Extended Research Statement

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1 Why Derived Algebraic Geometry?

Derived algebraic geometry refers to the study of geometric objects whose functions form not a sheaf of rings, but rather a sheaf of simplicial rings or, more generally, of E_∞ ring spectra. All of my work, in one way or another, stems from an effort to come to grips with various phenomena and concepts of derived algebraic geometry. My aim in this section is to motivate derived algebraic geometry by means of the two classes of problems that brought me to the subject — the former geometric, the latter topological. These two topics play a major role throughout this document.

Although I will describe the expectations of derived algebraic geometry as "conjectures" in this section, the success story of the subject is that, through the efforts of Toën, Vezzosi, Lurie, Rognes, and me, each of these assertions — with the obvious exceptions of 1.5 and 1.7 — has been at least partly established in the past five years.

Hidden quasismoothness

One of the first phenomena that pointed to the usefulness of derived algebro-geometric objects was the observation that certain familiar moduli stacks attached to smooth varieties are unexpectedly not locally complete intersections.

Example 1.1. A striking example of this phenomenon is the moduli stack $\mathbf{Vect}_n(X)$ of rank n vector bundles on a smooth projective complex variety X. For a given vector bundle $\mathscr{E} \in \mathbf{Vect}_n(X)(\mathbf{C})$, the tangent complex is a two-term truncation of the Zariski cohomology complex:[1]

$$T_{\mathscr{E}}\mathbf{Vect}_n(X) \simeq \tau^{\leq 0} C^\bullet_{\mathrm{Zar}}(X, \mathbf{End}\,\mathscr{E})[1].$$

The dimension of $\mathbf{Vect}_n(X)$ at $\mathscr{E}$ is thus the Euler characteristic of this truncated complex:

$$\dim T_{\mathscr{E}}\mathbf{Vect}_n(X) = h^1_{\mathrm{Zar}}(X, \mathbf{End}\,\mathscr{E}) - h^0_{\mathrm{Zar}}(X, \mathbf{End}\,\mathscr{E}).$$

As the total Euler characteristic $\chi_{\mathrm{Zar}}(X, \mathbf{End}\,\mathscr{E})$ is locally constant, if X is a curve, this stack has locally constant dimension; in fact it is smooth. If, however X is a surface, since the total Euler characteristic $\chi_{\mathrm{Zar}}(X, \mathbf{End}(\mathscr{E}))$ is locally constant, $\dim T_{\mathscr{E}}\mathbf{Vect}_n(X) = -\chi_{\mathrm{Zar}}(X, \mathbf{End}(\mathscr{E})) + h^2(X, \mathbf{End}\,\mathscr{E})$ is not, as $h^2_{\mathrm{Zar}}(X, \mathbf{End}(\mathscr{E}))$ may jump; thus the stack cannot be a locally complete intersection.

The *hidden quasismoothness philosophy* (once improperly dubbed the *hidden smoothness philosophy*) is the hope that one might find some kind of geometric object $\mathbf{RVect}_n(X)$ whose tangent complex is not necessarily concentrated in negative degrees, so that

$$T_{\mathscr{E}}\mathbf{RVect}_n(X) \simeq C^\bullet_{\mathrm{Zar}}(X, \mathbf{End}(\mathscr{E}))[1].$$

This object would have locally constant dimension and would ideally be LCI.

But what sort of object might $\mathbf{RVect}_n(X)$ be? Some clue comes from consideration of its tangent bundle; by analogy with classical algebraic geometry, one would anticipate that as a geometric object over $\mathbf{RVect}_n(X)$, the tangent bundle should be constructed as the relative spectrum of the symmetric algebra on the cotangent complex:

$$\mathbf{TRVect}_n(X) \simeq \operatorname{Spec}\operatorname{Sym} T^\vee\mathbf{RVect}_n(X).$$

Since $T^\vee\mathbf{RVect}_n(X)$ is a sheaf of complexes in degrees ≤ 1, and since the positive-dimensional part of these complexes arose from the "stackiness" of the moduli problem (i.e., from global endomorphisms of $\mathscr{E}$), it is natural to expect that $\mathbf{RVect}_n(X)$ is a stack that represents a moduli problem on commutative ring objects in the category $\mathbf{Cplx}^{\leq 0}(\mathbf{C})$, or, equivalently, simplicial $\mathbf{C}$-algebras.

[1] This is a negatively graded complex, in the style of algebraic geometry.

Conjecture 1.2. *Thus the hidden quasismoothness philosophy predicts an* LCI *or (in Lurie's terminology)* quasismooth derived moduli stack $\mathbf{RVect}_n(X)$ *of vector bundles of rank n, defined on simplicial* $\mathbf{C}$*-algebras, equipped with an equivalence between the restriction of* $\mathbf{RVect}_n(X)$ *to ordinary* $\mathbf{C}$*-algebras and the classical moduli stack* $\mathbf{Vect}_n(X)$.

Conjecture 1.3. *One can generalize the previous conjecture to ask, for any reductive* $\mathbf{C}$*-group G and any smooth projective* $\mathbf{C}$*-variety X, whether there is a quasismooth* derived moduli stack $\mathbf{RBun}_G(X)$ *of principal G-bundles, equipped with an equivalence between the restriction of* $\mathbf{RBun}_G(X)$ *to ordinary* $\mathbf{C}$*-algebras and the classical moduli stack* $\mathbf{Bun}_G(X)$.

Example 1.4. Even when X is a smooth projective curve (still over the complex numbers), there are nontrivial examples of derived moduli stacks associated with X. Indeed, suppose G a reductive complex group. by considering the *de Rham homotopy shape* X_{dR} of X, given by $X_{\mathrm{dR}}(A) := X(\pi_0(A)_{\mathrm{red}})$ for any simplicial $\mathbf{C}$-algebra A, one may define the derived moduli stack $\mathbf{RLoc}_G(X)$ of G-local systems as

$$\mathbf{RLoc}_G(X) := \mathbf{RBun}_G(X_{\mathrm{dR}}) := \mathbf{RMor}(X_{\mathrm{dR}}, \mathbf{RBun}_G(\star)).$$

When $G = \mathbf{G}_m$, for example, one can verify easily that

$$\mathbf{RLoc}_{\mathbf{G}_m}(X) \simeq \mathrm{Jac}(X)^{\dagger} \times B\mathbf{G}_m \times \operatorname{Spec}\mathbf{C}[\alpha],$$

where $\mathrm{Jac}(X)^{\dagger}$ is the universal extension of the Jacobian $\mathrm{Jac}(X)$ by $H^1(X,\Omega^1_X)$, and $\mathbf{C}[\alpha]$ is the free simplicial $\mathbf{C}$-algebra generated by an element α of degree 1.

It is important to note here that the derived part of the moduli stack of G-local systems plays a role in the Geometric Langlands Conjecture. Indeed, the naïve conjecture postulates an equivalence of triangulated categories

$$\mathbf{D}(\mathbf{Loc}_G(X), \mathscr{O}) \simeq \mathbf{D}(\mathbf{Bun}_{^LG}(X), \mathscr{D}),$$

under which skyscraper sheaves on the left correspond to Hecke eigen-$\mathscr{D}$-modules on the right. But this is plainly impossible as written, even for $G = {}^LG = \mathbf{G}_m$: indeed, one has

$$\mathbf{Bun}_{\mathbf{G}_m} \simeq \mathbf{Vect}_1(X) \simeq \mathrm{Jac}(X) \times \mathbf{Z} \times B\mathbf{G}_m.$$

There is an obvious equivalence $\mathbf{D}(B\mathbf{G}_m, \mathscr{O}) \simeq \mathbf{D}(\mathbf{Z}, \mathscr{D})$, and Laumon's geometric Fourier transform [31] provides an equivalence $\mathbf{D}(\mathrm{Jac}(X)^{\dagger}, \mathscr{O}) \simeq \mathbf{D}(\mathrm{Jac}(X), \mathscr{D})$, but the remaining factor $\mathbf{D}(B\mathbf{G}_m, \mathscr{D})$ is equivalent to $\mathbf{D}(\operatorname{Spec}\mathbf{C}[\alpha], \mathscr{O})$, which does not appear in the naïve formulation above.

There are two ways around this: one option is to impose level structures on $\mathbf{Bun}_{^LG}$ to kill the generic stabilizer; alternately, one can accept the generic stabilizers in $\mathbf{Bun}_{^LG}$ and embrace instead the following reformulation of (part of) the Geometric Langlands Conjecture.

Conjecture 1.5 (*cf.* 3.8)**.** *There is an equivalence of derived categories*

$$\mathbf{D}(\mathbf{RLoc}_G(X), \mathscr{O}) \simeq \mathbf{D}(\mathbf{Bun}_{^LG}(X), \mathscr{D}),$$

under which skyscraper sheaves on the left correspond to Hecke eigen-$\mathscr{D}$*-modules on the right, and the derived structure on the left is related to the generic stabilizers on the right.*

One might fantasize about a higher-dimensional version of this statement as well, but there is no chance for the conjecture to generalize as stated. Indeed, the case $G = \mathbf{G}_m$ shows this: Laumon's Fourier transformation is an equivalence between $\mathscr{D}$-complexes on an abelian variety A with $\mathscr{O}$-complexes on the universal extension of $A^{\vee}$; our argument for 1.5 for $G = \mathbf{G}_m$ relies upon the self-duality of the Jacobian. Thus there are two directions of possible generalizations of Geometric Langlands to higher dimensions: one may attempt either to describe $\mathscr{O}$-modules on $\mathbf{RLoc}_G(X)$ in terms of ${}^LG$ or to describe $\mathscr{D}$-modules on $\mathbf{RBun}_{^LG}(X)$ in terms of G. For the latter, see the discussion preceding 3.10.

Algebraic K-theory of E_∞ ring spectra

Another piece of evidence for the usefulness of derived algebraic geometry comes from the algebraic K-theory of E_∞-ring spectra.

Example 1.6. There is tremendous practical value in computing the algebraic K-theory of various E_∞ ring spectra. To illustrate this, suppose now X a pointed CW complex; then Waldhausen's algebraic K-theory spectrum $A(X)$ [51] is canonically equivalent to the algebraic K-theory $K(\mathbf{S}[\Omega X])$. Moreover, for any pointed space X, there is a split fiber sequence

$$Q(X_+) \longrightarrow \Omega^\infty K(\mathbf{S}[\Omega X]) \longrightarrow \mathrm{Wh}(X),$$

where $A(X)$ is the algebraic K-theory of X, and $\mathrm{Wh}(X)$ is the smooth Whitehead space. If X is a compact manifold, Waldhausen's stable parametrized h-cobordism theorem [52] expresses a remarkable equivalence between $\Omega\mathrm{Wh}(X)$ and the space of h-cobordisms on $X \times I^{\times N}$ for N sufficiently large. Furthermore, results of Farrell–Jones [18, 19] indicate that if X is a Riemannian manifold with nonpositive sectional curvature (say), then $\mathrm{Wh}(X)$ can be assembled from $\mathrm{Wh}(\{x\})$ for every point $x \in X$ and $\mathrm{Wh}(\gamma)$ for every closed geodesic $\gamma \subset X$. Putting all this together, if X is a closed Riemannian manifold with nonpositive sectional curvature, then the space of h-cobordisms on $X \times I^{\times N}$ for N sufficiently large is the loop space of a summand of a space constructed from $Q(X_+)$, $K(\mathbf{S})$, and $K(\mathbf{S}[\mathbf{Z}])$.

Thus let me here address an approach to computing $K(\mathbf{S})$, following insights of Rognes. The ring of integers $\mathbf{Z}$ is the initial object in the category of commutative unital rings, and, similarly, the sphere spectrum $\mathbf{S}$ is the initial object in the (homotopy) category of unital E_∞ ring spectra. Hence their K-theories are closely linked. In fact, the map $\mathbf{S} \to H\mathbf{Z}$ induces an equivalence $K(\mathbf{S}, H\mathbf{Q}) \to K(\mathbf{Z}, H\mathbf{Q})$, and moreover, after p-completion there is the Dundas homotopy pullback square [16]:

$$\begin{array}{ccc} K(\mathbf{S})_p & \longrightarrow & K(\mathbf{Z})_p \\ \downarrow & & \downarrow \\ TC(\mathbf{S})_p & \longrightarrow & TC(\mathbf{Z})_p \end{array}$$

where the vertical maps are the p-complete cyclotomic trace maps.

Unfortunately, the description of $K(\mathbf{S})$ provided by this relationship is computationally very intricate, and so Rognes proposes to organize the homotopy into periodic families by describing $K(\mathbf{S}_{(p)})$ via the chromatic tower, *viz.*:

$$\ldots \longrightarrow K(L_{E(n)}\mathbf{S}_{(p)}) \longrightarrow K(L_{E(n-1)}\mathbf{S}_{(p)}) \longrightarrow \ldots \longrightarrow K(L_{E(1)}\mathbf{S}_{(p)}) \longrightarrow K(L_{E(0)}\mathbf{S}_{(p)}) \simeq K(\mathbf{Q}).$$

One is thus led in particular to attempt to compute the algebraic K-theory $K(L_{K(n)}\mathbf{S}_{(p)})$ of the monochromatic layers. The hope is that, with suitable coefficients and in high enough degrees, this might be achieved by means of a Beilinson–Lichtenbaum spectral sequence, as follows.

Write E_n for the Morava E-theory spectrum given by the Landweber exact cohomology theory whose associated formal group is the Lubin–Tate universal deformation of the Honda formal group H_n of height n over $\mathbf{F}_{p^n}$, so that

$$E_{n,\star} \cong \mathbf{W}_{\mathbf{F}_{p^n}}[[u_1, \ldots, u_{n-1}]][u^{\pm}].$$

There is a canonical E_∞ ring structure on E, [34], [22], and there is a natural $\mathbf{G}_n$-Galois extension $L_{K(n)}\mathbf{S}_{(p)} \to E_n$ of $K(n)$-local E_∞ ring spectra (where $\mathbf{G}_n := \mathrm{Gal}(\mathbf{F}_{p^n}|\mathbf{F}_p) \ltimes \mathrm{Aut}\, H_n$ is the nth extended Morava stabilizer group), [15, 38]. Rognes proposed the following conjecture of Beilinson–Lichtenbaum type:

Conjecture 1.7. *If M is any $\mathbf{S}_{(p)}$-module of chromatic type $(n+1)$, then for sufficiently large N, the N-connected cover of the natural morphism*

$$K(L_{K(n)}\mathbf{S}_{(p)}, M) \to K(E_n, M)^{h\mathbf{G}_n}$$

is an equivalence. This gives rise to a Beilinson-Lichtenbaum spectral sequence

$$E^2_{s,t} := H^{-s}(\mathbf{G}_n; K_t(E_n, M)) \Longrightarrow K_{s+t}(L_{K(n)}\mathbf{S}_{(p)}, M),$$

which converges for $s + t > N$.

