PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 8, Pages 2319–2326
S 0002-9939(05)07957-8

ALMOST EVERYWHERE CONVERGENCE OF SERIES IN L^1

CIPRIAN DEMETER

(Communicated by Andreas Seeger)

Abstract. We answer positively a question of J. Rosenblatt (1988), proving the existence of a sequence (c_i) with $\sum_{i=1}^\infty |c_i| = \infty$, such that for every dynamical system (X, Σ, m, T) and $f \in L^1(X)$, $\sum_{i=1}^\infty c_i f(T^i x)$ converges almost everywhere. A similar result is obtained in the real variable context.

1. Introduction

Let T be a (not necessarily invertible) measure preserving transformation on the probability space (X, Σ, m). Given a sequence (c_i) we will state some mild conditions under which the series $\sum_{i=1}^\infty c_i f(T^i x)$ converges almost everywhere for every $f \in L^1(X)$. In [8] Rosenblatt proved that if $r_i(\omega)$ denotes the Rademacher sequence, then for almost every choice of ω one gets convergence of the above series with $c_i = \frac{r_i(\omega)}{i}$, for every $f \in L^p(X)$, $p > 1$. As a natural question, in the end of [8] it is asked whether there exists a sequence (c_i) with $\sum_{i=1}^\infty |c_i| = \infty$, such that for every $f \in L^1(X)$, $\sum_{i=1}^\infty c_i f(T^i x)$ converges almost everywhere. This question is also motivated by the fact that if one considers the same series associated with an invertible T and a two sided sequence $(c_i)_{i=-\infty}^\infty$, then the ergodic Hilbert Transform is an example for which the convergence is known to hold.

The purpose of this paper is twofold. On the one hand it gives a positive answer to the question above, as a consequence of Theorem 1 from [7]. On the other hand, the proof of this theorem (as presented in [7]) is quite long and makes use of the result concerning the convergence of the martingale transform from [4], which does not allow it to be extended to a larger setting. We will give a rather short proof here, based on a different type of argument, which will allow us in turn to prove a slightly more general result.

Given a sequence $C = (c_i)$ we will use the following notation:

$$A_{k,C} f(x) = \sum_{i=2^k+1}^{2^{k+1}} c_i f(T^i x).$$

When the sequence C is clear from the context, $A_k f(x)$ will be used instead.

Theorem 1.1. *Let (c_i) be a sequence of positive numbers with the following properties:*

(a) *The sequence (ic_i) is bounded.*

Received by the editors October 24, 2003.
2000 *Mathematics Subject Classification.* Primary 42B20, 28D05, 40A30, 26D15.

(b) *The sequence* (c_i) *is nonincreasing.*

(c) *The sequence* $s_k = \sum_{i=2^k+1}^{2^{k+1}} c_i$ *satisfies* $\sum_{k=0}^{\infty} |s_{k+1} - s_k| < \infty$.

Then for every bounded sequence (v_k)*, the operators*

$$S_n f(x) = \sum_{k=1}^{n} v_k \left(A_k f(x) - A_{k-1} f(x)\right)$$

converge a.e. for $f \in L^1(X)$*, and converge in norm for* $f \in L^p(X)$*,* $1 < p < \infty$.

Remark 1.2. This theorem remains valid if 2^k is replaced with an arbitrary lacunary sequence in the definition of A_k, and the proof does not suffer any serious modification. When $c_i = \frac{1}{2^{[\log_2 i]}}$, one recovers the result of Theorem 1 from [7].

From the above, one immediately gets the following:

Theorem 1.3. *Let* (c_i) *be a sequence of positive numbers with the following properties:*

(a) *The sequence* (ic_i) *converges to* 0.

(b) *The sequence* (c_i) *is nonincreasing.*

(c) *The sequence* $s_k = \sum_{i=2^k+1}^{2^{k+1}} c_i$ *satisfies* $\sum_{k=1}^{\infty} |s_{k+1} - s_k| < \infty$.

Define the sequence $d_i = c_i(-1)^k$*, when* $2^k + 1 \le i \le 2^{k+1}$*. Then the series*

$$S_n f(x) = \sum_{i=1}^{n} d_i f(T^i x)$$

converges a.e. for every $f \in L^1(X)$.

