

IGUSA ZETA FUNCTIONS AND DIOPHANTINE GEOMETRY

Antoine CHAMBERT-LOIR

Institut de recherche mathématique de Rennes, Université de Rennes 1

Joint work with Yuri TSCHINKEL

Göttingen University / Courant Institute, New-York University

*Motivic integration and its interactions
with model theory and non-archimedean analysis*

Edinburgh, May 16, 2008

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TWO VOLUME ESTIMATES FOR SEMI-SIMPLE GROUPS

1 TWO VOLUME ESTIMATES FOR SEMI-SIMPLE GROUPS

- Real semi-simple Lie groups
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3 VOLUMES AND DISTRIBUTION OF REAL OR p -ADIC HEIGHT BALLS

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REAL SEMI-SIMPLE LIE GROUPS

Let G be a semi-simple Lie group with trivial center, μ a Haar measure on G .

Let $\rho: G \rightarrow \text{GL}(V)$ be a finite dimensional faithful representation of G in a real vector space V . Let $\|\cdot\|$ be a norm on $\text{End}(V)$.

For any $T > 0$, let $B_T = \{g \in G; \|\rho(g)\| \leq T\}$ — compact in G .

THEOREM (MAUCOURANT, 2004)

When $T \rightarrow \infty$:

- 1 **volume estimate:** $\mu(B_T) \sim cT^d \log(T)^e$ for some real number c , some rational number d and some integer e ;
- 2 **convergence of measures:** there exists a measure μ_∞ on $\text{End}(V)$ such that for any function $f \in \mathcal{C}(\mathbb{P}\text{End}(V))$,

$$\frac{1}{\mu(B_T)} \int_{B_T} f(\rho(g)) \, d\mu(g) \rightarrow \int_{\mathbb{P}\text{End}(V)} f(g) \, d\mu_\infty(g).$$

THEOREM (MAUCOURANT)

$$\mu(B_T) \sim cT^d \log(T)^e$$
$$\mu(B_T)^{-1} \int_{B_T} f(\rho(g)) d\mu(g) \rightarrow \int_{\mathbb{P}\text{End}(V)} f(g) d\mu_\infty(g).$$

The numbers d and e are explicitly defined in terms of the relative root system of G and the weights of ρ .

One has $0 \leq e \leq \text{rank}_{\mathbb{R}}(G)$.

The measure μ_∞ is supported by a submanifold of $\mathbb{P}\text{End}(V)$ which is bi-invariant under G .

Principle of proof: $K\mathfrak{a}^+K$ -decomposition and integration formula.

ADELIC SEMI-SIMPLE ALGEBRAIC GROUPS

Let G be a semi-simple algebraic group over \mathbb{Q} .

Let $\rho: G \rightarrow \mathrm{GL}(V)$ be a faithful representation of G in a finite dimensional \mathbb{Q} -vector space V (with a unique highest weight).

For any $p \in \{\text{prime numbers}\} \cup \{\infty\}$, let $\|\cdot\|_p$ be a p -adic norm on $\mathrm{End}(V) \otimes \mathbb{Q}_p$.

Compatibility assumption: there exists a basis (e_i) of $\mathrm{End}(V)$ such that for almost all p : for any $u \in \mathrm{End}(V) \otimes \mathbb{Q}_p$ with coordinates (u_i) , one has $\|u\|_p = \max(|u_i|_p)$.

Adeles: $\mathbb{A} =$ restricted product $\prod'_p \mathbb{Q}_p$.

Then, $G(\mathbb{A})$ is a locally compact group — restricted product $\prod'_p G(\mathbb{Q}_p)$.

For any $T > 0$, let $B_T = \{g = (g_p) \in G(\mathbb{A}); \prod_p \|\rho(g_p)\|_p \leq T\}$ — **compact set in $G(\mathbb{A})$.**

$$B_T = \{g = (g_p) \in G(\mathbb{A}); \prod_p \|\rho(g_p)\|_p \leq T\}$$

Fix a Haar measure μ on $\mathbf{G}(\mathbb{A})$.