Admitting such a result, one is now reduced to computing the algebraic K-theory $K(E_n, M)$ as a $\mathbf{G}_n$-module. (More precisely, one should consider these objects as pro-spectra with compatible group actions of finite quotients of $\mathbf{G}_n$.)

Example 1.8. When $n = 0$, the algebraic K-theory (with suitable coefficients) of $e_0 = H\mathbf{Z}_p$ was computed by Bökstedt–Madsen and Rognes [12, 37]. When $n = 1$, one gets $E_1 = KU_p$, and Rognes asked whether there is a localization sequence:

$$K\mathbf{Z}_p \longrightarrow K(ku_p) \longrightarrow K(KU_p).$$

Such a localization sequence was established by Blumberg–Mandell [9]. Thus for the case $n = 1$, it remains only to compute $K(ku_p)$; Ausoni–Rognes [1, 39] have computed $K(ku_p)$ modulo p and v_1.

Conjecture 1.9. *Rognes proposed that by inverting p in KU_p in a suitable fashion, one might make sense of the K-theory of the fraction field $p^{-1}KU_p$ of KU_p. He further suggested that a three-by-three square of fiber sequences should exist:*

$$\begin{array}{ccccc} K\mathbf{F}_p & \longrightarrow & K\mathbf{Z}_p & \longrightarrow & K\mathbf{Q}_p \\ \downarrow & & \downarrow & & \downarrow \\ K(ku/p) & \longrightarrow & K(ku_p) & \longrightarrow & K(p^{-1}ku_p) \\ \downarrow & & \downarrow & & \downarrow \\ K(KU/p) & \longrightarrow & K(KU_p) & \longrightarrow & K(p^{-1}KU_p) \end{array}$$

Conjecture 1.10 (*cf.* 2.55)**.** *One might go further and propose a localization sequence:*

(1.10.1)
$$K(e_{n-1}) \longrightarrow K(e_n) \longrightarrow K(E_n)$$

and a three-by-three square of fiber sequences:

(1.10.2)
$$\begin{array}{ccccc} K(e_{n-1}/p) & \longrightarrow & K(e_{n-1}) & \longrightarrow & K(p^{-1}e_{n-1}) \\ \downarrow & & \downarrow & & \downarrow \\ K(e_n/p) & \longrightarrow & K(e_n) & \longrightarrow & K(p^{-1}e_n) \\ \downarrow & & \downarrow & & \downarrow \\ K(E_n/p) & \longrightarrow & K(E_n) & \longrightarrow & K(p^{-1}E_n) \end{array}$$

for any $n > 0$. (In fact, this is a face in what one may conjecture is an $(n+1)$-cube of fiber sequences.)

What contribution can derived algebraic geometry make to this story? In fact, it provides geometric interpretations for many of the spectra under consideration, and these geometric insights in turn provide general arguments for the behavior of algebraic K-theory. Let me quickly address some of the main ideas with a view toward computing $K(\mathbf{S})$.

The geometry of $\operatorname{Spec} E$ is relatively simple when E is a connective E_∞-ring spectrum, for then the geometry is entirely captured in $\operatorname{Spec} \pi_0 E$ equipped with the natural sheaf of E_∞ ring spectra. When E is not connective, there is still a spectrally ringed ∞-topos $\operatorname{Spec} E$, and one way to lend geometric meaning to the negative homotopy groups is to view them as indicating the presence of generic stabilizers, in analogy with the presence of positive cohomology in the cotangent complex of Artin stacks in classical algebraic geometry.

If the spectrum E is even periodic, so that there exists a Bott element $u \in \pi_2 E$ with the property that $\pi_\star E \cong \pi_0 E[u^{\pm}]$, then Rognes has proposed another, more transparent, way to conceptualize the geometry $\operatorname{Spec} E$: one may think instead of the spectrum $\operatorname{Spec} e$ of the connective cover e of E, equipped with a *logarithmic structure* generated (in a suitable sense) by the Bott element u. From this point of view, the geometry of even periodic spectra would be the logarithmic geometry of their connective covers. In particular, the geometry of the E_∞ ring spectrum E_n would be encoded in the derived logarithmic formal scheme $(\operatorname{Spf} e_n, \langle u \rangle)$. One can now combine these two geometric insights to describe $\operatorname{Spec} L_{K(n)}\mathbf{S}_{(p)}$ as the logarithmic derived ind-stack $[(\operatorname{Spf} e_n, \langle u \rangle)/\mathbf{G}_n]$.

Conjecture 1.11. *The prediction of the derived algebro-geometric perspective is that the K-theory spectrum $K(L_{K(n)}\mathbf{S}_{(p)})$ should be canonically equivalent to the equivariant K-theory spectrum $K(\mathbf{G}_n; (\operatorname{Spf} e_n, \langle u \rangle))$ of the derived logarithmic formal scheme $(\operatorname{Spf} e_n, \langle u \rangle)$ (where, again, this should be interpreted as a suitable equivariant pro-spectrum).*

Conjecture 1.12. *Moreover, the localization sequence* (1.10.1) *should follow from a general localization sequence: for any normal crossings divisor D in a derived scheme X, one should expect a localization sequence*

$$K(D) \longrightarrow K(X) \longrightarrow K(X, \langle D \rangle),$$

where $\langle D \rangle$ is the logarithmic structure generated by D in X.

Conjecture 1.13. *Still more should be true: one should have a similar localization sequence at the level of TC (with suitable coefficients), and such a localization sequence should be closely related to a localization sequence in logarithmic crystalline cohomology of derived schemes.*

2 The Story So Far

The foundations of derived algebraic geometry have, for the most part, been laid, and a great deal of progress toward the conjectures of the previous section has been made. In this section, I will review some (though certainly not all!) of my contributions.

Model categories, higher categories, and strictification

One of the primary technical difficulties in derived algebraic geometry is the need to work with "sheaves up to coherent homotopy." This leads rapidly to the development of higher categorical techniques.

Example 2.1. The issue arises in a nascent form when one considers the problem of gluing complexes up to quasi-isomorphism, mentioned in SGA 6, Exposé 0: "Il n'est pas possible de préceder par simple "recollement" à partir des constructions affines, car les objets de la catégorie derivée […] sont de nature essentiellement non recollables."

In other words, the assignment $\operatorname{Spec}(R) \longmapsto \mathbf{D}(R)$ is not a stack on the Zariski site of affine schemes. Indeed, suppose it were; then given an affine scheme X, an open cover $\{U_0, U_1, U_2, U_3\}$ thereof, and *descent data*:

— for each $0 \le i \le 3$, complexes of $\mathscr{O}_{U_i}$-modules $C_i^\bullet$ with quasicoherent cohomology,

— for each $0 \le i \le j \le 3$, a quasi-isomorphism $f_{ij} : C_i^\bullet|_{U_j} \xrightarrow{\sim} C_j^\bullet|_{U_i}$, and

— for each $0 \le i \le j \le k \le 3$, a chain homotopy $f_{jk} \circ f_{ij} \xrightarrow{\sim} f_{ik}$,

there exists a complex of $\mathscr{O}_X$-modules $C^\bullet$ and a quasi–isomorphism between $C^\bullet|_{U_i}$ and $C_i^\bullet$. If $C^\bullet$ exists, then the homotopies can be chosen so that the following square of chain homotopies commutes:

$$\begin{array}{ccccc} & & f_{23} \circ f_{12} \circ f_{01} & & \\ & \swarrow & & \searrow & \\ f_{13} \circ f_{01} & & & & f_{23} \circ f_{02} \\ & \searrow & & \swarrow & \\ & & f_{03} & & \end{array}$$

But in general, no such compatible choice of homotopies can be made from the data given.

However, if one includes the commutativity of this diagram of homotopies in the descent data, then such a global complex exists. For covers $\{U_i\}$ of higher cardinality, the descent data must include complicated coherence conditions to ensure the existence of a global complex. These conditions are encoded by the presence of certain simplices in the fibrant replacement of the nerve $\nu_\bullet w\mathbf{Cplx}(\mathscr{O}_{U_{i_0 \dots i_n}})$ of the category of quasi-isomorphisms of complexes with quasi-coherent cohomology on the multiple overlap $U_{i_0 \dots i_n}$. One can organize all this information into a *homotopy-coherent sheaf condition*, that the following natural morphism be a weak equivalence of simplicial sets:

$$\nu_\bullet w\mathbf{Cplx}(\mathscr{O}_X) \to \operatorname*{holim}_{\Delta} \left[\prod_i \nu_\bullet w\mathbf{Cplx}(\mathscr{O}_{U_i}) \rightrightarrows \prod_{i,j} \nu_\bullet w\mathbf{Cplx}(\mathscr{O}_{U_{i,j}}) \substack{\rightarrow\\\rightarrow\\\rightarrow} \prod_{i,j,k} \nu_\bullet w\mathbf{Cplx}(\mathscr{O}_{U_{i,j,k}}) \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots \right].$$

Definition 2.2. For any site (C, τ), the category of simplicial presheaves $C^{\mathrm{op}} \to s\mathbf{Set}$ has a *τ-local projective model structure*, in which the fibrant objects are those F such that each $F(X)$ is fibrant, and such that the homotopy-coherent sheaf condition is satisfied, so that for any cover $\{U_i \to X\}$, the natural morphism

$$F(X) \to \operatorname*{holim}_{\Delta} \left[\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_{i,j}) \substack{\rightarrow\\\rightarrow\\\rightarrow} \prod_{i,j,k} F(U_{i,j,k}) \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots \right]$$

is a weak equivalence. (To be precise, this is the condition that is satisfied when C has all fiber products, and the topology τ is generated by a pretopology.) Denote this model category $s\mathbf{Set}(C, \tau)$.

Example 2.3. Thus descent for complexes amounts to the assertion that the simplicial presheaf $\nu_\bullet w\mathbf{Cplx}$ is objectwise weakly equivalent to a fibrant object of $s\mathbf{Set}(X_{\mathrm{Zar}})$.

In derived algebraic geometry, it is necessary to generalize this picture in two directions. First, one must work with sites that already possess homotopical structure, such as an étale site of commutative simplicial rings or of E_∞ ring spectra. Second, one must also understand descent not just for quasi-isomorphisms of complexes, but also for ordinary morphisms of complexes. Both of these matters are nicely addressed by a good theory of higher categories.

I will now briefly outline a theory of weakly enriched categories that I developed as a generalization of work of C. Rezk. This particular model has very special properties, which do not appear in other (often equivalent) models.

Definition 2.4. An *enrichment model category* $\mathscr{E}$ is an internal, simplicial, left proper, tractable model category in which the terminal object $\star$ is cofibrant. Denote by **Enr** the $(2,1)$-category of *enrichment model categories* and *product-preserving, left Quillen functors.*

I have produced an enrichment model category of weakly $\mathscr{E}$-enriched categories for any enrichment model category $\mathscr{E}$ — hence an endofunctor on **Enr**.

One might try to approach this problem by constructing a model structure directly on the category of $\mathscr{E}$-enriched categories, where the weak equivalences are those $\mathscr{E}$-functors $A \to B$ that induce equivalences of the corresponding $\mathrm{Ho}\mathscr{E}$-enriched categories. Unfortunately, this cannot provide such an endofunctor, even when $\mathscr{E} = s\mathbf{Set}$.

Instead, I use a multi-object version of the theory of the Segal delooping machine [40], where one studies simplicial objects F of $\mathscr{E}$ satisfying a Segal condition. This works well to produce a theory of weak *internal categories* in $\mathscr{E}$, but to produce a theory of *weakly enriched* categories, some condition must be placed on the object $F(0)$. One option would be to force $F(0)$ to be discrete; this leads to the theory of Segal categories developed over time by Cooke, Dwyer, Kan, Smith, Corier, Porter, Tamsamani, Simpson, and Hirschowitz [13, 17, 14, 46, 43, 41, 45, 27, 44, 42]. Unfortunately, this leads to technical complications in understanding the homotopy theory of these gadgets that make them less well-adapted for the purposes of K-theory, for example. An alternative, whose idea is due to Rezk [33], is to place a condition on $F(0)$ that will effectively force it to be a kind of "interior" for F. This is the *completeness condition.*

Example 2.5. Before I proceed to the technical result, let me briefly describe what this completeness condition amounts to in the example of $\mathbf{Cplx}(\mathcal{O}_U)$. I will be able to turn the model category $\mathbf{Cplx}(\mathcal{O}_U)$ into a weakly $s\mathbf{Set}$-enriched category for each U in a functorial way, and in a manner that satisfies a descent condition, so one obtains a weakly $s\mathbf{Set}(X_{\text{Zar}})$-enriched category. (To be precise, one must use a Quillen equivalent τ-local injective model structure on the category of simplicial presheaves on C in order to get a true enrichment category.) This is a simplicial object F of $s\mathbf{Set}(X_{\text{Zar}})$ whose zeroth object $F(0)$ exactly the assignment $U \mapsto \nu_\bullet w\mathbf{Cplx}(\mathcal{O}_U)$ described above.

Theorem 2.6. *There is an endo-$(2,1)$-functor — the* Rezk categorification $\mathbf{Wk}(-)\mathbf{Cat} : \mathbf{Enr} \to \mathbf{Enr}$ — *equipped with a morphism of endo-$(2,1)$-functors* $\mathrm{id} \to \mathbf{Wk}(-)\mathbf{Cat}$ *satisfying the following conditions for any enrichment model category* $\mathcal{E}$.