Sequences such as $\left(\frac{(-1)^{[\log_2(i-1)]}}{i \log i}\right)$, $\left(\frac{(-1)^{[\log_2(i-1)]}}{i \log\log i}\right)$, etc. in which the logarithmic form is expanded satisfy the requirements (a), (b) and (c) of Theorem 1.3. This proves the following corollary:

Corollary 1.4. *There exists a nonsummable sequence* (c_i) *such that for every* $f \in L^1(X)$*,* $\sum_{i=1}^{\infty} c_i f(T^i x)$ *converges almost everywhere.*

Remark 1.5. An interesting question is whether there exists a choice of signs $r_i \in \{-1, 1\}$ such that the following modulated one-sided Hilbert Transform

$$Sf(x) = \sum_{i=1}^{\infty} \frac{r_i f(T^i x)}{i}$$

converges a.e. for $f \in L^1(X)$. It appears that this question cannot be addressed by the techniques employed in this paper, and here is the reason why: the proof (based on the machinery of Benedek, Calderón and Panzone) of the weak $(1,1)$ maximal inequality for $\sup_n |S_n|$ in Theorem 1.1 relies heavily on the fact that the summation index for A_k runs through a block of lacunary growth; on the other hand, a series such as

$$Sf(x) = \sum_{k=1}^{\infty} (-1)^k \sum_{i=n_k+1}^{n_{k+1}} \frac{f(T^i x)}{i}$$

diverges for constant functions when (n_k) is lacunary.

The real variable analogues of the above theorems also hold. For a given ψ defined on $(0,\infty)$ we will use the notation

$$D_{k,\psi}f(x)=\int_{\frac{1}{2^{k+1}}}^{\frac{1}{2^k}}\psi(y)f(x-y)dy.$$

Again when ψ is clear from the context, $D_kf(x)$ will be used instead.

Theorem 1.6. *Let $\psi:(0,\infty)\to[0,\infty)$ be a function satisfying the following:*

(a) *The function $x\psi(x)$ is bounded.*
(b) *The function ψ is nonincreasing.*
(c) *The sequence $s_k=\int_{1/2^{k+1}}^{1/2^k}\psi(x)dx$ satisfies $\sum_{k=1}^{\infty}|s_{k+1}-s_k|<\infty$.*

Then for every bounded sequence (v_k), the operators

$$S_nf(x)=\sum_{k=1}^{n}v_k\left(D_kf(x)-D_{k-1}f(x)\right)$$

converge a.e. for $f\in L^1(\mathbb{R})$, and converge in norm for $f\in L^p(\mathbb{R})$, $1<p<\infty$.

This immediately gives

Theorem 1.7. *Let $\psi:(0,\infty)\to[0,\infty)$ be a function satisfying the following:*

(a) *The function $\lim_{x\to0}x\psi(x)=0$.*
(b) *The function ψ is nonincreasing.*
(c) *The sequence $s_k=\int_{1/2^{k+1}}^{1/2^k}\psi(x)dx$ satisfies $\sum_{k=1}^{\infty}|s_{k+1}-s_k|<\infty$.*

Define the function $\theta(x)=(-1)^k\psi(x)$, when $x\in[\frac{1}{2^{k+1}},\frac{1}{2^k})$. Then the improper integral

$$I=\int_0^{\infty}\theta(y)f(x-y)dy$$

converges for a.e. x, for every $f\in L^1(\mathbb{R})$.

Remark 1.8. Note again that functions such as $\psi(x)=\frac{1}{x\log x}$ or $\psi(x)=\frac{1}{x\log\log x}$ satisfy the requirements of Theorem 1.7. Like in the ergodic theoretic setting, it remains open whether there exists a function θ with $|\theta(x)|=\frac{1}{x}$ for each $x\in(0,\infty)$, which satisfies the conclusion of Theorem 1.7.

Remark 1.9. An example of a function $\theta\notin L_1[0,\infty)$ such that

$$I=\int_0^{\infty}\theta(y)f(x-y)dy$$

converges for a.e. x, for every $f\in L^1(\mathbb{R})$, appears in [1]. The kernel there, $\theta(x)=\chi_{(0,\infty)}(x)\cdot\frac{1}{x}\frac{\sin(\log x)}{\log x}$, is a smooth variant of the one we are using here.