THEOREM (GORODNIK, MAUCOURANT, OH, 2007)

When $T \rightarrow \infty$:

- 1 **volume estimate:** $\mu(B_T) \sim cT^a \log(T)^b$ for some positive real number c , some rational number a and some non-negative integer b ;
- 2 **convergence of measures:** for any function $f \in \mathcal{C}(\mathbb{P}(\text{End}(V) \otimes \mathbb{A}))$,

$$\frac{1}{\mu(B_T)} \int_{B_T} f(\rho(g)) d\mu(g) \rightarrow \int_{\mathbb{P}(\text{End}(V) \otimes \mathbb{A})} f(g) d\mu_\infty(g).$$

THEOREM (GORODNIK, MAUCOURANT, OH)

$$\mu(B_T) \sim cT^a \log(T)^b$$
$$\frac{1}{\mu(B_T)} \int_{B_T} f(\rho(g)) d\mu(g) \rightarrow \int_{\mathbb{P}(\text{End}(V) \otimes \mathbb{A})} f(g) d\mu_\infty(g)$$

Again, a and b can be computed explicitly in terms of the weights of ρ , the root system of G and the action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ they possess.

The measure μ_∞ is supported by $X_\rho(\mathbb{A})$, where X_ρ is the Zariski closure of $\rho(G)$ in $\mathbb{P} \text{End}(V)$ — **DE CONCINI-PROCESI's wonderful compactification** of G .

MOTIVATION/CONSEQUENCE OF THESE ESTIMATES

These **volume estimates**, resp. **convergence of measures** are one step in understanding the number, resp. the distribution, of

- points in $\Gamma \cap B_T$, where Γ is a lattice of the Lie group G — **lattice points in balls**;
- points in $G(\mathbb{Q}) \cap B_T$ — **rational points of “bounded height”**.

When $T \rightarrow \infty$, and for adequate representations ρ , the obtained estimates are

- 1 $\#(\Gamma \cap B_T) \sim V(T)/\mu(G/\Gamma)$;
- 2 $\#(G(\mathbb{Q}) \cap B_T) \sim V(T)/\mu(G(\mathbb{A})/G(\mathbb{Q}))$ — with a deliberately ignored twist caused by automorphic characters.

Other actors of the play:

- 1 Duke, Rudnick, Sarnak; Eskin, McMullen;
- 2 Shalika, Tschinkel, Takloo-Biglash.

1 TWO VOLUME ESTIMATES FOR SEMI-SIMPLE GROUPS

2 CONJECTURES OF BATYREV, MANIN AND PEYRE

- Rapid preview
- Volumes

3 VOLUMES AND DISTRIBUTION OF REAL OR p -ADIC HEIGHT BALLS

4 TAMAGAWA MEASURES, ADELIC HEIGHT BALLS, AND THEIR VOLUMES

A basic problem in **diophantine geometry** consists in deciding whether **diophantine equations** have solutions or not, more generally, to tell as much as possible about the set of solutions.

From a **geometrical point of view**, describe the set of rational points of algebraic varieties defined over \mathbb{Q} , or the set of integral points of algebraic varieties over \mathbb{Z} .

We are interested in varieties whose rational points are dense for the Zariski topology. We thus have to sort them according to their “arithmetic complexity”, that is, their **height**.

Essential example: a point $P \in \mathbb{P}^n(\mathbb{Q})$, with homogeneous coordinates $[x_0 : \cdots : x_n]$ coprime integers, has height $H(P) = \max(|x_0|, \dots, |x_n|)$.

Finiteness property (NORTHCOTT): for any $B > 0$, there are only finitely many points $P \in \mathbb{P}^n(\mathbb{Q})$ such that $H(P) \leq B$.

Question: How many, when $B \rightarrow \infty$?