(2.6.1) The underlying category *of* $\mathbf{Wk}(\mathcal{E})\mathbf{Cat}$ *is the category* $s\mathcal{E}$ *of* simplicial objects *of* $\mathcal{E}$.

(2.6.2) The left Quillen functor $\mathcal{E} \to \mathbf{Wk}(\mathcal{E})\mathbf{Cat}$ *is the* diagonal *functor.*

(2.6.3) The cofibrations *of* $\mathbf{Wk}(\mathcal{E})\mathbf{Cat}$ *are the* Reedy *cofibrations.*

(2.6.4) An object $A \in \mathbf{Wk}(\mathcal{E})\mathbf{Cat}$ *is fibrant — a* weak $\mathcal{E}$-category — *if and only if it satisfies the following conditions.*

- *(2.6.4.1)* $A \in s\mathcal{E}$ *is* Reedy fibrant.
- *(2.6.4.2) The* Segal morphism $A_p \to A_1 \times^h_{A_0} \cdots \times^h_{A_0} A_1$ *is an isomorphism of* $\mathrm{Ho}\,\mathcal{E}$.
- *(2.6.4.3) The* Rezk morphism $A_0 \to \operatorname{holim}_{p \in (\Delta/\overline{1})^{\mathrm{op}}} A_p$ *is an isomorphism of* $\mathrm{Ho}\,\mathcal{E}$*, where* $\overline{1}$ *is the unique contractible groupoid with two objects, and* $(\Delta/\overline{1})$ *is the category of functors* $\mathbf{p} \to \overline{1}$.

(2.6.5) Weak equivalences between fibrant objects are objectwise.

This model structure on $s\mathcal{E}$ is constructed in a fashion similar to [33]. In effect, one forms *enriched* left Bousfield localizations of the Reedy model structure with respect to the Segal morphisms and the Rezk morphism.

Example 2.7. Define $\mathbf{Wk}(\infty,0)\mathbf{Cat} := s\mathbf{Set}$, and, for any $0 \le n \le \infty$ and any $0 \le m \le \infty$,

$$\begin{array}{rcl} \mathbf{Wk}(n,0)\mathbf{Cat} & := & L_{\{S^k \to \star \mid k>n\}}\mathbf{Wk}(\infty,0)\mathbf{Cat} \\ \mathbf{Wk}(n,m)\mathbf{Cat} & := & \mathbf{Wk}(\mathbf{Wk}(n,m-1)\mathbf{Cat})\mathbf{Cat}. \end{array}$$

These are *fantastic* models for *weak (n,m)-categories* — i.e., weak n-categories such that the i-morphisms for $i > m$ are weakly invertible.

We have a diagram of right Quillen functors:

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \mathbf{Wk}(\infty,2)\mathbf{Cat} & \longrightarrow & \mathbf{Wk}(\infty,1)\mathbf{Cat} & \longrightarrow & \mathbf{Wk}(\infty,0)\mathbf{Cat} \\
& \uparrow & & \uparrow & & \uparrow \\
& \vdots & & \vdots & & \vdots \\
& \uparrow & & \uparrow & & \uparrow \\
\cdots \xrightarrow[\sim]{} & \mathbf{Wk}(2,2)\mathbf{Cat} & \longrightarrow & \mathbf{Wk}(2,1)\mathbf{Cat} & \longrightarrow & \mathbf{Wk}(2,0)\mathbf{Cat} \\
& \uparrow & & \uparrow & & \uparrow \\
\cdots \xrightarrow[\sim]{} & \mathbf{Wk}(1,2)\mathbf{Cat} & \xrightarrow[\sim]{} & \mathbf{Wk}(1,1)\mathbf{Cat} & \longrightarrow & \mathbf{Wk}(1,0)\mathbf{Cat} \\
& \uparrow & & \uparrow & & \uparrow \\
\cdots \xrightarrow[\sim]{} & \mathbf{Wk}(0,2)\mathbf{Cat} & \xrightarrow[\sim]{} & \mathbf{Wk}(0,1)\mathbf{Cat} & \xrightarrow[\sim]{} & \mathbf{Wk}(0,0)\mathbf{Cat}
\end{array}$$

The right Quillen functors should be viewed as giving an "interior," i.e., the maximum subobject with the prescribed structure. In the case of the upward pointing maps, this reduces to a mere forgetful functor. As far as I know, no other theory of higher categories comes with a diagram of enrichment model categories like the one above.

Example 2.8. If $\mathcal{C}$ is a category with weak equivalences, we have the *Rezk nerve*:

$$\begin{array}{rrcl} N\mathcal{C} : & \Delta^{\mathrm{op}} & \longrightarrow & \mathbf{Wk}(\infty,0)\mathbf{Cat} \\ & p & \longrightarrow & \nu_\bullet w(\mathcal{C}^p) \end{array}$$

which is "close" to being fibrant in $\mathbf{Wk}(\infty,1)\mathbf{Cat}$. If $\mathscr{C}$ is a model category, for example, then an objectwise fibrant replacement of $N\mathscr{C}$ is fibrant.

If $\mathscr{C}$ is a model $\mathscr{E}$-category, then there is a weak $\mathscr{E}$-category $N_{\mathscr{E}}\mathscr{C}$ such that $N\mathscr{C}$ is the image under the right adjoint

$$\mathrm{Ho}(\mathbf{Wk}(\mathscr{E})\mathbf{Cat}) \to \mathrm{Ho}(\mathbf{Wk}(\infty,1)\mathbf{Cat}).$$

Moreover this association $\mathscr{C} \mapsto N_{\mathscr{E}}\mathscr{C}$ is functorial.

If (C,τ) is a site, one can define $\mathbf{Wk}(\infty,0)\mathbf{Cat}(C,\tau) := s\mathbf{Set}(C,\tau)$, and, for any $0 \le n \le \infty$ and any $0 \le m \le \infty$,

$$\begin{aligned}\mathbf{Wk}(n,0)\mathbf{Cat}(C,\tau) &:= L_{(\{S^k \to \star \ \mid\ k>n\}/\mathbf{Wk}(n,0)\mathbf{Cat}(C,\tau))}\mathbf{Wk}(\infty,0)\mathbf{Cat}(C,\tau)\\ \mathbf{Wk}(n,m)\mathbf{Cat}(C,\tau) &:= \mathbf{Wk}(\mathbf{Wk}(n,m-1)\mathbf{Cat}(C,\tau))\mathbf{Cat}.\end{aligned}$$

(Again, one should use the τ-local injective model structure on the category of simplicial presheaves on C.) These are excellent models for *weak* (n,m)*-stacks* on (C,τ) — i.e., homotopy-coherent sheaves of (n,m)-categories. Alternate models for weak (n,m)-stacks can be constructed directly, using techniques of enriched left Bousfield localizations, as τ-local projective model structures on the categories of presheaves $C^{\mathrm{op}} \to \mathbf{Wk}(n,m)\mathbf{Cat}$. Of course the resulting model structures are Quillen equivalent via the identity functor.

For the purposes of derived algebraic geometry, one wishes to consider not merely ordinary sites, but ∞*-sites*, i.e., $(\infty,1)$-categories C equipped with a topology τ on $\pi_0 C$. Associated with such a pair (C,τ) is its ∞*-topos* $\widetilde{C}^{\tau}$, which is the full subcategory of $\mathbf{RMor}_{\mathbf{Wk}(\infty,1)\mathbf{Cat}}(C^{\mathrm{op}}, N\mathbf{Wk}(\infty,0)\mathbf{Cat})$ comprised of those morphisms $C^{\mathrm{op}} \to \mathbf{Wk}(\infty,0)\mathbf{Cat}$ satisfying a homotopy-coherent sheaf condition similar to the one above. Lurie has undertaken a very thorough study of ∞-topoi in a different (but equivalent) $(\infty,1)$-categorical context.

In practice, virtually all of the $(\infty,1)$-categories with which one works arise via the application of homotopy limits, homotopy colimits, and derived internal Mors to Rezk nerves of full subcategories of various model categories. For the purpose of computation, it is frequently useful to know when the resulting $(\infty,1)$-category is itself the Rezk nerve of a full subcategory of a model category. This is provided by the Strictification Theorem, perhaps the most important result of $(\infty,1)$-category theory, originally proposed by C. Simpson, and proved in varying degrees of generality by Hirschowitz–Simpson [27], Toën–Vezzosi [49], Spitzweck, Lurie [32], and me.

Theorem 2.9 (Categorical Strictification)**.** *Suppose $\mathscr{M}$ a tractable left (respectively, right) Quillen presheaf on a category D; that is, $\mathscr{M}$ is a functor $D^{\mathrm{op}} \to$* **Cat** *along with a tractable model structure on each $\mathscr{M}_d$ such that for any morphism $f : d \to c$ of D, the induced morphism $f^{\star} : \mathscr{M}_c \to \mathscr{M}_d$ is a left (resp., right) Quillen functor.*

(2.9.1) There exists a tractable injective (resp., projective) model structure on the on the category $\mathbf{Sect}^L(\mathscr{M})$ of left sections of $\mathscr{M}$ (resp., on the on the category $\mathbf{Sect}^R(\mathscr{M})$ of right sections of $\mathscr{M}$) in which the weak equivalences and cofibrations (resp, the weak equivalences and fibrations) are defined objectwise.

(2.9.2) There is an equivalence of $\mathbf{Wk}(\infty,1)\mathbf{Cat}$:

$$N\mathbf{Sect}^L_{\mathrm{inj}}(\mathscr{M}) \xrightarrow{\sim} \mathrm{holim}^{\mathrm{lax}}_{d\in D^{\mathrm{op}}}\mathscr{M}_d \quad \textit{(resp.,}\quad N\mathbf{Sect}^R_{\mathrm{proj}}(\mathscr{M}) \xrightarrow{\sim} \mathrm{holim}^{\mathrm{colax}}_{d\in D^{\mathrm{op}}} N\mathscr{M}_d\,\textit{).}$$

(2.9.3) There exists a tractable right Bousfield localization $\mathbf{Sect}^L_{\mathrm{holim}}(\mathscr{M})$ of $\mathbf{Sect}^L_{\mathrm{inj}}(\mathscr{M})$ (resp., a left Bousfield localization $\mathbf{Sect}^R_{\mathrm{holim}}(\mathscr{M})$ of $\mathbf{Sect}^R_{\mathrm{proj}}(\mathscr{M})$) in which the cofibrant (resp., fibrant) objects are the cofibrant (resp., fibrant), homotopy cartesian objects of $\mathbf{Sect}(\mathscr{M})$.

(2.9.4) The equivalences of (2.9.2) restrict to equivalences:

$$\begin{array}{ccc} N\mathbf{Sect}^L_{\mathrm{holim}}(\mathscr{M}) & \xrightarrow{\sim} & \mathrm{holim}_{d\in D^{\mathrm{op}}} N\mathscr{M}_d \\ \downarrow & & \downarrow \\ N\mathbf{Sect}^L_{\mathrm{inj}}(\mathscr{M}) & \xrightarrow[\sim]{} & \mathrm{holim}^{\mathrm{lax}}_{d\in D^{\mathrm{op}}} N\mathscr{M}_d \end{array} \quad \textit{(resp.,} \quad \begin{array}{ccc} N\mathbf{Sect}^R_{\mathrm{holim}}(\mathscr{M}) & \xrightarrow{\sim} & \mathrm{holim}_{d\in D^{\mathrm{op}}} N\mathscr{M}_d \\ \downarrow & & \downarrow \\ N\mathbf{Sect}^R_{\mathrm{inj}}(\mathscr{M}) & \xrightarrow[\sim]{} & \mathrm{holim}^{\mathrm{colax}}_{d\in D^{\mathrm{op}}} N\mathscr{M}_d \end{array}\ \textit{).}$$

Corollary 2.10. *If $\mathscr{M}$ is a tractable model category, and D a category, then there is an equivalence of $\mathbf{Wk}(\infty,1)\mathbf{Cat}$:*

$$\mathbf{RMor}(D, N\mathscr{M}) \simeq N(\mathscr{M}^D)_{\mathrm{proj}}.$$

Using entirely abstract techniques, one can show that the lax homotopy limit and the homotopy limit of the theorem are Rezk nerves of *some* tractable model categories. But the advantage of this result is that it gives one an explicit — and very pleasant — model for the resulting $(\infty,1)$-category. The price one has to pay for this seems to be that in order to model homotopy limits of left Quillen presheaves one has to use right Bousfield localizations of model categories that are not right proper; these are no longer model categories, but *right semimodel categories.*

Operator categories, homotopy coherent algebra, and algebraic strictification

Operator categories are gadgets I invented to encode homotopy-coherent algebraic structures. In particular, operator categories provide a new, model-independent way of encoding A_n and E_n structures for $1 \leq n \leq \infty$. This is a very long story, and I will emphasize only the highlights.

Morally, operator categories are categories of indexing sets for sorts of multiplication laws, which I call Φ-monoids; hence operator categories provide a *rubric* to which a set of objects must conform in order for them to be multiplied.

Definition 2.11. An *operator category* is an essentially small category Φ satisfying the following conditions.

(2.11.1) Φ is locally finite; i.e., for any objects I and J of Φ, $\mathrm{Mor}_\Phi(J, I)$ is a finite set.

(2.11.2) There exists a terminal object $\star \in \Phi$.

(2.11.3) For any morphism $f : J \to I$ of Φ and any point $i : \star \to I$, the fiber of f over i exists.

Example 2.12. The following are all operator categories:

(2.12.1) the full subcategory $\mathbf{O}_{\leq n} \subset \mathbf{Cat}$ comprised of the linear categories $\mathbf{p} := [0 \longrightarrow 1 \longrightarrow \ldots \longrightarrow p]$ for $-1 \leq p \leq n$ — or, equivalently, totally ordered finite sets of cardinality $\leq n+1$,

(2.12.2) the full subcategory $\mathbf{O} \subset \mathbf{Cat}$ comprised of the categories $\mathbf{p}$ for $p \geq -1$ — or, equivalently, totally ordered finite sets,

(2.12.3) the full subcategory $\mathbf{F}_{\leq n} \subset \mathbf{Set}$ comprised of the sets $|\mathbf{p}| := \mathrm{Obj}\,\mathbf{p}$ for $-1 \leq p \leq n$ — or, equivalently, finite sets of cardinality $\leq n+1$, and

(2.12.4) the full subcategory $\mathbf{F} \subset \mathbf{Set}$ comprised of the sets $|\mathbf{p}|$ for $p \geq -1$ — or, equivalently, finite sets.

