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The Complexity of the Hamiltonian Circuit Problem for Maximal Planar Graphs

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Abstract. We show that the following two problems are NP-complete. 1) Given a maximal planar graph, is it Hamiltonian ? 2) Given a planar graph, does it have a Hamiltonian planar spanning supergraph ?

Key words. algorithms, computational complexity, graph theory, Hamiltonian circuit, maximal planar graph, NP-completeness.

Preliminaries

- All the graphs discussed in this paper are simple.
- The graph theoretic notation we use is from [4].
- For a detailed exposition of computational complexity and NP-completeness, the reader is referred to [1] and [6].
- There is an extensive literature on the Hamiltonian Circuit problem, including many survey articles, e.g. [3,5,10].

Background

The Hamiltonian Circuit (HC) problem is that of deciding whether a given graph contains a Hamiltonian Circuit. For more than a century graph theorists tried to find a "nice" characterization of Hamiltonian graphs and failed. From the computational complexity point of view, this failure was explained when Karp [9] showed that HC belongs to the notorious class of NP-complete problems. The problems in this class are generally believed to be computationally intractable.

Due to its strange connection to the Four Color Conjecture (4CC), a special interest was given to the restriction of the HC problem to the class of planar graphs, and in particular to two subclasses of it: the class of cubic, 3-connected planar graphs, which we denote by $3P$, and the class of maximal planar graphs, denoted by MP.

In 1880, Tait [13] conjectured that every graph in $3P$ is Hamiltonian, and showed that if true, this conjecture implies that the 4CC is true. Tutte [14] proved him wrong in 1946, constructing the first non-Hamiltonian graph in $3P$. Later, a simple method for generating many such graphs was discovered by Kozyrev and Grinberg (reported in [12]). In 1975, Garey et al [7] proved that distinguishing the Hamiltonian from the non-Hamiltonian graphs in $3P$ is also NP-complete, i.e. this restriction of HP is as hard as the general problem.

Hamiltonian circuits in maximal planar graphs became an important objective after Whitney [15] showed in 1931 that the 4CC is true iff it is true for Hamiltonian graphs in MP. For that purpose he proved that every 4-connected graph in MP is Hamiltonian. Since every maximal planar graph is 3-connected, it was left to characterize Hamiltonian graphs in

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MP that have separating triangles. This problem and related ones are still (50 years later !) being attacked by researchers (e.g. [2,8]). The complexity of this problem was open for a long time. We prove it is also NP-complete.

In the following section we study the structure and properties of a special planar graph. Then we use this graph in proving our main theorem and deduce from it some interesting corollaries. We conclude with a few related open problems.

A planar graph

The basic building block of the construction in the next section is a 55-node graph N, whose structure and special properties we turn now to describe.

Consider the maximal planar graph K, which is shown in figure 1.

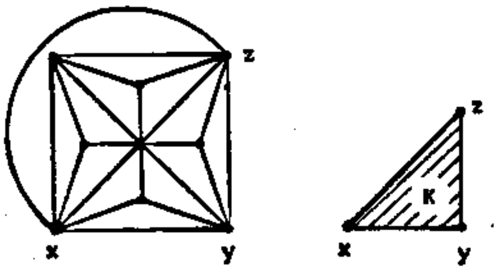

Figure 1
Graph K and abbreviation

Lemma 1: Let H be a graph which contains K as a vertex induced subgraph such that only the vertices x, y, z are incident on edges not in K. Then in any Hamiltonian circuit of H, the vertices of K appear consecutively.

Proof: A simple case analysis yields that any Hamiltonian circuit in H must appear locally in one of the six states given in Figure 2. □

Now take two copies of K and identify their z vertex. Complete the resulting graph to the maximal planar graph M given in Figure 3.

Lemma 2: Let H be a graph which contains M as a vertex induced subgraph, so that only vertices labeled x or z are incident on edges not in M. Then in any Hamiltonian circuit of H the vertices of M appear consecutively between the two vertices labeled x.

Proof: Let C be any Hamiltonian circuit in H. Note that H satisfies the conditions of lemma 1 w.r.t. each copy of K, so the vertices in each of the two copies appear consecutively in C. Since the vertex z is common to the two copies, and the vertices labeled y are not incident on any edge not in M, C must appear locally in M in the state given in Fig 4. □

Essential to the construction is the following observation which is a direct consequence of lemma 2.