2. PROOFS

We will use the fundamental results from [6] to get a maximal inequality for the operator $S^{\star}f(x)=\sup_n S_nf(x)$. Since these results are stated in the real variable context, we need to transfer them in the ergodic theoretic setting. For a measure μ on $\mathbb{Z}$ define the Borel measure w on $\mathbb{R}$ by the formula $w=\mu*\chi_{[0,1)}$ where

$$\mu*\chi_{[0,1)}(x)=\int_{\mathbb{Z}}\chi_{[0,1)}(x-y)d\mu(y)=\sum_{k=-\infty}^{\infty}\chi_{[k,k+1)}(x)\mu(k).$$

In the following, for any Borel measure w on $\mathbb{R}$ we will denote by $|w| = w^+ - w^-$ the total variation of w while $||w||_1$ will stand for the quantity $|w|(\mathbb{R})$. The same notation will be used for measures on $\mathbb{Z}$. Given a sequence (w_k) of Borel measures on $\mathbb{R}$, the associated maximal operator is defined as $w^*(\psi) = \sup_k ||w_k| * \psi|$, for each $\psi : \mathbb{R} \to \mathbb{R}$. The Dirac mass concentrated on $\{i\}$ will be denoted by δ_i . The following two lemmas are essentially contained in [2], but the proofs are slightly different in this context, so we will sketch them.

Lemma 2.1. *Assume that* (μ_k) *is a sequence of measures on* $\mathbb{Z}$ *satisfying*

$$|\hat{\mu}_k(\gamma)| \le C 2^k |\gamma - 1| \tag{2.1}$$

and

$$|\hat{\mu}_k(\gamma)| \le C(2^k|\gamma - 1|)^{-1}, \gamma \neq 1, \tag{2.2}$$

for some constant C *independent of* k*. Then for some constant* C' *we also have*

$$|\hat{w}_k(\xi)| \le C' 2^k |\xi| \tag{2.3}$$

and

$$|\hat{w}_k(\xi)| \le C'(2^k|\xi|)^{-1}, \xi \neq 0, \tag{2.4}$$

where (w_k) *are the corresponding measures on* $\mathbb{R}$*.*

Proof. The proof immediately follows from the identity

$$\hat{w}_k(\xi) = \hat{\mu}_k(e^{2\pi i \xi}) \frac{e^{2\pi i \xi} - 1}{2\pi i \xi},\ \xi \neq 0.$$

□

Lemma 2.2. *Let* (μ_k) *be a sequence of measures on* $\mathbb{Z}$ *satisfying*

$$\sum_{k=1}^{\infty} ||\mu_k - \mu_k * \delta_1||_1 < \infty, \tag{2.5}$$

and let (w_k) *denote the corresponding measures on* $\mathbb{R}$*. Define the integral operator* $T^*_{\mathbb{Z}}\phi(l) = \sup_k |T_{\mathbb{Z},k}\phi(l)|$ *with*

$$T_{\mathbb{Z},k}\phi(l) = \sum_{i=1}^{k} \mu_i * \phi(l) \tag{2.6}$$

and similarly the differential operator $T^*_{\mathbb{R}}\psi(x) = \sup_k |T_{\mathbb{R},k}\psi(x)|$ *with*

$$T_{\mathbb{R},k}\psi(x) = \sum_{i=1}^{k} w_i * \psi(x). \tag{2.7}$$

Then

(i) *if* $T^*_{\mathbb{R}}$ *is bounded in* $L^p(\mathbb{R})$ *for some* $p > 1$*, then* $T^*_{\mathbb{Z}}$ *is bounded in* $l^p(\mathbb{Z})$*;*
(ii) *if* $T^*_{\mathbb{R}}$ *satisfies a weak* $(1,1)$ *inequality, then so does* $T^*_{\mathbb{Z}}$*.*

Proof. We will only prove (i), since the second assertion follows similarly. We have that

$$\left|\left|\sup_k \left|\sum_{i=1}^{k} w_i * \psi\right|\right|\right|_p \le C_p ||\psi||_p,$$

for all $\psi \in L^p(\mathbb{R})$. From here we can prove our result on $l^p(\mathbb{Z})$. Given $\phi \in l^p(\mathbb{Z})$, let

$$\phi * \chi_{[0,1)} = \sum_{k=-\infty}^{\infty} \chi_{[k,k+1)}\phi(k).$$

Then $\phi * \chi_{[0,1)}$ is in $L^p(\mathbb{R})$ and in fact $||\phi||_{l^p(\mathbb{Z})} = ||\phi * \chi_{[0,1)}||_{L^p(\mathbb{R})}$. By the maximal inequality above,

$$\left|\left|\sup_k |\sum_{i=1}^{k} w_i * \phi * \chi_{[0,1)}|\right|\right|_p \le C_p||\phi||_p.$$