Answer (SCHANUEL): $\sim \frac{2^n}{\zeta(n+1)} B^{n+1}$.

Analytical tool: the “height zeta function”, *i.e.*, the generating series

$$Z_{\mathbb{P}^n}(s) = \sum_{P \in \mathbb{P}^n(\mathbb{Q})} H(P)^{-s}.$$

Understand abscissa of convergence, meromorphic continuation, location of poles,...

THE CONJECTURE OF BATYREV, MANIN, PEYRE

Question: What happens if one restricts to points lying in a subvariety X of \mathbb{P}^n ?

Conjectural answer (MANIN): If X is smooth, anticanonically embedded, it should be $\approx B(\log B)^{t-1}$, where $t = \text{rank Pic}(X)$, provided:

- you allow to enlarge the ground field;
- you exclude from X some strict algebraic subvarieties.

Refinement (PEYRE): it might even be $\sim cB(\log B)^{t-1}$, with an arithmetical description of the constant c .

Many, often non-trivial, examples:

- flag varieties (LANGLANDS's theory of Eisenstein series);
- toric varieties; equivariant compactifications of vector spaces;
- wonderful compactifications of adjoint semi-simple groups;
- Del Pezzo surfaces of degree ≥ 4 ...

... but a counter-example (total space of the family of diagonal cubic surfaces in \mathbb{P}^3).

In all cases where the conjectures have been established for the variety X , the constant c in front of the asymptotic expansion can be expressed as the product of 4 factors:

- some (uninteresting) rational numbers;
- some rational number related to the position of the anticanonical class in the effective cone;
- the cardinality of a Galois cohomology group;
- the volume of (an adequate part of) the adelic space $X(\mathbb{A})$ with respect to a suitable measure.

Assume $K_X = \mathcal{O}_X(-d)$ for some integer d .

- for each prime $p \leq \infty$, p -adic measure defined by suitable local gauge forms: since $K_X \sim \mathcal{O}_X(-d)$, one can take a meromorphic differential form ω with $\text{div}(\omega) = -dH_0 \cap X$ and set

$$\tau_p = |\omega|_p \left(\frac{|x_0|_p}{\max(|x_0|_p, \dots, |x_n|_p)} \right)^d$$

- find suitable convergence factors λ_p — Peyre takes $\tau_p = L_p(1, \text{Pic}(\bar{X}))^{-1}$;
- define τ as the absolutely convergent product $L^*(1, \text{Pic}(\bar{X})) \prod_p (\lambda_p \tau_p)$.

EXAMPLE: PEYRE'S MEASURE FOR \mathbb{P}^n

For $X = \mathbb{P}^n$, $d = n + 1$ — Homogeneous coordinates $[x_0 : \cdots : x_n]$

On the chart $x_0 = 1$ and $|x_j| \leq 1$ for all j , one has

$$\tau_p = \frac{|dx_1 \cdots dx_n|_p}{\max(|x_0|_p, \dots, |x_n|_p)^d} = |dx_1|_p \cdots |dx_n|_p.$$

One has $\tau_\infty(\mathbb{P}^n(\mathbb{R})) = (n+1)2^n$.

For p prime, τ_p is nothing but the canonical measure on

$\mathbb{P}^n(\mathbb{Q}_p) = \mathbb{P}^n(\mathbb{Z}_p)$, hence:

$$\tau_p(\mathbb{P}^n(\mathbb{Q}_p)) = p^{-n} \#\mathbb{P}^n(\mathbb{F}_p) = 1 + p^{-1} + \cdots + p^{-n} = \frac{1 - p^{-n-1}}{1 - p^{-1}}.$$

Convergence factors: $\lambda_p = L_p(s, \text{Pic}(\overline{\mathbb{P}^n})) = (1 - p^{-s})^{-1}$; hence