Corollary 1: Let H satisfy the conditions in lemma 2, and let e be an edge touching a z-vertex of an M-subgraph of H. If e is not in this M-subgraph, then it cannot participate in any Hamiltonian circuit of H.

Finally, use three copies of M to construct the graph N, which is given in Fig 5. Note that except for the outer face of N, all faces are triangles.

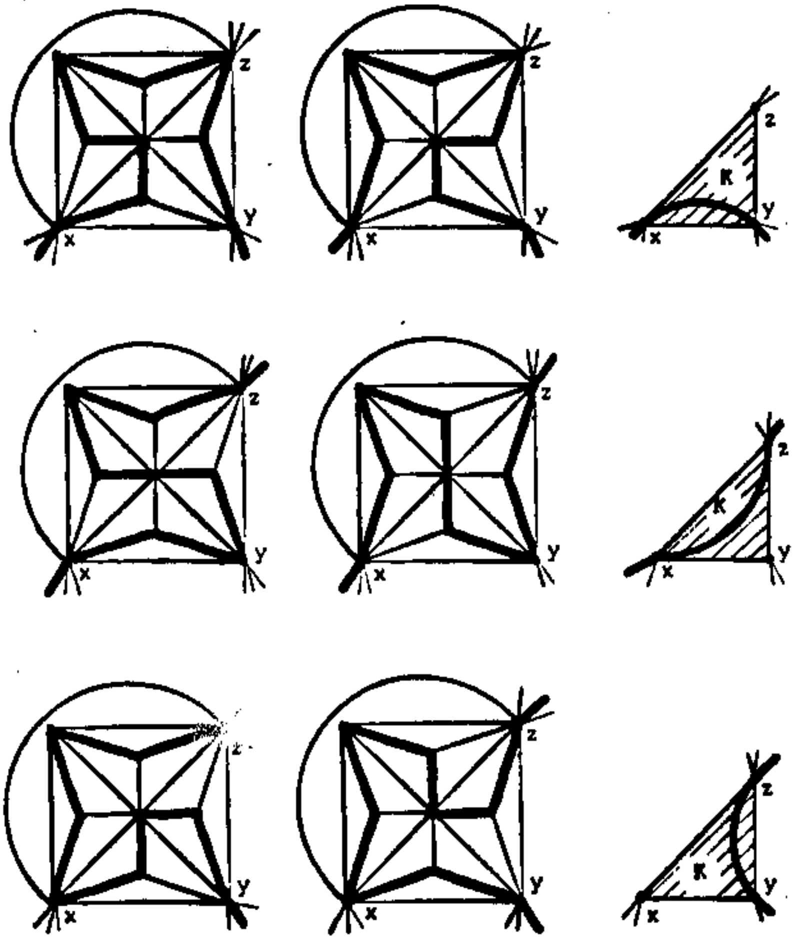

Figure 2
Possible local states and their abbreviation

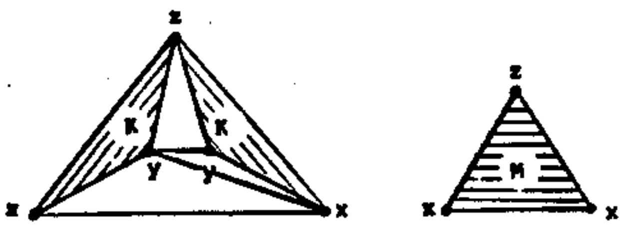

Figure 3
Graph M and abbreviation

Lemma 3: Let H be a graph containing N as a vertex induced subgraph s.t. only vertices labeled z or w lie on edges not in N. Then in any Hamiltonian circuit of H the vertices of N appear consecutively between two vertices labeled w.

Proof: Let C be any Hamiltonian circuit in H. Note that H satisfies the conditions of lemma 2 w.r.t. each copy of M. Therefore C appears locally in each copy of M as described in Figure 4. Since each x-vertex is adjacent only to u and exactly one w-vertex, it is easy to see that the only local state (up to rotation) in which C can appear in N is the one given in