All that remains to be proven now is that

$$\left|\left|\sup_k |\sum_{i=1}^{k} \mu_i * \phi * \chi_{[0,1)}|\right|\right|_p \le C_p' \left\{ ||\phi||_p + \left|\left|\sup_k |\sum_{i=1}^{k} w_i * \phi * \chi_{[0,1)}|\right|\right|_p \right\}.$$

Note that

$$\left|\left|\sup_k |\sum_{i=1}^{k} \mu_i * \phi * \chi_{[0,1)}|\right|\right|_p \le \left|\left|\sup_k |\sum_{i=1}^{k} w_i * \phi * \chi_{[0,1)}|\right|\right|_p + \sum_{i=1}^{\infty} \left|\left|\mu_i * \phi * \chi_{[0,1)} - w_i * \phi * \chi_{[0,1)}\right|\right|_p.$$

Now if $l \le x < l+1$ for some $l \in \mathbb{Z}$, say $x = l + \epsilon$, then

$$w_i * \phi * \chi_{[0,1)}(x) = \sum_k (1-\epsilon)\mu_i(k-1)\phi(l-k) + \sum_k \epsilon\mu_i(k)\phi(l-k),$$

while

$$\mu_i * \phi * \chi_{[0,1)}(x) = \sum_k \mu_i(k)\phi(l-k).$$

This immediately proves that

$$|\mu_i * \phi * \chi_{[0,1)}(x) - w_i * \phi * \chi_{[0,1)}(x)| \le |\mu_i - \mu_i * \delta_1| * |\phi|(l),$$

and hence

$$\left|\left|\mu_i * \phi * \chi_{[0,1)} - w_i * \phi * \chi_{[0,1)}\right|\right|_p \le ||(|\mu_i - \mu_i * \delta_1|) * (|\phi|)||_p \le ||\mu_i - \mu_i * \delta_1||_1 ||\phi||_p.$$

The result now follows from (2.5). ☐

The main ingredient of our proofs is the following fundamental lemma:

Lemma 2.3. *Let $dw_k = f_k dx$ be a sequence of measures on $\mathbb{R}$ and let $T^*_{\mathbb{R}}$ be as above. Assume the following are satisfied:*

$$\int_{|x|>4|y|} \sup_k |\sum_{i=1}^{k} (f_i(x-y) - f_i(x))| dx \le C', \tag{2.8}$$

$$||w_k||_1 < M, \tag{2.9}$$

$$|\hat{w_k}(\xi)| \le C' 2^k |\xi|, \tag{2.10}$$

$$|\hat{w_k}(\xi)| \le C'(2^k|\xi|)^{-1}, \ \xi \ne 0, \tag{2.11}$$

$$||w^*(\psi)||_2 \le C'||\psi||_2, \tag{2.12}$$

$$\mathrm{supp}(w_k) \subset \{x \in \mathbb{R} : |x| < 2^{k+1}\} \tag{2.13}$$

for some constants M and C' independent of k, y and ψ. Then $T^*_{\mathbb{R}}$ is bounded in $L^p(\mathbb{R})$ for $1 < p < \infty$ and satisfies a weak $(1,1)$ inequality.

Proof. Conditions (2.9), (2.10), (2.11), (2.12) and (2.13) are the ones used by Duoandikoetxea and Rubio de Francia in Theorem E of [6]. Using their result, we have $||T^*_{\mathbb{R}}||_2 \le C$. This fact together with (2.8) are the conditions needed in Theorem 2 from [3], with $B_1 = \mathbb{R}$ and $B_2 = l^\infty$. The result follows immediately. □

Here is the proof of Theorem 1.1:

Proof. Without loss of generality we can assume that $||(v_k)||_{l^\infty} \le 1$. Define the measures μ_k on $\mathbb{Z}$ by

$$\mu_k = v_k \left(\sum_{i=2^k+1}^{2^{k+1}} c_i \delta_i - \frac{s_k}{s_{k-1}} \sum_{i=2^{k-1}+1}^{2^k} c_i \delta_i \right),$$

and let w_k denote the corresponding measures on $\mathbb{R}$. We will first show that the operator $T^*_{\mathbb{R}}$ associated to these measures is bounded in $L^p(\mathbb{R})$, $p > 1$, and satisfies a weak $(1,1)$ maximal inequality, as a consequence of Lemma 2.3. Conditions (a) and (b) from Theorem 1.1 are easily seen to imply (2.9) and (2.13). Also, since $s_k \le 2s_{k-1}$, (2.12) follows as a consequence of (a) and the boundedness of the Hardy-Littlewood maximal operator. In order to prove (2.10) and (2.11) it suffices (according to Lemma 2.1) to prove that $|\hat{\mu}_k(\gamma)| \le C2^k|\gamma - 1|$ and $|\hat{\mu}_k(\gamma)| \le C(2^k|\gamma-1|)^{-1}, \gamma \ne 1$. But