For any symmetric monoidal category $\mathscr{E}$ and any operator category Φ, there is a category of Φ-monoids in $\mathscr{E}$. Here's a table of these operator categories and their associated rubrics in the category of sets.

Φ	Φ-monoids in **Set**
$\mathbf{0}$	sets
$\mathbf{O}_{\leq 1}$	unital magmas
$\mathbf{O}_{\leq n}$ $(n > 1)$	monoids
$\mathbf{O}$	monoids
$\mathbf{F}_{\leq 1}$	commutative, unital magmas
$\mathbf{F}_{\leq n}$ $(n > 1)$	commutative monoids
$\mathbf{F}$	commutative monoids

There is a useful notion of *operator morphism*. In effect, this is a functor $f : \Psi \to \Phi$ of operator categories that respects terminal objects and fibers such that for any object I of Ψ, the induced morphism $|I| \to |fI|$ is a *surjection*. This yields a $(2,1)$-category **Op** of operator categories. The key point here is that since **Op** is a $(2,1)$-category, it is sensible (and correct) to assert that many key constructions are fully functorial in **Op**.

The operator category $\mathbf{0}$ is homotopy initial, and $\mathbf{F}$ is homotopy terminal in **Op**; hence for any Φ, a Φ-monoid in a symmetric monoidal category $\mathscr{E}$ is in particular an object of $\mathscr{E}$, and a commutative monoid in $\mathscr{E}$ is a Φ-monoid.

There are also notions of Φ*-multicategory*, *colored* Φ*-operad* and Φ*-operad*, all of which are fully functorial in Φ.

Theorem 2.13. *Suppose $\mathscr{M}$ a tractable, symmetric monoidal model category satisfying the monoid axiom. Then the category* $\mathbf{Operad}^\Phi(\mathscr{M})$ *is a tractable model category with a projective model structure.*

This model category is $(2,1)$-functorial in both the operator category Φ and the symmetric monoidal model category $\mathscr{M}$, and it respects Quillen equivalences of the latter.

Theorem 2.14. *For any set S, the category* $\mathbf{Col}(S)\mathbf{Operad}^\Phi(\mathscr{M})$ *of S-colored Φ-operads in $\mathscr{M}$ — i.e., Φ-multicategories enriched in $\mathscr{M}$ with object set S with morphisms* preserving the colors — *is a tractable model category with a projective model structure.*

Note that these results do *not* provide a model category of Φ-multi-$\mathscr{M}$-categories. The weak equivalences of such a model structure should be essentially surjective multifunctors that induce weak equivalences on all polymorphism objects. For a long time such a structure seemed elusive, but thanks to terribly useful conversations with Lurie and Bergner, I believe I have at last worked out how to produce such a model structure, at least in some key cases.

Example 2.15. For any operator category Φ, the unit I of any symmetric monoidal category $\mathscr{M}$ inherits a canonical Φ-operad structure. If $\mathbf{M}$ is a tractable symmetric monoidal model category satisfying the monoid axiom, denote by $\mathscr{Q}^{\Phi}$ or simply $\mathscr{Q}$ a cofibrant replacement of I in $\mathbf{Operad}^{\Phi}(\mathscr{E})$.

Theorem 2.16. *Suppose $(\Phi, \mathscr{E}, S, \mathscr{P}, \mathscr{M})$ a tuple in which Φ is an operator category, $\mathscr{E}$ is a tractable symmetric monoidal model category satisfying the monoid axiom, S is a set, $\mathscr{P}$ is a cofibrant S-colored Φ-operad in $\mathscr{E}$, and $\mathscr{M}$ is a tractable Φ-monoidal model $\mathscr{E}$-category. Then the category*

$$\mathbf{Alg}^{\Phi}_{\mathscr{E},\mathscr{P}}(\mathscr{M}) := \mathbf{Mor}_{\mu^{\Phi}(\mathscr{E})\mathbf{Cat}}(\mathscr{P}, \mathscr{M})$$

of Φ-algebras over $\mathscr{P}$ in $\mathscr{M}$ (i.e., Φ-multifunctors $\mathscr{P} \to \mathscr{M}$) has a tractable projective model structure.

Once again, this is fully functorial in every possible way. This result is general enough to produce model categories of algebras over ordinary operads as well as modules over these algebras with any chirality.

Example 2.17. Suppose $\mathscr{E}$ a tractable symmetric monoidal model category satisfying the monoid axiom. Then the model category of *(weak) Φ-monoids in $\mathscr{E}$* is the model category $\mathbf{Mon}^{\Phi}(\mathscr{E}) := \mathbf{Alg}^{\Phi}_{\mathscr{E},\mathscr{Q}}(\mathscr{E})$.

Let us see how this definition plays out in the category of spaces.

Φ	$\mathbf{Mon}^{\Phi}(s\mathbf{Set})$
$\mathbf{0}$	$s\mathbf{Set}$
$\mathbf{O}_{\leq n}$ $(n > 0)$	$\mathbf{Alg}(A_n)$
$\mathbf{O}$	$\mathbf{Alg}(A_{\infty})$
$\mathbf{F}_{\leq n}$ $(n > 0)$	$\mathbf{Alg}(F_n)$
$\mathbf{F}$	$\mathbf{Alg}(E_{\infty})$

Here of course $\mathbf{Alg}(A_n)$ is the model category of A_n-algebras, and $\mathbf{Alg}(E_{\infty})$ is the model category of E_{∞}-algebras in spaces. The category $\mathbf{Alg}(F_n)$ of F_n-algebras is closely related to a filtration of the little cubes operad introduced by Robinson [36] and used by Richter [35].

Another excellent source of examples of operator categories is a monoidal structure on $\mathbf{Op}$, which I have dubbed the *wreath product*. Loosely speaking, a $(\Phi \wr \Psi)$-monoid is a Ψ-monoid in Φ-monoids. One way to look at this monoidal structure is as a recontextualization of the Boardman–Vogt tensor product [10, 11].

Definition 2.18. Suppose Φ and Ψ two operator categories. Define the *wreath product operator category* as the category of pairs $((K_I), I)$ consisting of an object $I \in \Psi$ and an I-tuple of objects $(K_I) \in \Phi^{\times |I|}$. A morphism $((L_J), J) \to ((K_I), I)$ is a morphism $\psi : J \to I$ of Ψ, and a J-tuple of morphisms $(L_j \to K_{\psi(j)})_{j \in |J|}$ of Φ.

The wreath product makes $\mathbf{Op}$ into a monoidal $(2,1)$-category, wherein $\mathbf{0}$ is the unit. There is no commutativity here: the wreath product is not even a braided monoidal structure.

Example 2.19. This wreath product provides a host of new operator categories. In particular, the iterated wreath product of $\mathbf{O}$ with itself gives rise to a sequence of operator categories

$$\mathbf{0} \longrightarrow \mathbf{O} \longrightarrow \mathbf{O} \wr \mathbf{O} \longrightarrow \mathbf{O} \wr \mathbf{O} \wr \mathbf{O} \longrightarrow \cdots \longrightarrow \mathrm{Wr}^{(n)}(\mathbf{O}) \longrightarrow \cdots.$$

Suppose $\mathscr{E}$ a tractable symmetric monoidal model category satisfying the monoid axiom; then there is the associated *Eckman-Hilton tower*

$$\mathbf{EH}_{\mathscr{E}} := [\ \mathscr{E} \longleftarrow \mathbf{Mon}^{\mathbf{O}}(\mathscr{E}) \longleftarrow \mathbf{Mon}^{(\mathbf{O} \wr \mathbf{O})}(\mathscr{E}) \longleftarrow \mathbf{Mon}^{(\mathbf{O} \wr \mathbf{O} \wr \mathbf{O})}(\mathscr{E}) \longleftarrow \cdots \longleftarrow \mathbf{Mon}^{\mathrm{Wr}^{(n)}(\mathbf{O})}(\mathscr{E}) \longleftarrow \cdots\ ],$$

a tower of right Quillen functors.

A related gadget — which I call the Leinster category of $\mathrm{Wr}^{(n)}(\mathbf{O})$ — seems to have been arrived at independently by Joyal and Berger [6], and it may even be that the following pair of results has been established, independently and in a completely different context, by Batanin [2, 3].

Theorem 2.20. *In the model category $s\mathbf{Set}$ of spaces, there is, for any $1 \leq n \leq \infty$, a Quillen equivalence*

$$\mathbf{Mon}^{\mathrm{Wr}^{(n)}(\mathbf{O})}(s\mathbf{Set}) \simeq \mathbf{Alg}(E_n).$$

That is, the symmetric operad in spaces freely generated by the $\mathrm{Wr}^{(n)}(\mathbf{O})$-operad $\mathscr{Q}$ is canonically equivalent to the little n-cubes operad.

Theorem 2.21. *For any tractable symmetric monoidal model category $\mathscr{E}$ satisfying the monoid axiom, the homotopy limit of the Eckmann-Hilton tower is*

$$\operatorname{holim} N\mathbf{EH}_{\mathscr{E}} = \operatorname{holim}_{n\to\infty} N\mathbf{Mon}^{\mathrm{Wr}^{(n)}(\mathbf{O})}(\mathscr{E}) \simeq N\mathbf{Mon}^{\mathbf{F}}(\mathscr{E}).$$

One can now apply this formalism to the theory of higher categories itself, to yield the following result.

Theorem 2.22. *Suppose $0 \le m \le n$ nonnegative integers, and suppose $k > n+1$. Then there is a Quillen equivalence*

$$\mathbf{Mon}^{\mathrm{Wr}^{(k)}(\mathbf{O})}(\mathbf{Wk}(n,m)\mathbf{Cat}) \simeq \mathbf{Mon}^{\mathbf{F}}(\mathbf{Wk}(n,m)\mathbf{Cat}).$$

This is the Stabilization Effect expected by Breen–Baez–Dolan. Here is a table to illustrate.

	0	1	2	3	∞
0	**Set**	**Cat**	$\mathbf{Wk}(2,m)\mathbf{Cat}$	$\mathbf{Wk}(3,m)\mathbf{Cat}$	$\mathbf{Wk}(\infty,m)\mathbf{Cat}$
1	**Mon**	**MonCat**	$\mathbf{Mon}^{\mathbf{O}}(\mathbf{Wk}(2,m)\mathbf{Cat})$	$\mathbf{Mon}^{\mathbf{O}}(\mathbf{Wk}(3,m)\mathbf{Cat})$	$\mathbf{Mon}^{\mathbf{O}}(\mathbf{Wk}(\infty,m)\mathbf{Cat})$
2	**ComMon**	**BrMonCat**	$\mathbf{Mon}^{(\mathbf{O}\wr\mathbf{O})}(\mathbf{Wk}(2,m)\mathbf{Cat})$	$\mathbf{Mon}^{(\mathbf{O}\wr\mathbf{O})}(\mathbf{Wk}(3,m)\mathbf{Cat})$	$\mathbf{Mon}^{(\mathbf{O}\wr\mathbf{O})}(\mathbf{Wk}(\infty,m)\mathbf{Cat})$
3	"	**SymMonCat**	$\mathbf{Mon}^{(\mathbf{O}\wr\mathbf{O}\wr\mathbf{O})}(\mathbf{Wk}(2,m)\mathbf{Cat})$	$\mathbf{Mon}^{(\mathbf{O}\wr\mathbf{O}\wr\mathbf{O})}(\mathbf{Wk}(3,m)\mathbf{Cat})$	$\mathbf{Mon}^{(\mathbf{O}\wr\mathbf{O}\wr\mathbf{O})}(\mathbf{Wk}(\infty,m)\mathbf{Cat})$
4	"	"	$\mathbf{Mon}^{\mathbf{F}}(\mathbf{Wk}(2,m)\mathbf{Cat})$	$\mathbf{Mon}^{\mathrm{Wr}^{(4)}(\mathbf{O})}(\mathbf{Wk}(3,m)\mathbf{Cat})$	$\mathbf{Mon}^{\mathrm{Wr}^{(4)}(\mathbf{O})}(\mathbf{Wk}(\infty,m)\mathbf{Cat})$
5	"	"	"	$\mathbf{Mon}^{\mathbf{F}}(\mathbf{Wk}(3,m)\mathbf{Cat})$	$\mathbf{Mon}^{\mathrm{Wr}^{(5)}(\mathbf{O})}(\mathbf{Wk}(\infty,m)\mathbf{Cat})$
∞	"	"	"	"	$\mathbf{Mon}^{\mathbf{F}}(\mathbf{Wk}(\infty,m)\mathbf{Cat})$

In practice, constructions at the $(\infty,1)$-category level (such as the K-theory functor discussed below) tend not to produce output with canonical actions by a cofibrant Φ-operad. Instead, the output of such a functor will have the structure of a weak algebra over a weak operad. I do not have the space to address this phenomenon precisely here, but I have developed (in part with Spitzweck) a notion of a *weak Φ-multi-$\mathscr{E}$-category* for any enrichment category $\mathscr{E}$. This also provides notions of *weak Φ-operad in $\mathscr{E}$* and *colored weak Φ-operad in $\mathscr{E}$*. I have shown that these form tractable model categories $\mathbf{Wk}\mu^{\Phi}(\mathscr{E})\mathbf{Cat}$, $\mathbf{WkOperad}^{\Phi}(\mathscr{E})$, and $\mathbf{Col}(S)\mathbf{WkOperad}^{\Phi}(\mathscr{E})$, respectively. The model category $\mathbf{Wk}\mu^{\Phi}(\mathscr{E})\mathbf{Cat}$ is naturally a $\mathbf{Wk}(\mathscr{E})\mathbf{Cat}$-category.

I consider the following theorem my deepest result in the realm of higher categories and homotopy coherent algebra. It provides an incredibly powerful sort of strictification procedure whereby one recovers strict models of weak algebras over a colored Φ-operad.