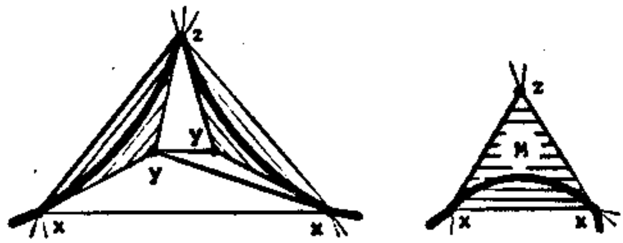

Figure 4
Local state and abbreviation

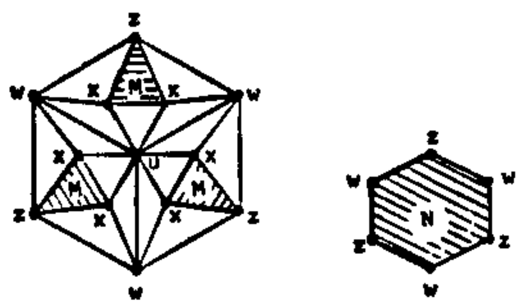

Figure 5
Graph N and abbreviation

Figure 6. □

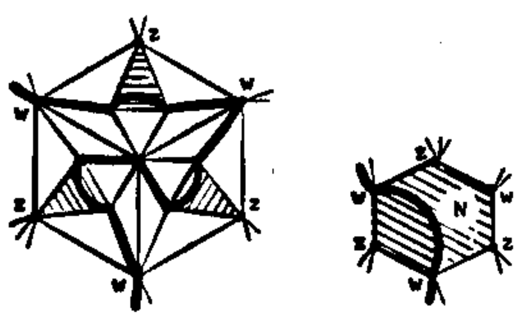

Figure 6
Local state and abbreviation

In a similar way to corollary 1 we may conclude:

Corollary 2: Let H satisfy the conditions of lemma 3, and let e be an edge touching a z-vertex of an N-subgraph of H. If e is not in this subgraph, then it cannot participate in any Hamiltonian circuit of H.

Lemma 4: There are exactly $2^6=64$ Hamiltonian paths in N between any two w-vertices.

Proof: This is immediate from the fact that there are six copies of K in N, each admits two Hamiltonian paths between its x and z vertices (Figure 2). □

Main results

Theorem: The Hamiltonian circuit problem for maximal planar graphs is NP-complete.

Proof: Garey, Johnson and Tarjan [7] proved, using a beautiful construction, that the Hamiltonian circuit problem for 3-connected cubic planar graphs (3PHC) is NP-complete. We give a polynomial time transformation from 3PHC to MPHC; Given a graph G in $3P$, an instance of 3PHC, we show how to construct a maximal planar graph G', such that G' has a Hamiltonian circuit if and only if G has one.

Let $G(V,E)$ be a graph in $3P$, an instance of 3PHC. Replace each vertex $v \in V$ by a copy of N, N_v, letting each of the three edges incident on v touch a different w-vertex in N_v. (Figure 7).

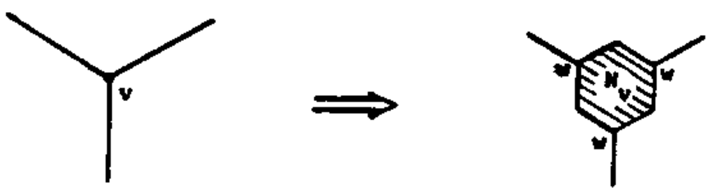

Figure 7

Let the resulting graph be $G_1(V_1,E_1)$. G_1 is planar, and for each face of size k in G we have a face of size $3k$ in G_1 (Figure 8a).

Connect the z-vertices inside each face of G_1 so that a k-cycle, "parallel" to the original one is created (Fig. 8b).

Now triangulate each face (in any way) to obtain the maximal planar graph $G'(V_1,E')$ (Figure 8c). Since every copy of N has 55 vertices, $|V_1| = 55|V|$, and since G' is maximal planar, $|E'| = 3|V_1|-6 = 165|V|-6$. Therefore, the transformation can be done in linear time in the size of G. To show that G is Hamiltonian if and only if G' is, we prove that each of them is Hamiltonian iff G_1 is.

Figure 9 explains how to construct a Hamiltonian circuit in G_1 from a given one in G and vice versa. Given a Hamiltonian circuit in G, we expend each vertex v to a Hamiltonian path in N_v between the two appropriate w-vertices. Conversely, since G_1 satisfies the conditions of lemma 3 w.r.t. each copy of N, every Hamiltonian circuit in G_1 appears locally in each N_v as in Figure 6. Therefore it enters and leaves each N_v exactly once via two of the three edges touching w-vertices of N_v. To obtain a Hamiltonian circuit in G, we simply shrink each N_v into one vertex, v.