$$\begin{aligned} |\hat{\mu}_k(\gamma)| &\le \left| \sum_{i=2^k+1}^{2^{k+1}} c_i(\gamma^i - 1) \right| + \frac{s_k}{s_{k-1}} \left| \sum_{i=2^{k-1}+1}^{2^k} c_i(\gamma^i - 1) \right| \\ &\le \sum_{i=2^k+1}^{2^{k+1}} i c_i |\gamma - 1| + \frac{s_k}{s_{k-1}} \sum_{i=2^{k-1}+1}^{2^k} i c_i |\gamma - 1| \\ &\le C2^k |\gamma - 1|, \end{aligned}$$

while by using Abel's summation, (a) and (b) we get

$$\begin{aligned} |\hat{\mu}_k(\gamma)(\gamma - 1)| &\le \left| \sum_{i=2^k+2}^{2^{k+1}} (c_{i-1} - c_i)\gamma^i \right| + |c_{2^{k+1}}\gamma^{2^{k+1}+1} - c_{2^k+1}\gamma^{2^k+1}| \\ &+ \left| \sum_{i=2^{k-1}+2}^{2^k} (c_{i-1} - c_i)\gamma^i \right| + |c_{2^k}\gamma^{2^k+1} - c_{2^{k-1}+1}\gamma^{2^{k-1}+1}| \\ &\le C2^{-k}. \end{aligned}$$

It only remains to prove (2.8). Obviously

$$f_k(x) = v_k \left(\sum_{i=2^k+1}^{2^{k+1}} c_i \chi_{[i,i+1)}(x) - \frac{s_k}{s_{k-1}} \sum_{i=2^{k-1}+1}^{2^k} c_i \chi_{[i,i+1)}(x) \right).$$

Fix a y. Note that since $f_k(x) = 0$ when $x < 2$, the integral in (2.8) is only over the set $\{x > 1\}$, so we can assume x is positive and hence $0 < x - y < x < 2(x-y)$. Moreover, for each such x there are at most 2 values of k such that $f_k(x) \ne f_k(x-y)$. Define the sets $D = \{x \in [1, \infty) : x > 4|y|\}$, $A_1 = \{x \in D : \exists\, k \ge 0 \text{ s.t. } 2^k + 1 \le$

$x, x-y < 2^{k+1}+1\}$ and $A_2 = D \setminus A_1$. Since $4|y| < x$, it follows that any k that is used in the definition of A_1 must be greater than $\log_2 |y|$. Note that if $x \in A_1$, then

$$\sup_k \Big|\sum_{i=1}^{k}(f_i(x-y)-f_i(x))\Big| \leq 4|c_{[x-y]}-c_{[x]}| \leq 4|c_{[x]-[y]-1}-c_{[x]}| + 4|c_{[x]-[y]}-c_{[x]}|.$$

Hence

$$\begin{aligned}\int_{A_1}\sup_k \Big|\sum_{i=1}^{k}(f_i(x-y)-f_i(x))\Big|dx &< 4\sum_{i\geq|y|+1}|c_{i-[y]-1}-c_i| + 4\sum_{i\geq|y|+1}|c_{i-[y]}-c_i| \\ &< C_1' \text{ by (a) and (b).}\end{aligned}$$

Consider now an $x \in A_2$. There will exist an $l \in \mathbb{N}$ such that $x - y < 2^l + 1 \leq x$, and from the same reasons described above, $l \geq \log_2 |y|$. Note also that

$$A_2 \subset \bigcup_{k\geq 0}[2^k+1, 2^k+1+|y|)$$

and

$$\sup_k \Big|\sum_{i=1}^{k} f_i(x)\Big| < 4c_{[x]} < 4c_{2^l}.$$

This implies that

$$\begin{aligned}\int_{A_2}\sup_k \Big|\sum_{i=1}^{k} f_i(x)\Big|dx &< 4|y|\sum_{l>\log_2|y|} c_{2^l} \\ &< C_2' \text{ by (a).}\end{aligned}$$

Similarly one finds that

$$\int_{A_2}\sup_k \Big|\sum_{i=1}^{k} f_i(x-y)\Big|dx < C_3'.$$

This proves that (2.8) is satisfied with $C' = C_1' + C_2' + C_3'$.