$\lambda_p \tau_p(\mathbb{P}^n(\mathbb{Q}_p)) = 1 - p^{-n-1}$, and finally:

$$\tau(X(\mathbb{A})) = L^*(1, \text{Pic}(\overline{X})) \prod (\lambda_p \tau_p(\mathbb{P}^n(\mathbb{Q}_p))) = (n+1)2^n / \zeta(n+1).$$

This is essentially **SCHANUEL'S constant!**

HEIGHT BALLS AND THEIR VOLUMES

To study more general varieties, in any projective embedding, a convenient language is that of “adelic metrics” on line bundles:

Measures: If K_X is endowed with an adelic metric, one obtains a measure on $X(\mathbb{Q}_p)$ by glueing the local gauge forms $|\omega|_p / \|\omega\|_p$ where ω is a local non-vanishing top-diff. form.

Heights: for $\bar{L} = (L, (\|\cdot\|_p))$, define a height

$$H(P) = \prod_{p \leq \infty} \|f\|_p^{-1}, \quad P \in X(\mathbb{Q}), \quad 0 \neq f \in L(P).$$

More generally, for $f \in \Gamma(X, L)$, one can define a height function on the adelic space of $X_f := X \setminus \text{div}(f)$:

$$H_f(P) = \prod_{p \leq \infty} \|f(P_p)\|_p^{-1}, \quad P = (P_p) \in X_f(\mathbb{A}).$$

GOAL

Generalize the results recalled at the beginning (X_f = semi-simple group) and understand the volume of height balls, or the measure-theoretical behaviour of these height balls, when $T \rightarrow \infty$. This requires to define convergence factors for the (essentially affine) variety X_f .

Limit measure: of the type introduced by Peyre.

Also: real/ p -adic case — in all the products above, take only the corresponding factor.

Limit measure is supported on $\text{div}(f)(\mathbb{Q}_p)$.

Tool: analytic properties of the Mellin transforms

$$\int_{X_f(\mathbb{Q}_p)} \|f\| (P)^s d\tau_p(P), \quad \int_{X_f(\mathbb{A})} H_f(P)^{-s} d\tau(P).$$

Basic remark: The first one is a kind of “global” local Igusa zeta function, and the second one is an adelic version.

VOLUMES AND DISTRIBUTION OF REAL OR p -ADIC HEIGHT BALLS

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- 2 CONJECTURES OF BATYREV, MANIN AND PEYRE
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 - Measures: gauge forms vs. metrics
- 4 TAMAGAWA MEASURES, ADELIC HEIGHT BALLS, AND THEIR VOLUMES

MEASURES: GAUGE FORMS VS. METRICS

F , local field with a fixed Haar measure.

Let X be a smooth projective variety over a local field F , purely of dimension n ,

D divisor on X , $U = X \setminus |D|$.

How to define measures on U ?

- 1 a gauge form $\omega \in K_X(U) = \Omega_X^n(U)$ defines a measure $|\omega|$;
- 2 given a metric on the line bundle K_X , one may take local forms ω and define τ_X by glueing the measures $|\omega| / \|\omega\|$.
- 3 metric on $K_X(D)$: take local forms ω and glue the measures $\omega / \|\omega f_D\|$ to define $\tau_{(X,D)}$.

If X is an equivariant compactification of a group G , then $K_X = \mathcal{O}_X(-D)$ with $|D| = X \setminus G$,

pick $\omega \in K_X(G)$ a (left-)invariant differential form,

Then, $\text{div}(\omega) = -D$ and $\|\omega f_D\| = \text{cst} = 1$, hence $\tau_{(X,D)} = \frac{|\omega|}{\|\omega f_D\|} = |\omega|$ is a Haar measure on G .

HEIGHT BALLS

L effective divisor with support $|D|$, metric on L , f_L canonical section of $\mathcal{O}_X(L)$;

for $T > 0$, the inequality $\|f_L\| \leq 1/T$ defines a compact subset B_T in $U(F)$;

fix a metric on $K_X(D)$, gets a measure $\tau_{(X,D)}$ on $U(F)$,

volume of B_T : $V(T) = \tau_{(X,D)}(B_T)$.