Theorem 2.23 (General Algebraic Strictification)**.** *Suppose $(\Phi,\mathscr{E},S,\mathscr{P},\mathscr{M})$ a tuple in which Φ is an operator category, $\mathscr{E}$ an enrichment model category, S is a set, $\mathscr{P}$ a cofibrant S-colored Φ-operad in $\mathscr{E}$, and $\mathscr{M}$ a tractable Φ-monoidal model $\mathscr{E}$-category satisfying the monoid axiom. Then there is a canonical equivalence of weak $\mathscr{E}$-categories:*

$$N_{\mathscr{E}}\mathbf{Alg}^{\Phi}_{\mathscr{E},\mathscr{P}}(\mathscr{M}) := N_{\mathscr{E}}\,\mathbf{Mor}_{\mu^{\Phi}(\mathscr{E})\mathbf{Cat}}(\mathscr{P},\mathscr{M}) \xrightarrow{\sim} \mathbf{RMor}^{\mathbf{Wk}(\mathscr{E})\mathbf{Cat}}_{\mathbf{Wk}\mu^{\Phi}(\mathscr{E})\mathbf{Cat}}(\mathscr{P},\mathscr{M}) =: \mathbf{WkAlg}^{\Phi}_{\mathscr{E},\mathscr{P}}(N_{\mathscr{E}}\mathscr{M}).$$

A special case of this result was (at least partially) demonstrated some time ago by Spitzweck and me, but the method of proof required for the result above is different from the technique used in our previous work.

The General Algebraic Strictification Theorem implies a host of surprising corollaries, including the Categorical Strictification Theorem above. Some of these corollaries were conjectured by Toën [48], and some of these have been shown (by different methods) by Lurie. The first of these is a surprising strictification theorem for weak Φ-operads, provided Φ satisfies a particular condition — satisfied in all useful cases — which I call *perfection.*

Corollary 2.24 (Operadic Strictification)**.** *Suppose Φ, $\mathscr{E}$, and $\mathscr{M}$ as above, and suppose Φ perfect. Then there is a canonical equivalence of weak $\mathscr{E}$-categories*

$$N_{\mathscr{E}}\mathbf{Operad}^{\Phi}(\mathscr{M}) \xrightarrow{\sim} N_{\mathscr{E}}\mathbf{WkOperad}^{\Phi}(\mathscr{M}).$$

The next corollary is a strictification result for algebras over an operad.

Corollary 2.25 (Algebraic Strictification)**.** *Suppose Φ, $\mathscr{E}$, and $\mathscr{M}$ as above, and suppose again that Φ is perfect. Suppose $\mathscr{P}$ a cofibrant weak Φ-operad in $\mathscr{E}$, and suppose $\mathscr{P}^{\mathrm{str}}$ a cofibrant strictification of $\mathscr{P}$. Then there is a canonical equivalence of weak $\mathscr{E}$-categories*

$$N_{\mathscr{E}}\mathbf{Alg}^{\Phi}_{\mathscr{E},\mathscr{P}^{\mathrm{str}}}(\mathscr{M}) \xrightarrow{\sim} \mathbf{WkAlg}^{\Phi}_{\mathscr{E},\mathscr{P}}(N_{\mathscr{E}}\mathscr{M}).$$

In particular, one has the following result.

Corollary 2.26. *Suppose $\mathscr{M}$ a tractable simplicial symmetric monoidal category satisfying the monoid axiom. There are canonical equivalences of $(\infty,1)$-categories*

$$N\mathbf{Alg}_{A_\infty}(\mathscr{M}) \xrightarrow{\sim} \mathbf{RMor}^{\mathbf{Wk}(\infty,1)\mathbf{Cat}}_{\mathbf{Mon}^{\mathbf{O}}(\mathbf{Wk}(\infty,1)\mathbf{Cat})}(\mathbf{O}, N\mathscr{M}) \quad \textit{and} \quad N\mathbf{Alg}_{E_\infty}(\mathscr{M}) \xrightarrow{\sim} \mathbf{RMor}^{\mathbf{Wk}(\infty,1)\mathbf{Cat}}_{\mathbf{Mon}^{\mathbf{F}}(\mathbf{Wk}(\infty,1)\mathbf{Cat})}(\mathbf{F}, N\mathscr{M}).$$

The General Algebraic Strictification Theorem also implies strictification corollaries that permit one to strictify any kind of module, but the precise formulation makes use of my notion of a Φ-*chirality*, whose explication here would take us too far afield. I will instead be satisfied with stating the following corollary, which I suspect is widely believed but till now not satisfactorily proved.

Corollary 2.27. *Suppose $\mathscr{M}$ as above. Suppose A an A_∞-algebra in $\mathscr{M}$. Write $\mathbf{Mod}^r(A)$ for the category of right A-modules, and write $\mathbf{Mod}(A)$ for the category of A-bimodules. Then the $(\infty,1)$-category $N\mathbf{Mod}(A)$ inherits the structure of a weak $\mathbf{O}$-monoidal $(\infty,1)$-category. There are equivalences among the spaces of weak $\mathrm{Wr}^{(k+1)}(\mathbf{O})$-monoidal structures on $N\mathbf{Mod}^r(A)$ recovering the weak $\mathbf{O}$-monoidal structure on A, of weak $\mathrm{Wr}^{(k)}(\mathbf{O})$-monoidal structures on $N\mathbf{Mod}(A)$ recovering the weak $\mathbf{O}$-monoidal structure on A, and of E_k-algebra structures on A recovering the A_∞-algebra structures on A:*

$$(\mathbf{WkMonStr}^{\mathrm{Wr}^{(k+1)}(\mathbf{O})}(N\mathbf{Mod}^r(A))/A) \simeq (\mathbf{WkMonStr}^{\mathrm{Wr}^{(k)}(\mathbf{O})}(N\mathbf{Mod}(A))/A) \simeq (\mathbf{WkAlgStr}_{E_k}(A)/A).$$

The following subcorollary answers a question of Miller. In effect, it supplies an E_{k+1} algebra structure on the "E_k topological Hochschild cohomology" of an E_k ring spectrum A.

Corollary 2.28. *Suppose A an E_k ring spectrum for $1 \leq k \leq \infty$. Then the endomorphism spectrum $\mathbf{End}(A)$ of A in $\mathbf{Mod}(A)$ inherits a natural structure as an E_{k+1} ring spectrum.*

Additive theories for Waldhausen and stable $(\infty,1)$-categories

Waldhausen K-theory is ordinarily defined via the $S_\bullet$ construction [51] applied to a Waldhausen category — i.e., a category with cofibrations and weak equivalences. This has proved to be a very powerful construction, which has led to numerous insights into the role of K-theory in both algebraic geometry and algebraic topology. Unfortunately, the construction is very rigid: it is functorial only in exact functors of Waldhausen categories. This makes it difficult, for example, to determine what sorts of structure the K-theory might inherit.

Some influential work of Toën–Vezzosi [50] has lent credence to the idea that this problem might be circumvented by constructing K-theory at the level of $(\infty,1)$-categories. In effect, they constructed a K-theory of $\mathbf{S}$-categories (i.e., categories enriched in $s\mathbf{Set}$) and showed that the K-theory of a certain broad class of Waldhausen categories (which they dubbed *good*) is the same as that of their simplicial localizations. They further suggest that it should be possible to perform the $S_\bullet$ construction at the $(\infty,1)$-categorical level.

I have shown that such a construction is indeed possible, and the result, unsurprisingly, can be shown to coincide with the Toën-Vezzosi construction, and hence to the classical Waldhausen construction when the $(\infty,1)$-category arises as the Rezk nerve of a good Waldhausen category. But, inspired by a conversation with Rognes, I discovered a much more powerful and conceptual characterization of K-theory.

Definition 2.29. A *Waldhausen $(\infty,1)$-category* is an $(\infty,1)$-category containing a zero object 0 and all finite homotopy colimits. An *exact functor* of Waldhausen $(\infty,1)$-categories is a morphism of $(\infty,1)$-categories preserving finite homotopy colimits. Denote by **Wald** the sub-$\mathbf{S}$-category of $\mathbf{Wk}(\infty,1)\mathbf{Cat}_f$ whose objects are Waldhausen $(\infty,1)$-categories and whose morphism spaces are the connected components of the morphism spaces of $\mathbf{Wk}(\infty,1)\mathbf{Cat}_f$ corresponding to exact functors.

The aim is to characterize K-theory as an $\mathbf{S}$-functor $\mathbf{Wald} \to \mathbf{Alg}_{E_\infty}(\star/s\mathbf{Set})$. Observe that there is an obvious such $\mathbf{S}$-functor — the direct-sum K-theory:

$$\begin{aligned} K^\oplus : \mathbf{Wald} &\longrightarrow \mathbf{Alg}_{E_\infty}(\star/s\mathbf{Set}) \\ A &\longmapsto (A_0, \sqcup^h). \end{aligned}$$

The claim is that K is a kind of additive hull of $K^\oplus$. The aim of the following pair of theorems is to make this precise.

Theorem 2.30. *Consider the category of $\mathbf{S}$-functors $\mathbf{Wald} \to \mathbf{Alg}_{E_\infty}(\star/s\mathbf{Set})$ with its projective model structure. There exists a left Bousfield localization* **Add** *of this model category with the property that a functor $F : \mathbf{Wald} \to \mathbf{Alg}_{E_\infty}(\star/s\mathbf{Set})$ is fibrant in* **Add** *if and only if F* is an additive theory, *in the sense that the following conditions are satisfied.*

(2.30.1) For every $A \in \mathbf{Wald}$, the E_∞ space FA is grouplike.

(2.30.2) The natural morphism $FE(A', A, A'') \to F(A') \times F(A'')$ is a weak equivalence of E_∞ spaces for any pair of exact functors $A' \to A$ and $A'' \to A$ of Waldhausen $(\infty,1)$-categories.

In other words, additive theories are spectrum-valued **S**-functors that satisfy the additivity theorem of Waldhausen.

Theorem 2.31. *The $S_\bullet$ construction produces a fibrant replacement of $K^\oplus$ in* **Add**.

In other words, K-theory is the homotopy initial additive theory receiving a map from $K^\oplus$. However, K is not the only interesting additive theory. Indeed, I have constructed (suitably derived versions of) THH and TC as additive theories as well, each receiving maps from direct-sum K-theory in a natural manner. This provides a unique characterization of both the Dennis trace and the cyclotomic trace.

Here is another key result, which depends heavily on the Strictification Theorems of the previous section.

Theorem 2.32. *Suppose A an E_k ring spectrum for $1 \leq k \leq \infty$, and suppose F an additive theory. Then $F^r(A) := F(N\mathbf{Mod}^r(A))$ is an E_{k-1} ring spectrum, and $F(A) := F(N\mathbf{Mod}(A))$ is an E_k ring spectrum.*

Variants of this result have been asserted for K-theory and for THH in a number of places, but I have not seen a complete proof in print thus far.

Given an additive theory F, one can ask whether F satisfies localization.

Definition 2.33. A *localization* of an $(\infty,1)$-category A consists of a functor $L\colon A \to C$ with a fully faithful right adjoint. If A is a Waldhausen $(\infty,1)$-category, the full subcategory $\mathrm{Acyc}_A L$ of *L-acyclic* objects of A consists of those objects X such that $LX \simeq 0$. An additive theory F is a *localization theory* if for any such localization $L\colon A \to C$, the sequence

(2.33.1)
$$F(\mathrm{Acyc}_A L) \longrightarrow FA \longrightarrow FC$$

is a homotopy fiber sequence of connective spectra.

Observe that if F is a localization theory, it does *not* follow that (2.33.1) is a fiber sequence in all spectra.

Theorem 2.34. *The additive theories K, THH, and TC are all localization theories.*

An important class of Waldhausen $(\infty,1)$-categories is that of *stable* $(\infty,1)$-categories — i.e., Waldhausen $(\infty,1)$-categories with the property that the suspension functor $X \mapsto X[1] := 0 \sqcup^{h,X} 0$ is an autoequivalence. Denote by **Stab** the full subcategory of **Wald** comprised thereof. What follows now is an $(\infty,1)$-categorical version of the Bass Fundamental Theorem.

Theorem 2.35. *If F is a localization theory, then the restriction of F to* **Stab** *has an essentially unique delooping F^B to the category of all spectra such that F^B :* **Stab** → **Spectra** *is both additive and satisfies localization; that is, the following conditions hold.*

(2.35.1) For any localization $L\colon A \to C$ of stable $(\infty,1)$-categories, the sequence

$$F^B(\mathrm{Acyc}_A L) \longrightarrow F^B A \longrightarrow F^B C$$

is a homotopy fiber sequence of spectra.

(2.35.2) The natural morphism $F^B E(A', A, A'') \to F^B(A') \vee F^B(A'')$ is a weak equivalence of spectra for any pair of exact functors $A' \to A$ and $A'' \to A$ of stable $(\infty,1)$-categories.

This provides a universal characterization of K^B, THH^B, and TC^B.

Of course one wishes to apply localization theories F to closed immersions $i : Z \to X$ of derived stacks with open complements $j : U \to X$, to obtain fiber sequences

$$F^B(Z) \longrightarrow F^B(X) \longrightarrow F^B(U) ,$$

but the fiber term may be difficult to identify in this manner, as the $(\infty,1)$-category of modules (with whatever finiteness condition) on Z cannot be identified with the $(\infty,1)$-category of modules on X acyclic on U. This tends to be something of a pernicious point.

The homotopy theory of divided power algebras over operads

Motivated by the problem of devising a theory of differential operators on derived stacks that would "explain" computations of TR by Hesselholt–Madsen [24, 25], I was led to devise a homotopy theory of *divided power E_∞ algebras.* At first blush, this seems an outlandish thing to hope for.

