mathbb{P}^{(n-1)/2}$ and H is contractible.

When n is even then $Y^{ss} \neq Y^s$ and $X//H = Y//G$ is not an orbifold, which means that there is more work to do to compute the intersection Betti numbers of $X//H$. We first find the ordinary Betti numbers of a ‘partial desingularization’ $\widetilde{X//H}$ of $X//H$ (see [31]) obtained by blowing up the image in $X//H = Y//G$ of the subvariety Z of Y consisting of those $(w, x) \in \mathbb{P}^2 \times \mathbb{P}^n$ with $w = [1 : 0 : 0]$ (the origin in $\mathbb{C}^2 \subseteq \mathbb{P}^2$) and x representing n points on \mathbb{P}^1 , exactly half of which coincide at some $u \in \mathbb{P}^1$ and the remaining half coincide at some $v \neq u$. This partial desingularization $\widetilde{X//H}$ (which has only orbifold singularities) is itself a projective completion of the geometric quotient X^s/H . It can be represented as $\widetilde{X//H} = \widetilde{Y//G}$ where $\widetilde{Y}^{ss} = \widetilde{Y}^s$ is obtained from Y^{ss} by blowing up along its intersection with Z and removing the proper transform of the subvariety consisting of those $(w, x) \in \mathbb{P}^2 \times \mathbb{P}^n$ with x representing n points on \mathbb{P}^1 exactly half of which coincide at some $u = [a : b] \in \mathbb{P}^1$, and with $w = [1 : ta : tb]$ for some $t \in \mathbb{C}$. We obtain

$$\begin{aligned} P_t(\widetilde{X//H}) &= P_t^G(\widetilde{Y}^{ss}) \\ &= P_t^G(Y^{ss}) + (t^2 + t^4 + \dots + t^{2n-2})(1 - t^4)^{-1} - t^n(1 + t^2 + \dots + t^n)(1 - t^2)^{-1} \\ &= 1 + 2t^2 + 3t^4 + \dots + (n/2)t^{n-2} + (n/2)t^n + \dots + t^{2n-2}. \end{aligned}$$

Finally from this, using the decomposition theorem of [4], we can obtain the intersection Poincaré polynomial $IP_t(X//H) = \sum_{i \geq 0} t^i \dim IH(X//H; \mathbb{Q})$ as

$$\begin{aligned} IP_t(X//H) &= P_t(\widetilde{X//H}) - (t^2 + t^4 + 2t^6 + \dots + [\frac{n}{4}]t^{n-2} + [\frac{n}{4}]t^n + \dots + t^{2n-6} + t^{2n-4}) \\ &= 1 + t^2 + 2t^4 + 2t^6 + \dots + [(n+2)/4]t^{n-2} + [(n+2)/4]t^n + \dots + 2t^{2n-6} + t^{2n-4} + t^{2n-2}. \end{aligned}$$

For more details see [32]. The Poincaré polynomial of the geometric quotient X^s/H when n is even is given by

$$P_t(X^s/H) = P_t(\widetilde{X//H}) - t^4 P_t(\widetilde{X//G}) = 1 + t^2 + t^4 + \cdots + t^{n-2}$$

as we can also see directly, since X^s retracts onto $\mathbb{P}^{(n-2)/2}$ and H is contractible.

We can also compute intersection pairings in $IH(X//H; \mathbb{Q})$ and $H^*(\widetilde{X//H}; \mathbb{Q})$, and the ring structure of $H^*(\widetilde{X//H}; \mathbb{Q})$ and of $H^*(X//H; \mathbb{Q})$ when n is odd, using the methods of [27, 28].

Remark 6.2.1. One can work similarly with other (generalized) cohomology theories, like K -theory, or motivic cohomology in the sense of Voevodsky [2], at least when n is odd.

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MATHEMATICAL INSTITUTE, OXFORD OX1 3BJ, UK