DEFINITION

Mellin transform:

$$Z(s) = \int_0^\infty t^{-s} dV(t) = \int_{U(F)} \|f_L\|^s \tau_{(X,D)}.$$

Tauberian theory relates analytic properties of $Z(s)$ to the asymptotic behaviour of $V(T)$.

Then, the detailed asymptotic behaviour of $V(T)$ can be used to study the convergence of the probability measures $V(T)^{-1} \tau_{(X,D)}|_{B_T}$.

$$Z(s) = \int_0^\infty t^{-s} dV(t) = \int_{U(F)} \|f_L\|^s \tau_{(X,D)}.$$

Simple remark: $Z(s)$ is a kind of Igusa zeta function; it should not be looked as in integral on $U(F)$ but computed using the projective compactification $X(F)$.

Geometric assumption: over \bar{F} , the irreducible components D_α of D are smooth, and intersect transversally.

To simplify the exposition, I pretend here that the irreducible components of D over F are geometrically irreducible.

Consider the corresponding stratification (D_A) of X , hence D_A is a smooth subvariety of X of codimension $\#A$ (or empty).

Decomposition of divisors: $D = \sum \rho_\alpha D_\alpha$, $L = \sum \lambda_\alpha D_\alpha$.

LOCAL COMPUTATION “AT INFINITY”

We may use finitely many local charts on $X(F)$ to study the integral $Z(s)$.

The part “around” a point $x \in D_A^\circ(F)$ can be computed as

$$\int \prod_{\alpha} \|f_{D_{\alpha}}\| (x)^{\lambda_{\alpha} s - \rho_{\alpha}} d\tau_X(x) = \int \prod_{\alpha \in A} |x_{\alpha}|^{\lambda_{\alpha} s - \rho_{\alpha}} \varphi(x; y; s) \prod_{\mathfrak{d}} x_{\alpha} dy.$$

Then, the analytic properties of $Z(s)$ are completely clear and can be expressed in terms of the combinatorics of the stratification (D_A) .

For example, **abscissa of convergence** =

$$\max_{\substack{D_{\alpha}(F) \neq \emptyset \\ \lambda_{\alpha} > 0}} \frac{\rho_{\alpha} - 1}{\lambda_{\alpha}} ;$$

order of pole = numbers of α that achieve equality;

leading coefficient = sum of integrals over all minimal stratas A consisting of such α s.

TAMAGAWA MEASURES, ADELIC HEIGHT BALLS, AND THEIR VOLUMES

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- 4 TAMAGAWA MEASURES, ADELIC HEIGHT BALLS, AND THEIR VOLUMES
 - Definition of a Tamagawa measure
 - The adelic zeta function

LOCAL MEASURES AND CONVERGENCE FACTORS

Let X be a smooth projective variety over \mathbb{Q} ,

D effective divisor on X , $U = X \setminus |D|$.

Fix an adelic metric on $K_X(D)$; this defines measures $\tau_{(X,D),p}$ on $U(\mathbb{Q}_p)$ for all p .

To define a measure on $U(\mathbb{A})$ from these τ_p , one needs convergence factors λ_p such that the infinite product

$$\prod_p \lambda_p \tau_p(U(\mathbb{Z}_p))$$

converges absolutely.

Examples:

- X equivariant compactification of a semi-simple algebraic group G , $\tau_p = \text{Haar measure}$: one may take $\lambda_p = 1$;
- same, but G unipotent: $\lambda_p = 1$;
- same, but G is a torus, $\lambda_p = L_p(1, X^*(G_{\overline{\mathbb{Q}}}))$;
- if $D = \emptyset$, $\lambda_p = L_p(1, \text{Pic}(X_{\overline{\mathbb{Q}}}))^{-1}$.