Indeed, recall that a *divided power* or *PD structure* on a ring R is an ideal $I \lhd R$ and endomorphisms γ_n of I, one for every positive integer n, such that for any $x \in I$, the elements $\gamma_n(x)$ have all the properties possessed by $x^n/n!$. More precisely, we have $\gamma_1 = \mathrm{id}$, and, for any $x, y \in I$,

$$\begin{aligned} \gamma_n(x+y) &= \sum_{0\le i\le n} \gamma_i(x)\gamma_{n-i}(y); \\ \gamma_n(xy) &= n!\gamma_n(x)\gamma_n(y); \\ m!n!\gamma_m(x)\gamma_n(x) &= (m+n)!\gamma_{m+n}(x); \\ m!(n!)^m\gamma_m(\gamma_n(x)) &= (mn)!\gamma_{mn}(x). \end{aligned}$$

Of course if R is a $\mathbf{Q}$-algebra, then one can take $\gamma_n(x) = x^n/n!$, and this is the only PD structure. If R is more general, it may not be possible to divide by $n!$, so neither the existence nor the uniqueness of PD structures is guaranteed.

Example 2.36. Any mixed characteristic discrete valuation ring with ramification index $e < p$ admits a unique PD structure on its maximal ideal. If k is a field of characteristic p, then the Cohen ring $\mathbf{W}_k$ of k satisfies this condition.

With such an explicit, element-based definition, what can be meant by PD structures for more general ring objects (e.g., algebras over an operad)? And how can its homotopy theory be understood?

The first question was addressed by Fresse [20]. Suppose k a field, and consider the category $\mathbf{Vec}(k)$ of k-vector spaces. One has the following monoidal structures on the category of symmetric sequences in $\mathbf{Vec}(k)$:

$$M \circ N := \coprod_{n\ge 0} (M(n) \otimes N^{\otimes n})_{\Sigma_n} \quad \text{and} \quad M \oslash N := \coprod_{n\ge 0} (M(n) \otimes N^{\otimes n})^{\Sigma_n}.$$

As observed by Kelly [30], operads (by which I mean $\mathbf{F}$-operads — fully symmetric operads) are precisely monoids for the monoidal structure $\circ$, and algebras over operads are nothing more than algebras over these monoids; that is, an operad $\mathscr{P}$ gives rise to a monad $T(\mathscr{P}, -) := \mathscr{P} \circ -$ on $\mathbf{Vec}(k)$. Fresse's observation is that, as long as the operad $\mathscr{P}$ is *connected* (so that $\mathscr{P}(0) = 0$) one may define a new monad $\Gamma(\mathscr{P}, -) := \mathscr{P} \oslash -$, whose algebras will be called *divided power* of *PD $\mathscr{P}$-algebras.*

Example 2.37. This name is reasonable: if $\mathscr{C}$ denotes the nonunital commutative operad, PD $\mathscr{C}$-algebras are precisely the same thing as nonunital PD k-algebras in the classical sense.

Example 2.38. Consider the Lie operad $\mathscr{L}$ in $\mathbf{Vec}(k)$. Then PD $\mathscr{L}$-algebras are precisely the same thing as restricted Lie algebras. This is true even in characteristic 2, where ordinary algebras over the Lie operad $\mathscr{L}$ do not coincide with Lie algebras.

The norm map $M \circ N \to M \oslash N$ induces, for any connected operad $\mathscr{P}$, a morphism of monads $T(\mathscr{P}, -) \to \Gamma(\mathscr{P}, -)$, which gives rise to a forgetful functor from PD $\mathscr{P}$-algebras to $\mathscr{P}$-algebras. Hence one may view PD $\mathscr{P}$-algebras as $\mathscr{P}$-algebras equipped with an additional structure. Left adjoint to this is the functor that assigns to any $\mathscr{P}$-algebra A the PD $\mathscr{P}$-algebra freely generated by A.

In order to apply this idea to derived algebraic geometry, I have devised an $(\infty,1)$-categorical version of the above story. Suppose that M is a pointed, locally presentable, symmetric monoidal *additive* $(\infty,1)$-category so that the tensor product commutes with finite homotopy limits. A key case for us will be $M = N\mathbf{Mod}(A)$ for some E_∞ ring spectrum A.

On the $(\infty,1)$-category $\mathbf{RMor}(\Sigma, M)$ of symmetric sequences, there are two monoidal structures, $\circ$ and $\oslash$. The monoids for the product $\circ$ are the weak operads of M, and once again one can construct monads $T(\mathscr{P}, -)$ and $\Gamma(\mathscr{P}, -)$ for any weak connected operad $\mathscr{P}$. The $(\infty,1)$-category of algebras for $T(\mathscr{P}, -)$ is canonically equivalent to the $(\infty,1)$-category $\mathbf{WkAlg}_{\mathscr{P}}(M)$ of weak $\mathscr{P}$-algebras, and the algebras for $\Gamma(\mathscr{P}, -)$ form an $(\infty,1)$-category $\mathbf{WkPDAlg}_{\mathscr{P}}(M)$ of weak PD $\mathscr{P}$-algebras in M.

The additivity of M again provides a norm map $T(\mathscr{P}, -) \to \Gamma(\mathscr{P}, -)$ of monads, whence one has an adjunction

$$E^{\mathrm{PD}}_{\mathscr{P}} : \mathbf{WkAlg}_{\mathscr{P}}(M) \rightleftarrows \mathbf{WkPDAlg}_{\mathscr{P}}(M) : U^{\mathrm{PD}}_{\mathscr{P}}.$$

The structure of this adjunction can be used to develop an obstruction theory to the existence of a PD structure on a given $\mathscr{P}$-algebra. To explain all this precisely here would take us rather far afield; let me instead give a few examples.

Example 2.39. Suppose $1 \le n < \infty$, $\mathscr{P}$ an E_n operad in $s\mathbf{Set}$ — i.e., the symmetrization of $\mathscr{Q}^{\mathrm{Wr}^{(n)}(\mathbf{O})}$ —, and M the $(\infty,1)$-category of spectra. Then the adjunction $(E^{\mathrm{PD}}_{\mathscr{P}}, U^{\mathrm{PD}}_{\mathscr{P}})$ is an equivalence of $(\infty,1)$-categories. Thus every E_k ring spectrum is a PD E_k ring spectrum in a unique way. This is patently false for $n = \infty$!

Example 2.40. The connective spectra e_n have canonical PD E_∞ algebra structures, and the quotient map $e_n \to H\mathbf{W}_{\mathbf{F}_{p^n}}$ is a PD homomorphism.

Now suppose $1 \le n \le \infty$, and suppose k a field. One can define the $(\infty,1)$-category $\mathbf{Frm}_{E_k}(Hk)$ of *formal derived E_n moduli problems* as that of morphisms of $(\infty,1)$-categories from Artin local E_n Hk-algebras to simplicial sets, satisfying certain exactness criteria. As Lurie suggested, one has the following generalization of a theorem of Hinich [26].

Theorem 2.41. *There is an adjoint pair $(\infty,1)$-categories*

$$T : \mathbf{Frm}_{E_n}(Hk) \rightleftarrows \mathbf{WkPDAlg}_{L_n}(\mathbf{Mod}(Hk)) : MC,$$

where the left adjoint is a tangent space functor, and the right adjoint is a Maurer-Cartan functor. Here $L_n = E_n$ when n is finite, but L_∞ is a cofibrant model of the Lie operad.

This adjunction is "almost" an equivalence. I suspect Lurie has a more general result that supersedes this.

$\mathscr{D}$-crystals

Suppose k a field of characteristic zero. One of the most important results in the theory of $\mathscr{D}$-modules is the following theorem of Kashiwara, which permits one to extend $\mathscr{D}$-modules (that are quasicoherent as $\mathscr{O}$-modules) along any nilpotent thickening of k-varieties.

Theorem (Kashiwara, [29])**.** *Suppose $i : Z \to X$ a closed immersion of smooth varieties, and suppose $j : U \to X$ its open complement. Then the category $\mathbf{Mod}^r(\mathscr{D}_{Z/k})$ is canonically equivalent to the category of right $\mathscr{D}_{X/k}$-modules set-theoretically supported on Z, i.e., those right $\mathscr{D}_{X/k}$-modules M such that $j^! M = 0$.*

If, however, the variety Z is singular, then the sheaf $\mathscr{D}_{X/k}$ of linear differential operators can be a very unpleasant ring, and Kashiwara's Theorem will *fail* for right $\mathscr{D}_{Z/k}$-modules.

Example (Bernstein-Gelfand-Gelfand, [7])**.** To illustrate this phenomenon, let us consider the following well-known example: suppose that C is the affine cone over the Fermat curve $x^3 + y^3 + z^3 = 0$ over an algebraically closed field k of characteristic 0. This variety is normal, with a single isolated, Gorenstein singularity at the origin. Despite the mild nature of this singularity, the ring $\mathscr{D}_{C/k}$ of differential operators is neither left nor right nœtherian: if e denotes the *Euler operator*

$$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z},$$

and if $\mathscr{D}^{(j)}_{C/k}$ (respectively, $\mathscr{D}^{(j)}_{m,C/k}$) is the $\mathscr{O}_C$-module of homogenous differential operators of degree j (resp., and of order m), then the two-sided ideals

$$\mathscr{J}_m := \sum_{j>1} \mathscr{D}^{(j)}_{C/k} + \sum_{n\ge 0} e^n \mathscr{D}^{(1)}_{m,C/k}$$

form an ascending chain that does not stabilize.

It now follows that Kashiwara's Theorem fails for the embedding $C \to \mathbf{A}^3_k$; for if it held, $\mathbf{Mod}^r(\mathscr{D}_{C/k})$ would be a colocalization of $\mathbf{Mod}^r(\mathscr{D}_{\mathbf{A}^3_k/k})$, hence a nœtherian abelian category, which we have just seen it is not.

The standard method for rectifying this is *defining deviancy down* by *forcing* Kashiwara's Theorem; namely, for a singular scheme Z, one embeds Z (at least locally) into a smooth scheme X and defines the category of right $\mathscr{D}_{Z/k}$-modules to be the full subcategory of right $\mathscr{D}_{X/k}$-modules set-theoretically supported along Z. Of course, one must then show that the resulting category is well-defined up to a canonical equivalence of categories.

At the level of stable $(\infty,1)$-categories, one can prove the following.

Corollary 2.42. *Suppose $i : Z \to X$ a closed immersion of smooth varieties, and suppose $j : U \to X$ its open complement. Then $j^! : \mathbf{Cplx}^r(\mathscr{D}_{X/k}) \to \mathbf{Cplx}^r(\mathscr{D}_{U/k})$ is a localization of $\mathbf{Cplx}^r(\mathscr{D}_{X/k})$, and $\mathbf{Cplx}^r(\mathscr{D}_{Z/k})$ is canonically equivalent to the $(\infty,1)$-category $\mathrm{Acyc}_{\mathbf{Cplx}^r(\mathscr{D}_{X/k})}(j^!)$ of $j^!$-acyclics.*

Once again, this is false if Z is singular, and, once again, a popular strategy to rectify this is a *defining deviancy down* technique. Note, however, that defining deviancy down at the level of complexes and at the level of modules are not in general compatible: if Z is singular and X is smooth in the above set-up, the triangulated category $\mathrm{Ho}\,\mathrm{Acyc}_{\mathbf{Cplx}^r(\mathscr{D}_{X/k})}(j^!)$ of $j^!$-acyclics has a canonical t-structure, but it need not be the derived category of its heart.

Kashiwara's Theorem has a kind of converse: a $\mathscr{D}$-module is completely determined by its underlying $\mathscr{O}$-module and the underlying underlying $\mathscr{O}$-module of its extensions to all nilpotent thickenings. This leads to the following remarkable idea of Grothendieck, reformulated slightly by Beilinson and Drinfeld.

Definition (Grothendieck, [23, 4.1])**.** The *infinitesimal site* (X_{inf}/k) of X/k is the category of diagrams $X \leftarrow S \rightarrow T$ in which the morphism $S \to T$ is a closed nilimmersion of k-schemes, and the morphism $S \to X$ is étale. There is a natural forgetful functor $(S,T) \mapsto T$ to the category of k-schemes; pull back the étale topology along this functor.

There is now a stack in categories on the infinitesimal site of X:

$$\begin{aligned} \mathbf{Mod}^!_{X/k} : (X_{\mathrm{inf}}/k)^{\mathrm{op}} &\longrightarrow \mathrm{Cat} \\ (S,T) &\longmapsto \mathbf{Mod}(\mathcal{O}_T) \\ (f,g) &\longmapsto H^0 g^!. \end{aligned}$$

Definition (Beilinson–Drinfeld, [4, Definition 7.10.3])**.** A $\mathcal{D}$*-crystal* on X is a cartesian section of the stack $\mathbf{Mod}^!_{X/k}$.

So a $\mathcal{D}$-crystal M assigns to every object (S,T) of the infinitesimal site a quasicoherent $\mathcal{O}_T$-module $M_{(S,T)}$ and to every morphism $(f,g) : (S,T) \to (S',T')$ an isomorphism $M_{(S,T)} \to H^0 g^! M_{(S',T')}$. The category of $\mathcal{D}$-crystals on X/k will be denoted $\mathbf{Cris}^!(X/k)$; this is the category as the infinitesimal cohomology of X with coefficients in the stack $\mathbf{Mod}^!_{X/k}$:

$$\mathbf{Cris}^!(X/k) \simeq \operatorname{holim} \mathbf{Mod}^!_{X/k}.$$

Example 2.43. Suppose X a smooth k-scheme. Then for any object $(S,T) \in (X_{\mathrm{inf}}/k)$, let $p_T : T \to \operatorname{Spec} k$ denote the structure morphism of T, and set

$$\widetilde{\omega}_{X/k}(S,T) := H^n p_T^! \mathcal{O}_{\operatorname{Spec} k}.$$

It follows from the smoothness of X that there exists a morphism $q : T \to X$ of k-schemes such that $H^n p_T^! \mathcal{O}_{\operatorname{Spec} k} \cong H^0 q^! \omega_{X/k}$, where $\omega_{X/k}$ is the dualizing sheaf of top-degree differential forms.[2] Thus $\widetilde{\omega}_{X/k}$ is a $\mathcal{D}$-crystal.

The following pair of propositions suggest that the category of $\mathcal{D}$-crystals has ideal formal properties.