To see that G_1 is Hamiltonian if and only if G' is, note that G_1 is a spanning subgraph of G' (the vertex set of both is V_1). This immediately shows that every Hamiltonian circuit in G_1 is a Hamiltonian circuit in G'. We constructed G' from G so that every edge in $E'-E_1$ touches a z-vertex of some N_v. By corollary 2, none of these edges may participate in any Hamiltonian circuit in G', and therefore every such circuit in G' is a Hamiltonian circuit in G_1. □

Corollary 3: Suppose we add the following restriction to the MPHC problem: every instance which *is* Hamiltonian must have an exponential (in the number of vertices) number of Hamiltonian circuits. Even then the problem remains NP-complete.

Proof: Using the notation of the last proof, every Hamiltonian circuit in $G(V,E)$ determines a Hamiltonian circuit in $G'(V_1,E')$ up to the Hamiltonian path between two w-

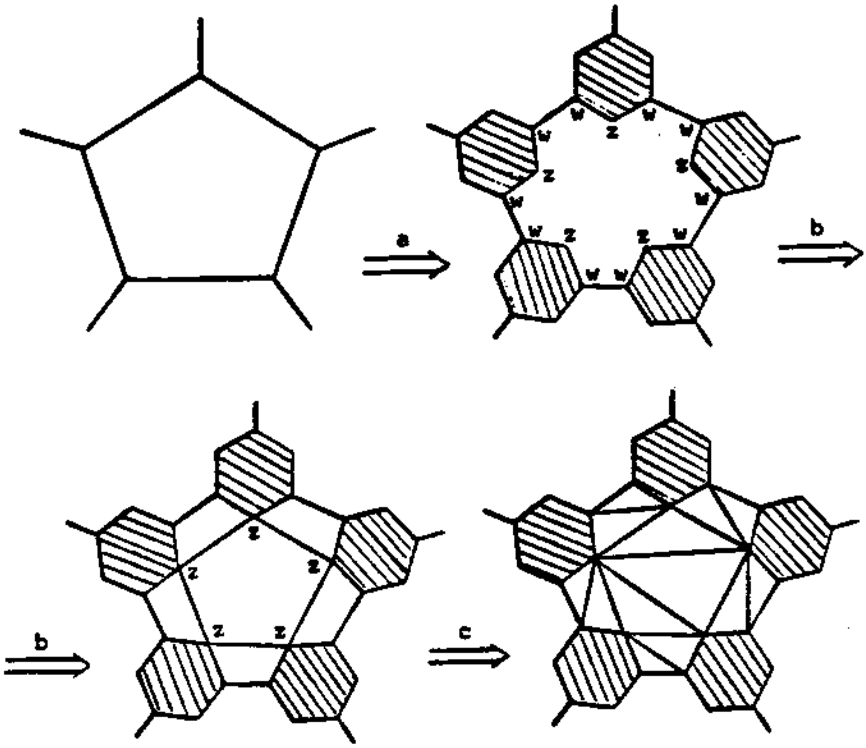

Figure 8

Figure 9

vertices in each copy of N, N_v, as shown in Figure 9. By Lemma 4, for each copy we have 64 choices for this Hamiltonian path. Therefore, every Hamiltonian circuit in G determines $64^{|V|}$ Hamiltonian circuits in G'. Since $|V_1| = 55|V|$, this number is exponential in $|V_1|$. □

Corollary 4: The Planar Hamiltonian Completion problem is defined as follows: Given a planar graph, does it have a spanning supergraph which is both planar and Hamiltonian. This problem is NP-complete.

Proof: It is sufficient to show that a subproblem is NP-complete. Suppose that all instances are maximal planar graphs. If G is such a graph, then the only planar spanning subgraph of G is G itself. Therefore the problem reduces to deciding whether G is Hamiltonian. But this is the MPHC problem. □

Open problems.

1) The Hamiltonian circuit problem restricted to maximal planar graphs was shown to be NP-complete. On the other hand, it is easy to see that the 3-colorability problem (NP-complete for arbitrary planar graphs) is solvable in linear time for maximal planar graphs. In general, it may be interesting to consider this restriction on any problem which is NP-complete for planar graphs, e.g. Vertex Cover and Maximum Stable Set. Which of these problems are made easier (computationally) by the special structure of maximal planar graphs ?

2) Note that every graph in *MP* (except the triangle) has a dual in *3P* and vice versa. The Hamiltonian circuit problem restricted to either of these classes is NP-complete. Given a maximal planar graph, what is the complexity of deciding whether it *or* its dual is Hamiltonian ?

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