Equation (2.5) is easily seen to be satisfied for the measures μ_k, based on (a) and (b). Hence according to Lemma 2.2 the operator $T^*_{\mathbb{Z}}$ is also bounded on $l^p(\mathbb{Z})$ and satisfies a weak $(1,1)$ type inequality. Define now the operators $S_{\mathbb{Z},n}$ on $\mathbb{Z}$ by

$$S_{\mathbb{Z},n}\phi(l) = \sum_{k=1}^{n} v_k\left(\sum_{i=2^k+1}^{2^{k+1}} c_i\phi(i+l) - \sum_{i=2^{k-1}+1}^{2^k} c_i\phi(i+l)\right).$$

Note that

$$\begin{aligned}S^*_{\mathbb{Z}}\phi(l) &\leq T^*_{\mathbb{Z}}\phi(l) + \sum_{k=1}^{\infty}\left(\frac{s_k}{s_{k-1}}-1\right)\sum_{i=2^{k-1}+1}^{2^k} c_i\phi(l+i) \\ &= T^*_{\mathbb{Z}}\phi(l) + M\phi(l).\end{aligned}$$

Since

$$||M\phi||_1 \leq ||\phi||_1 \sum_{k=0}^{\infty}|s_{k+1}-s_k|,$$

it follows immediately that $S^*_{\mathbb{Z}}$ is bounded in $l^p(\mathbb{Z})$, $p > 1$, and satisfies a weak $(1, 1)$ maximal inequality. By using Calderón's standard transfer principle, see for example [5], we get the same results for the ergodic operator S^* of Theorem 1.1.

Condition (c) proves that $S_n f(x)$ converges a.e. for T invariant functions, while (a) and (b) prove the convergence for every coboundary $f(x) = g(Tx) - g(x)$. Since these functions span a dense subclass of $L^1(X)$, convergence on the whole $L^1(X)$ follows. The norm convergence follows as a consequence of the Dominated Convergence Theorem. □

Proof of Theorem 1.3. Note that (a) implies that

$$\sup_{2^k+1\le j\le 2^{k+1}} \left| \sum_{i=2^k+1}^{j} c_i f(T^i x) \right| = o(1) \frac{1}{2^{k+1}} \sum_{i=1}^{2^{k+1}} |f| (T^i x);$$

hence

$$\lim_{k\to\infty} \sup_{2^k+1\le j\le 2^{k+1}} \left| \sum_{i=2^k+1}^{j} c_i f(T^i x) \right| = 0$$

for a.e. x and all $f \in L^1(X)$. Using this, the conclusion of Theorem 1.3 follows now as an application of Theorem 1.1 with $v_k = (-1)^k$. □

The proofs of Theorems 1.6 and 1.7 are very similar. The argument is simpler in this case since the transfer lemmas 2.1 and 2.2 are no longer needed.

References

[1] H. Aimar, L. Forzani and F. J. Martin-Reyes, *On weighted inequalities for singular integrals*, Proc. Amer. Math. Soc. **125** (1997) no. 7, 2057-2064. MR1376747 (97i:42012)

[2] A. Bellow, R. L. Jones and J. Rosenblatt, *Almost everywhere convergence of weighted averages*, Math. Ann. **292** (1992), 399-426. MR1170516 (93e:28019)

[3] A. Benedek, A. P. Calderón and R. Panzone, *Convolution operators on Banach space valued functions*, Proc. Natl. Acad. of Sciences, USA **48** (1962), 356-365. MR0133653 (24:A3479)

[4] D. L. Burkholder, *Martingale transforms*, Ann. Math. Statist., **37** (1966), 1494-1504. MR0208647 (34:8456)

[5] A. P. Calderón, *Ergodic theory and translation invariant operators*, Proc. Natl. Acad. of Sciences, USA **59** (1968), 349-353. MR0227354 (37:2939)

[6] J. Duoandikoetxea and J. Rubio de Francia, *Maximal and singular integral operators via Fourier transform estimates*, Invent. Math. **84** (1986), 541-561. MR0837527 (87f:42046)

[7] R. L. Jones and J. Rosenblatt, *Differential and ergodic transforms*, Math. Ann. **323** (2002), 525-546. MR1923696 (2003g:37003)

[8] J. Rosenblatt, *Almost everywhere convergence of series*, Math. Ann. **280** (1988), 565-577. MR0939919 (89g:47012)

Department of Mathematics, University of California Los Angeles, Los Angeles, California 90095-1555

E-mail address: `demeter@math.ucla.edu`