Proposition (Beilinson–Drinfeld, [4, Proposition 7.10.12])**.** *If X is a smooth k-scheme, then the category $\mathbf{Cris}^!(X/k)$ is canonically equivalent to the category $\mathbf{Mod}^r(\mathcal{D}_{X/k})$.*

Proposition (Beilinson–Drinfeld, [4, Lemma 7.10.11])**.** *Kashiwara's Theorem holds for $\mathcal{D}$-crystals; i.e., for any closed immersion $Z \to X$ of schemes (not necessarily smooth), the category of $\mathcal{D}$-crystals on Z is naturally equivalent to the category of $\mathcal{D}$-crystals on X set-theoretically supported on Z.*

The appropriate functorialities of $\mathcal{D}$-crystals do not exist in general. It is more natural not to truncate $g^!$, and to consider instead the following $(\infty,1)$-stack:

$$\begin{aligned} \mathbf{Cplx}^!_{X/k} : (X_{\mathrm{inf}}/k)^{\mathrm{op}} &\to \mathbf{Wk}(\infty,1)\mathbf{Cat} \\ (S,T) &\longmapsto \mathbf{Cplx}(\mathcal{O}_T) \\ (f,g) &\longmapsto g^!. \end{aligned}$$

One can then define a *homotopy $\mathcal{D}$-crystal* as a homotopy cartesian section of this $(\infty,1)$-stack.

A still better strategy is to use derived algebraic geometry. Let us work with derived scheme over k, modeled on simplicial commutative k-algebras or, equivalently, on E_∞ Hk-algebras. A morphism $S \to T$ of derived k-schemes is a *nilimmersion* if the morphism of the truncations $\pi_0 S \to \pi_0 T$ is a nilimmersion. The *infinitesimal ∞-site* (X_{inf}/k) is a defined as a suitable homotopy limit, so that (by the Categorical Strictification Theorem) the objects are uniquely represented, up to equivalence, by diagrams of derived k-schemes

$$X \longleftarrow S \longrightarrow T,$$

where $S \to X$ is an étale morphism, and $S \to T$ is a nilimmersion. Once again one pulls back the étale topology along the forgetful functor $(S,T) \mapsto T$. One now considers the following $(\infty,1)$-stack:

$$\begin{aligned} \mathbf{Mod}^!_{X/k} : (X_{\mathrm{inf}}/k)^{\mathrm{op}} &\to \mathbf{Wk}(\infty,1)\mathbf{Cat} \\ (S,T) &\longmapsto \mathbf{Mod}(\mathcal{O}_T) \\ (f,g) &\longmapsto g^!. \end{aligned}$$

Definition 2.44. The $(\infty,1)$*-category of $\mathcal{D}$-crystals* on a derived k-scheme is the infinitesimal cohomology of X with coefficients in the $(\infty,1)$-stack $\mathbf{Mod}^!_{X/k}$:

$$\mathbf{Cris}^!(X/k) := \operatorname{holim} \mathbf{Mod}^!_{X/k}$$

[2] Observe however that $\omega_{X/k}(T)$ is only a truncation of the dualizing complex $\omega_{T/k}$.

If X is an ordinary k-scheme, then a homotopy $\mathscr{D}$-crystal on X is the same thing as a $\mathscr{D}$-crystal on $H_!(X)$. (Here $H_!(X)$ is X viewed as a derived scheme, so that for any commutative simplicial k-algebra A, $H_!(X)(A) := X(\pi_0 A)$.)

Example 2.45. The assignment $(S,T) \mapsto \omega_{T/k}$ is a homotopy $\mathscr{D}$-crystal on X.

The following pair of theorems guarantees that the theory of $\mathscr{D}$-crystals has the right formal properties.

Theorem 2.46. *If X is an ordinary smooth k-variety, then the $(\infty,1)$-category $\mathrm{Cris}^!(X/k)$ is equivalent to the $(\infty,1)$-category $\mathbf{Cplx}^r(\mathscr{D}_{X/k})$.*

Theorem 2.47. *Kashiwara's Theorem holds for $\mathscr{D}$-crystals on derived k-schemes; i.e., if $i : Z \to X$ a closed immersion of derived schemes, and suppose $j : U \to X$ its open complement. Then $j^! : \mathbf{Cris}^!(X/k) \to \mathbf{Cris}^!(U/k)$ is a localization of $\mathbf{Cris}^!(X/k)$, and $\mathbf{Cris}^!(Z/k)$ is canonically equivalent to the $(\infty,1)$-category $\mathrm{Acyc}_{\mathbf{Cris}^!(X/k)}(j^!)$ of $j^!$-acyclics.*

Corollary 2.48. *A $\mathscr{D}$-crystal on a derived k-scheme X is precisely the same data as a $\mathscr{D}$-crystal on $\pi_0 X$.*

This seems a little silly at first blush; after all, I went to all the trouble to extend the definition of $\mathscr{D}$-crystals to derived schemes, but the result only depends upon the truncation in the end. However, the notion of $\mathscr{D}$-crystal discussed here is only reasonable when k is a field of characteristic zero.

When the base k is more general, one does not ask that a $\mathscr{D}$-crystal extend uniquely over nilpotent thickenings; rather, one only asks for unique extensions over PD nilpotent thickenings.

So suppose k a connective PD E_∞ ring spectrum. Then for any derived k-scheme X, one can define the *crystalline ∞-site* (X_{cris}/k) as the $(\infty,1)$-category of PD nilpotent schemes (S,T) over k with an étale morphism $S \to X$. Again the étale topology is pulled back along the forgetful functor $(S,T) \to T$. One defines the $(\infty,1)$-stack $\mathbf{Mod}^!_{X/k}$ just as above; the $(\infty,1)$-category of $\mathscr{D}$-crystals on X/k is then the crystalline cohomology of X with coefficients in $\mathbf{Mod}^!_{X/k}$:

$$\mathbf{Cris}^!(X/k) := \operatorname{holim} \mathbf{Mod}^!_{X/k}.$$

Theorem 2.49. *Kashiwara's Theorem holds for $\mathscr{D}$-crystals on derived k-schemes; i.e., if $i : Z \to X$ a closed PD immersion of derived schemes, and $j : U \to X$ is its open complement, then $j^! : \mathbf{Cris}^!(X/k) \to \mathbf{Cris}^!(U/k)$ is a localization of $\mathbf{Cris}^!(X/k)$, and $\mathbf{Cris}^!(Z/k)$ is canonically equivalent to the $(\infty,1)$-category $\mathrm{Acyc}_{\mathbf{Cris}^!(X/k)}(j^!)$ of $j^!$-acyclics.*

It is not the case that the closed immersion $\pi_0 X \to X$ is a PD immersion, and the result above does not appear to hold for this closed immersion. Hence $\mathscr{D}$-crystals on derived schemes indeed seem to "see" the derived structure.

An interesting corollary of Kashiwara's Theorem is that for any localization theory F, the functor $F^B(\mathbf{Cris}^!(-/k))$ satisfies (PD) localization for derived schemes.

Corollary 2.50. *If $i : Z \to X$ a closed (PD) immersion of derived schemes, and $j : U \to X$ is its open complement, then the sequence*

$$F^B(\mathbf{Cris}^!(Z/k)) \longrightarrow F^B(\mathbf{Cris}^!(X/k)) \longrightarrow F^B(\mathbf{Cris}^!(U/k))$$

is a fiber sequence of spectra.

In the special case of K-theory, one can combine this with localization and Zariski descent for G-theory of ordinary schemes to obtain the following.

Theorem 2.51. *Suppose k any field, X a coherent k-scheme, locally of finite presentation. Then there is a natural equivalence of spectra*

$$G^B(X) \simeq K^B(\mathbf{Cris}^!(X/k)).$$

There is a similar equivalence for certain derived schemes. One defines a notion of *pseudocoherent* module, and one defines the G-theory to be the K-theory of the $(\infty,1)$-category of *cohomologically bounded* (in a suitable sense) pseudocoherent modules. Since coherent cohomologically bounded $\mathscr{D}$-crystals themselves satisfy localization, the $(\infty,1)$-category of $\mathscr{D}$-crystals can be viewed as an $(\infty,1)$-categorification of the G-theory of a derived scheme.

For any gerbe α on X_{cris} with band $\mathbf{G}_m$, one can study the $(\infty,1)$-category $\mathbf{Cris}^!_\alpha(X/k)$ of α-twisted $\mathscr{D}$-crystals. This also works in the derived setting. Some piece of this is relevant below, where I discuss Quantum Geometric Langlands.

Logarithmic structures in derived algebraic geometry

The significance of log poles in K-theory was first suggested by the following example.

Example (Hesselholt–Madsen, [24])**.** Suppose A a mixed characteristic discrete valuation ring with perfect residue field k of odd characteristic and fraction field K. Fix a uniformizer π. Hesselholt and Madsen define $THH(A|K)$ as the cofiber of the natural morphism $THH(k) \to THH(A)$; they show that

$$V(0)_\star THH(A|K) \cong A/p \otimes \Lambda\{d\log\pi\} \otimes \mathrm{S}\{\kappa_0\},$$

where $d\log\pi$ is an element of degree 1 with the property that the image of $d\pi$ under the natural map $V(0)_1 THH(A) \to V(0)_1 THH(A|K)$ is $\pi d\log\pi$, and κ_0 is an element of degree 2 that Bocksteins to $d\log p$. Hence one may consider (by the Hochschild–Kostant–Rosenberg Theorem [28]) $THH(A|K)$ as the ring of differentials on $\operatorname{Spec} A$ with logarithmic poles along $\operatorname{Spec} k$. The Dennis trace map gives a morphism of fiber sequences

$$\begin{array}{ccccc} K(k) & \longrightarrow & K(A) & \longrightarrow & K(K) \\ \downarrow & & \downarrow & & \downarrow \\ THH(k) & \longrightarrow & THH(A) & \longrightarrow & THH(A|K). \end{array}$$

One verifies easily that the trace map $K(K) \to THH(A|K)$ sends $\{\pi\} \mapsto d\log\pi$.

This induces a morphism of fiber sequences

$$\begin{array}{ccccc} K(k) & \longrightarrow & K(A) & \longrightarrow & K(K) \\ \downarrow & & \downarrow & & \downarrow \\ TC(k) & \longrightarrow & TC(A) & \longrightarrow & TC(A|K), \end{array}$$

and Hesselholt–Madsen show that the trace morphism induces an isomorphism

$$K_i(K, \mathbf{Z}/p^\nu) \xrightarrow{\sim} TC_i(A|K, \mathbf{Z}/p^\nu),$$

whence they obtain a description of the K-theory of K with $\mathbf{Z}/p^\nu$-coefficients in terms of the de Rham–Witt complex of $\operatorname{Spec} A$ with logarithmic poles along $\operatorname{Spec} k$.

Rognes asked whether one might also describe the K-theory of the "fraction field" $p^{-1}ku_p$ in terms of a de Rham–Witt complex on $\operatorname{Spec} ku_p$ with logarithmic poles along the derived subscheme corresponding to the ideal (p). To make this precise, I proposed that one should define a notion of *logarithmic structure* on derived schemes, and to study the log-cristalline cohomology of $\mathscr{D}$-crystals. As Rognes later observed, such a notion of log structure might provide a geometric meaning to both the rows and the columns of (1.10.2).

I constructed a model category of derived log affines in the following manner: one begins with the model category **PreLog** of right sections of the one-arrow right Quillen presheaf

$$\mathbf{M}_m : \mathbf{Alg}_{E_\infty}(\mathbf{Sp}^\Sigma, \wedge) \to \mathbf{Alg}_{E_\infty}(\star/s\mathbf{Set}, \wedge)$$

(which just remembers the multiplicative monoid). These are pairs (A, E) consisting of an E_∞ ring spectrum A and a morphism of E_∞ spaces $E \to \mathbf{M}_m(A)$. One now forms the left Bousfield localization **Log** of **PreLog** with respect to maps to guarantee the log condition for fibrant objects (A, E): that

$$\mathbf{G}_m(A) \times_{\mathbf{M}_m(A)} E \to \mathbf{G}_m(A)$$

is a weak equivalence. The localization **PreLog** $\to$ **Log** is the logification functor.

Example 2.52. The connective Morava E-theory spectra e_n has a log structure $\langle p \rangle$ generated by p.

Example 2.53. The connective cover e of any periodic E_∞ ring spectrum E has a log structure $\langle u \rangle$ generated by the Bott element u.

Pulling back a topology τ that is at least as coarse as the fppf topology along the functor $(A, E) \mapsto A$, one can define the *log-τ ∞-site of derived log affines*, and gluing such objects together, one defines the notion of *derived log scheme*.

In particular, one can pull back the crystalline topology to give the log-crystalline topology on a derived log scheme $(X, \mathscr{M}_X)$. The log-crystalline cohomology of $(X, \mathscr{M}_X)$ with coefficients in the $(\infty,1)$-stack $\mathbf{Mod}^!$ is the $(\infty,1)$-category $\mathbf{Cris}^!(X, \mathscr{M}_X)$ of *$\mathscr{D}$-crystals with logarithmic poles on* $(X, \mathscr{M}_X)$. One can then demonstrate the following extension of Kashiwara's Theorem to derived log schemes.

Theorem 2.54. *If $(X, \mathscr{M}_X)$ is derived log scheme, then $j^! : \mathbf{Cris}^!(X) \to \mathbf{Cris}^!(X, \mathscr{M}_X)$ is a localization of $\mathbf{Cris}^!(X)$. If the log structure on X is generated by an effective normal crossings divisor D, then $\mathbf{Cris}^!(D)$ is canonically equivalent to the $(\infty,1)$-category $\mathrm{Acyc}_{\mathbf{Cris}^!(X)}(j^!)$ of $j^!$-acyclics.*

This leads to a kind of categorification of the three-by-three square (1.10.2) predicted by Rognes.