A CHOICE OF CONVERGENCE FACTORS

Same notations: X, D, U, τ_p .

Geometric assumption: $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$.

Two free \mathbb{Z} -modules with a $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -action:

- $\Gamma(U_{\overline{\mathbb{Q}}}, \mathcal{O}_X^*)/\overline{\mathbb{Q}}^*$;
- $\text{Pic}(U_{\overline{\mathbb{Q}}})/\text{torsion}$.

Virtual $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -module: $\text{EP}(U) = \Gamma(U_{\overline{\mathbb{Q}}}, \mathcal{O}_X^*)/\overline{\mathbb{Q}}^* - \text{Pic}(U_{\overline{\mathbb{Q}}})/\text{torsion}$.

THEOREM

One may take $\lambda_p = L_p(1, \text{EP}(U))$ for all $p < \infty$.

DEFINITION

Global measure on $U(\mathbb{A})$:

$$\tau_{(X,D)} = L^*(1, \text{EP}(U))^{-1} \prod_{p < \infty} (L_p(1, \text{EP}(U)) \tau_{(X,D),p}) \tau_{(X,D),\infty}.$$

HEIGHT ON THE ADELIC SPACE $U(\mathbb{A})$

Let L be an effective divisor supported on $|D|$

f_L the canonical section of $\mathcal{O}_X(L)$

Choosing an adelic metric on $\mathcal{O}_X(L)$, one get a **height function on the adelic space** $U(\mathbb{A})$ defined by

$$H_L((x_p)) = \prod_{p \leq \infty} \|f_L(x_p)\|_p^{-1}.$$

PROPOSITION

The function H_L defines a continuous exhaustion of $U(\mathbb{A})$.

Height ball: compact subset $B_T = \{x \in U(\mathbb{A}); H_L(x) \leq T\}$.

Volume and zeta function:

$$V(T) = \tau_{(X,D)}(B_T), \quad Z(s) = \int_0^\infty t^{-s} dV(t) = \int_{U(\mathbb{A})} H_L(x)^{-s} d\tau_{(X,D)}(x).$$

PRODUCT OF p -ADIC ZETA FUNCTIONS

Modulo absolute convergence, one has

$$Z(s) = L^*(1, \text{EP}(U))^{-1} \prod_{p < \infty} (L_p(1, \text{EP}(U)) Z_p(s)) Z_\infty(s),$$

where for $p \leq \infty$,

$$Z_p(s) = \int_{U(\mathbb{Q}_p)} \|f_L(x)\|^s d\tau_{(X,D),p}(x)$$

is the p -adic Igusa zeta function described previously.

Recall the decomposition $D = \sum \rho_\alpha D_\alpha$, $L = \sum \lambda_\alpha D_\alpha$, as well as the transversality assumption on the D_α . Then, choosing compatibly adelic metrics on $\mathcal{O}(D_\alpha)$, one has:

$$Z_p(s) = \int_{X(\mathbb{Q}_p)} \prod_\alpha \|f_{D_\alpha}\|^{s\lambda_\alpha - \rho_\alpha} d\tau_{X,p}(x).$$

The previous computation in charts shows that it converges absolutely for $\Re(s) > \max((\rho_\alpha - 1)/\lambda_\alpha)$.

DENEFF'S FORMULA

For almost all p , one can give a precise formula for $Z_p(s)$ in terms of the reduction mod. p of the whole situation. This done by adapting the method used by **J. DENEFF** to prove that the degrees of the local zeta functions are bounded when one makes the prime number p vary.

PROPOSITION

For p large enough, and for any complex number s such that $\Re(s) > (\rho_\alpha - 1) / \lambda_\alpha$, one has

$$Z_p(s) = \sum_A p^{-\dim X} \#D_A^\circ(\mathbb{F}_p) \prod_{\alpha \in A} \frac{p-1}{p^{s\lambda_\alpha - \rho_\alpha + 1} - 1}.$$

This follows from the fact that the local computation in charts around $x \in D_A^\circ$ can be done using étale coordinates $((x_\alpha)_{\alpha \in A}, y)$ such that $\|f_{D_\alpha}\| = |x_\alpha|$, etc., and from the explicit computation:

$$\int_{p\mathbb{Z}_p} |x|^s dx = \sum_{n=1}^{\infty} \int_{p^n\mathbb{Z}_p \setminus p^{n+1}\mathbb{Z}_p} p^{-ns} dx = \sum_{n=1}^{\infty} p^{-ns-n} \left(1 - \frac{1}{p}\right) = p^{-1} \frac{p-1}{p^{s+1} - 1}.$$

MEROMORPHIC CONTINUATION OF AN EULER PRODUCT

Let $\sigma = \max(\rho_\alpha / \lambda_\alpha)$, let $A(L, D)$ be the set of α where equality is achieved

One can deduce from Denef's formula that for $\Re(s) > \sigma - \varepsilon$,

$$Z_p(s) = p^{-\dim X} \#U(\mathbb{F}_p) \prod_{\alpha \in A(L, D)} (1 + p^{-s\lambda_\alpha - \rho_\alpha + 1})$$

hence

- 1 $\prod L_p(1, \text{EP}(U)) Z_p(s)$ converges absolutely for $\Re(s) > \sigma$;
- 2 $\prod L_p(1, \text{EP}(U)) Z_p(s) \prod_{\alpha \in A(L, D)} (1 - p^{-s\lambda_\alpha + \rho_\alpha - 1})$ converges absolutely for $\Re(s) > \sigma - \varepsilon$.

Consequently, one obtains a meromorphic continuation of the form

$$Z(s) = L^*(1, \text{EP}(U))^{-1} \prod_p (L_p(1, \text{EP}(U)) Z_p(s)) = \varphi(s) \prod_{\alpha \in A(L, D)} \zeta(\lambda_\alpha(s - \sigma) + 1),$$

with

$$\varphi(1) = \prod_{\alpha \in A(L, D)} \zeta^*(1) \int_{X(\mathbb{A}_F)} \prod_{\alpha \notin A(L, D)} H_{D_\alpha}(x)^{\rho_\alpha - \sigma \lambda_\alpha} d\tau_X(x).$$

CONCLUSION

Let E be the divisor $\sigma L - D$; it is effective and its support is contained in $|D|$.

Let $t = \#A(L, D)$.

Some more calculation implies:

$$\lim_{s \rightarrow \sigma} Z(s)(s - \sigma)^t \prod_{\alpha \in A(L, D)} \lambda_\alpha = \int_{X(\mathbb{A})} H_E(x)^{-1} d\tau_X(x).$$

Using tauberian theorems, we deduce:

THEOREM

When $T \rightarrow \infty$,

- 1 one has the asymptotic expansion $V(T) = \tau_{(X, D)}(B_T) \sim B^\sigma (\log B)^{t-1} (\sigma(t-1)! \prod_{\alpha \in A(L, D)} \lambda_\alpha)^{-1} \int_{X(\mathbb{A})} H_E(x)^{-1} d\tau_X(x)$;
- 2 the probability measures $V(T)^{-1} \tau_{(X, D)}|_{B_T}$ equidistribute to the measure

$$\frac{1}{\int_{X(\mathbb{A})} H_E(x)^{-1} d\tau_X(x)} H_E(x)^{-1} d\tau_X(x).$$

- 1 The comparison of the geometric estimates we obtained with those of MAUCOURANT *et al.* is an exercise for specialists of wonderful compactifications of algebraic groups.
- 2 The question, raised by A. MACINTYRE at the beginning of the conference, of quantifier elimination in adèle rings prompted to me the analogy with results of COMTE/LION/ROLIN concerning the behaviour of volumes of parametrized subsets of \mathbb{R}^m .
Do they have an adelic analogue?