Theorem 2.55. *One has a three-by-three square of localization sequences*

(2.55.1)
$$\begin{array}{ccccc}
\mathrm{Acyc}_{\mathbf{Cris}^!(\operatorname{Spec} e_n)}(i^!, j^!) & \longrightarrow & \mathrm{Acyc}_{\mathbf{Cris}^!(\operatorname{Spec} e_n)}(i^!) & \longrightarrow & \mathrm{Acyc}_{\mathbf{Cris}^!(\operatorname{Spec} e_n, \langle p \rangle)}(I^!) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Acyc}_{\mathbf{Cris}^!(\operatorname{Spec} e_n)}(j^!) & \longrightarrow & \mathbf{Cris}^!(\operatorname{Spec} e_n) & \xrightarrow{j^!} & \mathbf{Cris}^!(\operatorname{Spec} e_n, \langle p \rangle) \\
\downarrow & & \downarrow{\scriptstyle i^!} & & \downarrow{\scriptstyle I^!} \\
\mathrm{Acyc}_{\mathbf{Cris}^!(\operatorname{Spec} e_n, \langle u \rangle)}(J^!) & \longrightarrow & \mathbf{Cris}^!(\operatorname{Spec} e_n, \langle u \rangle) & \xrightarrow[J^!]{} & \mathbf{Cris}^!(\operatorname{Spec} e_n, \langle p, u \rangle)
\end{array}$$

This leads to localization sequences for G-theory, but it is unfortunately not clear to me yet how to identify the G-theory of the fiber terms of (2.55.1) with the G-theory of $\mathscr{D}$-crystals on $\operatorname{Spec} e_{n-1}$; this point seems to require a kind of dévissage argument that does not seem particularly straightforward.

3 What Lies Ahead

In this section I briefly describe the two most inspiring sources of new ideas for my future research.

TC, crystalline cohomology, and syntomic cohomology in derived algebraic geometry

The results of Hesselholt–Madsen [24, 25] suggest that there is a close relationship between topological cyclic homology TC and crystalline cohomology. Moreover, there is a close relationship between the cyclotomic trace $K \to TC$ and the cycle class map $K \to H_{\mathrm{cris}}$. One of my (only partially realized) aims in developing the theory of $\mathscr{D}$-crystals in the context of derived algebraic geometry was to give a fully conceptual explanation of this relationship.

One very basic point that requires a great deal more elucidation is the model-dependent nature of the cyclotomic structure on THH used by Hesselholt–Madsen to construct the pro-spectrum TR. The S^1-equivariant structure is used to provide a Dieudonné module structure on the homotopy of THH, but critical to the construction of TR are the restriction morphisms, which arise as a result of the cyclotomic structure.

Question 3.1. Is it possible to develop a reasonable homotopy theory of cyclotomic spectra, in a manner sufficiently transparent that certain functors (such as THH and crystalline cohomology) can be seen to inherit such a structure?

Over the $E(n)$-local sphere spectrum $\mathbf{S}_{(p,n)} := L_{E(n)}\mathbf{S}_{(p)}$, there exists a syntomic site whose cohomology plays a role similar to that of the syntomic site over a complete valuation ring of mixed characteristic in classical algebraic geometry. There is also a syntomic sheaf $\mathscr{S}_{L_{K(n)}\mathbf{S}_{(p)}}(j)$ on the big syntomic site of $\operatorname{Spec}\mathbf{S}_{(p,n)}$ whose syntomic cohomology on each derived scheme over $\operatorname{Spec}\mathbf{S}_{(p,n)}$ plays the role of (p,n)-adic absolute Hodge cohomology of weight j.

Conjecture 3.2. *In particular, for any smooth and proper derived varieties X over* $\operatorname{Spec}\mathbf{S}_{(p,n)}$*, there should exist "regulator" morphisms*

$$K_{2j-i}(X, \mathbf{S}_{(p,n)})^{(j)} \to H^i_{\mathrm{syn}}(X, \mathscr{S}_{L_{K(n)}\mathbf{S}_{(p)}}(j)),$$

from suitable weight j eigenspaces of the (p,n)-local K-theory to the syntomic cohomology of $\mathscr{S}_{L_{K(n)}\mathbf{S}_{(p)}}(j)$.

The construction of such regulators would be very interesting indeed, as they should carry a great deal of information about the K-theory of X with $\mathbf{S}_{(p,n)}$-coefficients.

Conjecture 3.3. *In particular, for $2j > i$ (and perhaps with some restriction on the pole of the Kubota–Leopoldt L-function described in the next paragraph), it seems reasonable to conjecture that the regulator induce an isomorphism after completion:*

$$K_{2j-i}(X, L_{K(n)}\mathbf{S}_{(p)})^{(j)} \simeq H^i_{\mathrm{syn}}(X, \mathscr{S}_{L_{K(n)}\mathbf{S}_{(p)}}(j)).$$

I have virtually no evidence for or against such a conjecture, however.

Additionally, by using higher Tannakian duality applied to unipotent isocrystals on certain derived rigid analytic spaces, it should be possible to follow [8] to define a version of Coleman functions on derived schemes X over $\operatorname{Spec}\mathbf{S}_{(p,n)}$. It seems quite clear how to go about this. One can thus define *(p,n)-adic Kubota–Leopoldt L-functions* on such an X. One can even attempt to give a description of the special values of these L-functions for smooth and proper derived varieties as determinants of the regulator morphism above; regardless of whether such a Beilinson conjecture is true, these L-functions should nevertheless capture very interesting cohomological information.

Quantum Geometric Langlands

The most exciting ideas I have investigated recently are the Quantum Geometric Langlands Correspondences conjectured by Gaitsgory and Lurie.

Suppose G a complex reductive group; write ${}^LG$ for its Langlands dual. Fix a level $\kappa = (c - \frac{1}{2})\kappa_{\mathrm{Kill}}$ for G — which I denote simply by c —, and denote by $\frac{1}{c}$ the dual level for ${}^LG$.

It is widely understood that the Beilinson–Drinfeld notion [5] of a chiral algebra on a curve X (smooth, projective, complex) should have an $(\infty,1)$-categorification; this is the notion of a *chiral* $(\infty,1)$*-category* or *chiral category* for short. It is relatively clear how to give a precise definition in (at least) two different ways. First, one can use the theory of multi-$(\infty,2)$-categories as above to develop a notion the chiral operations of *crystalline categories*. Then a chiral category is a kind of Lie algebra in the resulting compound multi-$(\infty,2)$-category of crystalline categories. Second, one can use the factorization algebra model to give a definition that is simpler in some respects. The equivalence of the two definitions should be an instance of derived Koszul duality.

Example 3.4. Once the definition and basic homotopy theory of these objects has been determined, one finds a stockpile of examples at the ready.

(3.4.1) Indeed, for any chiral algebra A, the category $\mathbf{Mod}(A)$ is naturally a chiral category. In particular, the category $\mathbf{Mod}_c(\widehat{\mathfrak{g}})$ of smooth Kac–Moody modules at level c is naturally a chiral category.

(3.4.2) Also, the full sub-$(\infty,1)$-category $\mathbf{KL}(\mathbf{Mod}_c(\widehat{\mathfrak{g}}))$ comprised of Kazhdan–Lusztig objects (i.e., $G[[t]]$-integrable Kac–Moody modules at level c) is a chiral category.

(3.4.3) The $(\infty,1)$-category $\mathbf{Cris}^!_c(\mathrm{Gr}_G)$ of c-twisted $\mathscr{D}$-crystals on the affine Grassmannian Gr_G is a chiral category, via the Beilinson–Drinfeld construction.

(3.4.4) The Whittaker reduction $\mathbf{Whit}(\mathbf{Cris}^!_c(\mathrm{Gr}_G))$ of the above (comprised of the $(N((t)),\chi)$-equivariant objects for some nondegenerate character χ of the loop group $N((t))$ of the radical of the Borel) is also a chiral category.

There is additionally a notion of *module* over a chiral category, which is relatively straightforward, given the groundwork that has been laid above.

Example 3.5. (3.5.1) The $(\infty,1)$-category $\mathbf{Cris}^!_c(G((t)))$ of c-twisted $\mathscr{D}$-crystals on the loop group $G((t))$ is a $\mathbf{Cris}^!_c(\mathrm{Gr}_G)$-module.

(3.5.2) The Whittaker reduction $\mathbf{Whit}(\mathbf{Cris}^!_c(G((t))))$ of the above is a $\mathbf{Whit}(\mathbf{Cris}^!_c(\mathrm{Gr}_G))$-module.

Gaitsgory generalizes the Beilinson–Drinfeld construction to give any stable $(\infty,1)$-category with an action of the loop group $G((t))$ at level c the structure of a $\mathbf{Cris}^!_c(\mathrm{Gr}_G)$-module. He postulates the following.

Conjecture 3.6. *This gives an* $(\infty,2)$*-equivalence*

$$\mathbf{Stab}_{G((t)),c} \to \mathbf{Mod}(\mathbf{Cris}^!_c(\mathrm{Gr}_G)),$$

This is far from a mere formality; even the claim that it is fully faithful seems to be a nontrivial computation. I hope to address this question carefully soon. It will be convenient to assume this conjecture in what follows.

Theorem (Gaitsgory, [21])**.** *Suppose c irrational; then there exists an equivalence of chiral categories*

$$\phi_{G,c} : \mathbf{Whit}(\mathbf{Cris}^!_c(\mathrm{Gr}_G)) \to \mathbf{KL}(\mathbf{Mod}_{\frac{1}{c}}(\widehat{{}^L\mathfrak{g}})).$$

This equivalence is conjectured for c rational and nonnegative as well; it is known in the classical limit ($c \in \{0,\infty\}$).

Conjecture 3.7 (Local Quantum Geometric Langlands)**.** *Suppose c irrational; then there exists a commutative diagram of equivalences of* $(\infty,2)$*-categories*

$$\begin{array}{ccc}
\mathbf{Mod}(\mathbf{KL}(\mathbf{Mod}_{-c}(\widehat{\mathfrak{g}}))) & \xrightarrow[\sim]{\phi^{\star}_{{}^LG,-\frac{1}{c}}} & \mathbf{Mod}(\mathbf{Whit}(\mathbf{Cris}^!_{-\frac{1}{c}}(\mathrm{Gr}_G))) \\
\Big\uparrow {\scriptstyle (-)\otimes_{\mathbf{Cris}^!_c(\mathrm{Gr}_G)}\mathbf{Mod}_{-c}(\widehat{\mathfrak{g}})} & & \Big\uparrow {\scriptstyle (-)\otimes_{\mathbf{Cris}^!_{-\frac{1}{c}}(\mathrm{Gr}_{{}^LG})}\mathbf{Whit}(\mathbf{Cris}^!_{-\frac{1}{c}}(\mathrm{Gr}_G))} \\
\mathbf{Mod}(\mathbf{Cris}^!_c(\mathrm{Gr}_G)) & \xrightarrow{\Psi_{G,c}} & \mathbf{Mod}(\mathbf{Cris}^!_{-\frac{1}{c}}(\mathrm{Gr}_{{}^LG})) \\
\Big\downarrow {\scriptstyle (-)\otimes_{\mathbf{Cris}^!_c(\mathrm{Gr}_G)}\mathbf{Whit}(\mathbf{Cris}^!_c(\mathrm{Gr}_G)} & & \Big\downarrow {\scriptstyle (-)\otimes_{\mathbf{Cris}^!_{-\frac{1}{c}}(\mathrm{Gr}_{{}^LG})}\mathbf{Mod}_{\frac{1}{c}}(\widehat{{}^L\mathfrak{g}})} \\
\mathbf{Mod}(\mathbf{Whit}(\mathbf{Cris}^!_c(\mathrm{Gr}_G))) & \xleftarrow[\phi^{\star}_{G,c}]{\sim} & \mathbf{Mod}(\mathbf{KL}(\mathbf{Mod}_{\frac{1}{c}}(\widehat{{}^L\mathfrak{g}}))).
\end{array}$$

Conjecture 3.8 (Global Unramified Quantum Geometric Langlands)**.** *Suppose $c \notin \mathbf{Q}_{<0}$; then there exists an equivalence of $(\infty, 1)$ categories*

$$\Phi_{G,c} : \mathbf{Cris}^!_c(\mathbf{Bun}_G(X)) \to \mathbf{Cris}^!_{-\frac{1}{c}}(\mathbf{Bun}_{{}^LG}(X))$$

such that for any nonnegative integer n and any n-tuple (x_i) of points of X, the diagram

$$\begin{array}{ccc} \prod_{1\leq i\leq n} \mathbf{KL}(\mathbf{Mod}_c(\widehat{\mathfrak{g}}))_{x_i} & \longrightarrow & \prod_{1\leq i\leq n} \mathbf{Whit}(\mathbf{Cris}^!_{\frac{1}{c}}(\mathrm{Gr}_{{}^LG})) \\ {\scriptstyle \mathrm{Loc}}\downarrow & & \downarrow{\scriptstyle \mathrm{Poin}} \\ \mathbf{Cris}^!_c(\mathbf{Bun}_G(X)) & \xrightarrow[\Phi_{G,c}]{} & \mathbf{Cris}^!_{-\frac{1}{c}}(\mathbf{Bun}_{{}^LG}(X)), \end{array}$$

where Loc *is a localization functor, and* Poin *is a Poincaré series functor.*

Conjecture 3.9 (Global Ramified Quantum Geometric Langlands)**.** *Suppose c irrational, and suppose x a point of X. Then there is an equivalence of chiral categories*

$$\Psi_{G,c}(\mathbf{Cris}^!_c(\mathbf{Bun}_{G,x}(X))) \simeq \mathbf{Cris}^!_{-\frac{1}{c}}(\mathbf{Bun}_{{}^LG,x}(X)).$$

These conjectures are known to imply many of the local and global Geometric Langlands conjectures.

Notice that the equivalence $\Psi_{G,c}$ posited by the Global Unramified Quantum Geometric Langlands Conjecture (first proposed by Drinfeld and Stoyanovsky [47]) is sensible regardless of the dimension of X, provided one works with the derived moduli stacks $\mathbf{RBun}_G(X)$ and $\mathbf{RBun}_{{}^LG}(X)$; however, the the functors Loc and Poin are ostensibly the result of one-dimensional phenomena.

Question 3.10. What data is needed to specify a unique equivalence of $(\infty, 1)$-categories

$$\Psi_{G,c} : \mathbf{Cris}^!_c(\mathbf{RBun}_G(X)) \to \mathbf{Cris}^!_{-\frac{1}{c}}(\mathbf{RBun}_{{}^LG}(X))?$$

I have only just learned of many of these ideas, but it seems probable that the algebraic tools I have developed will provide a number of useful insights, especially in the irrational level case.

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